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The Shadow Operator Formalism for Conformal Algebra. Vacuum Expectation Values and Operator Products.

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1. – Introduction.

Conformal covariance [1-7] has recently become of physical interest after the success of the scaling hypothesis in deep inelastic electroproduction [8,9] and the theoretical arguments showing that the small-distance behaviour of operator products may be obtained from limits of zero mass [10,11].

In this note we present a set of general properties of conformal covariant theories. The formalism is entirely formulated in space-time and the requirement of conformal symmetry is simply achieved by adding to the Poincaré algebra plus dilatation a non-linear discrete operation, to be called co-ordinate inversion, R, defined on space-time as $Rx_\mu = x_\mu/x^2$ [1,12].

The formalism allows for an easy derivation of two-point and three-point correlation functions.

The two-point correlation function for any local tensor operator is shown to be strictly related to the matrix transformation of the local operator under the action of R.

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(1) H. A. KASTRUP: *Phys. Rev.*, **142**, 1060 (1966); **143**, 1041 (1966).
(2) G. MACK: *Nucl. Phys.*, **5** B, 499 (1968).
(3) G. MACK and A. SALAM: *Ann. of Phys.*, **53**, 174 (1969).
(4) J. WESS and D. GROSS: *Phys. Rev. D*, **2**, 753 (1970).
(5) S. FERRARA, R. GATTO and A. F. GRILLO: *Nucl. Phys.*, **34** B, 349 (1971).
(6) A. A. MIGDAL: *Phys. Lett.*, **37** B, 98 (1971).
(7) G. MACK and I. T. TODOROV: Trieste preprint IC/71/139 (to be published).
(8) J. D. BJORKEN: *Phys. Rev.*, **179**, 1547 (1967).
(9) See: H. W. KENDALL: in *Proceedings of the 1971 International Symposium on Electron and Photon Interactions* (to be published).
(10) K. WILSON: *Phys. Rev.*, **179**, 1499 (1969); *Phys. Rev. D*, **2**, 1473 (1970).
(11) K. SYMANZIK: *Commun. Math. Phys.*, **23**, 49 (1971).
(12) E. J. SCHREIER: *Phys. Rev. D*, **3**, 980 (1971).

An identity, satisfied by the conformal invariant three-point function [13,14], allows us to derive a projection operator formalism which enables one to solve, in a closed form, the constraints of full conformal covariance on the operator-product expansion of any two tensors.

We stress that the above results strictly apply to local fields transforming according to irreducible representations of the conformal algebra of space-time which are induced by particular representations of the stability subalgebra at $x = 0$ [3,5].

It is known that the interesting irreducible representations of the stability subalgebra (in particular all finite-dimensional ones) are essentially of two types:

a) with $K_\lambda = 0$;

b) with $K_\lambda^p = 0$,

the latter being finite-dimensional for p integer (K_λ are the generators of special conformal transformations).

The representations of type *a*) are only those which behave irreducibly also with respect to the Lorentz algebra plus dilatations.

2. – The co-ordinate inversion operation.

Let us consider the Poincaré algebra+dilatations on space-time, generated by $M_{\mu\nu}$, D and P_μ, and let us define a discrete (nonlinear) operation, defined on co-ordinates as follows (a part from a sign):

$$Rx_\mu = \frac{x_\mu}{x^2} \equiv \left(\frac{1}{x}\right)_\mu , \qquad R^2 = I . \tag{2.1}$$

Then it is easy to show that

$$RM_{\mu\nu}R = M_{\mu\nu} , \qquad RDR = -D , \tag{2.2}$$

$$RP_\mu R = K_\mu , \tag{2.3}$$

where K_μ generate the four-parameter Abelian subalgebra of the special conformal transformations, which act on the co-ordinates as

$$x_\mu^c = \frac{x_\mu + c_\mu x^2}{1 + 2c\cdot x + c^2 x^2} . \tag{2.4}$$

From (2.1), (2.2), (2.3) and (2.4) it follows that covariance (invariance) under the conformal algebra is equivalent to covariance (invariance) under the Poincaré algebra+ +dilatation and R. Moreover, we remark that on space-time it is easier, and therefore convenient, to impose covariance under the operator R rather than under the complicated action of the transformation (2.4); so it is natural to investigate the action of this oper-

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([13]) M. D'Eramo, G. Parisi and L. Peliti: *Lett. Nuovo Cimento*, **2**, 878 (1971).
([14]) S. Ferrara and G. Parisi: Frascati preprint LNF 72/1 and to appear in *Nucl. Phys.*

ation on conformal irreducible tensors. To this purpose we note that [12]

$$\frac{\partial}{\partial x_\mu} = \frac{1}{x^2} M_{\mu\nu}(x) \frac{\partial}{\partial(Rx_\nu)}, \tag{2.5}$$

where

$$M_{\mu\nu}(x) \equiv g_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}. \tag{2.6}$$

For arbitrary conformal tensor fields we have

$$RO_{\alpha_1\ldots\alpha_n}(x) = \left(\frac{1}{x^2}\right)^{l_n} M^{\beta_1}_{\alpha_1}(x) \ldots M^{\beta_n}_{\alpha_n}(x)\, O_{\beta_1\ldots\beta_n}\left(\frac{1}{x}\right). \tag{2.7}$$

We write some remarkable properties of the «metric» matrix [15] $M_{\mu\nu}(x)$:

$$M_{\mu\nu}(x) = M_{\mu\nu}\left(\frac{1}{x}\right), \tag{2.8}$$

$$M_{\mu\sigma}(x)\, M^\sigma_\nu(x) = g_{\mu\nu}, \tag{2.9}$$

$$M_{\mu\nu}(x)\, x^\nu = -x_\mu, \tag{2.10}$$

$$M_{\mu\nu}(x-y) = M^\varrho_\mu(x)\, M^\sigma_\nu(y)\, M_{\varrho\sigma}\left(\frac{1}{y} - \frac{1}{x}\right). \tag{2.11}$$

For a spin-$\frac{1}{2}$ field we have (apart from a possible phase)

$$R\psi_\alpha(x) = \left(\frac{1}{x^2}\right)^l S^\beta_\alpha(x)\, \psi_\beta\left(\frac{1}{x}\right), \tag{2.12}$$

where

$$S^\beta_\alpha(x) = x^\mu(\gamma_\mu)^\beta_\alpha(x^2)^{-\frac{1}{2}},$$

note that (2.8), (2.9) and (2.11) still hold for the metric $S_{\alpha\beta}(x)$.

3. – Shadow operators and projection operators.

In ref. [14] it was shown, using a vertex identity for the three-point conformal scalar function, that one can put

$$A(x)B(y) = \int d^4t \langle 0|A(x)B(y)O^*(t)|0\rangle O(t), \tag{3.1}$$

to single out the contribution of the representation induced from the operator $O(0)$ to the operator product $A(x)B(y)$. $O^*(x)$ stands for a conventional local operator of scale dimension $l^* = 4 - l$. We shall call $O^*(x)$ the «shadow operator» of $O(x)$ [16]. Equa-

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[15] D. G. Boulware, L. S. Brown and R. D. Peccei: *Phys. Rev. D*, **2**, 293 (1970).

[16] The conjugation property $l \leftrightarrow -l+4$ occurring here is quite analogous to the conjugation $L \leftrightarrow -L-1$ leading to twin Regge trajectories and to the conjugation $\nu \leftrightarrow -\nu$ (ν = four-dimensional angular momentum) leading to gemel Toller trajectories.

tion (3.1) suggests the definition of the following projection operator:

$$\mathscr{P}_l = \int \mathrm{d}^4 t O^*(t)|0\rangle\langle 0|O(t) \,, \tag{3.2}$$

which is conformal scalar of dimension zero. Note in fact that

$$R\mathscr{P}_l R^{-1} = \mathscr{P}_l \,, \tag{3.3}$$

since

$$RO(x) = (x^2)^{-l} O\left(\frac{1}{x}\right) . \tag{3.4}$$

The following properties hold:

$$\mathscr{P}_l \mathscr{P}_l = \mathscr{P}_l \,, \tag{3.5}$$

$$\mathscr{P}_l \mathscr{P}_{l'} = 0 ; \tag{3.6}$$

eq. (3.5) is a consequence of the conformal covariant statement for the vacuum expectation value of the product of $O(x)$ with its « shadow »

$$\langle 0|O^*(x) O(0)|0\rangle = \delta^4(x) \,, \tag{3.7}$$

and eq. (3.6) is a consequence of the orthogonality relation between two point functions ([17]). Equation (3.6) also holds for $l' = l^*$ in the sense of distributions.

The above considerations allow us to define, for any conformal irreducible tensor $O_{\alpha_1 \dots \alpha_n}(x)$, the projection operator

$$\mathscr{P}_{l_n, n} = \int \mathrm{d}^4 t O^*_{\alpha_1 \dots \alpha_n}(t)|0\rangle\langle 0|O^{\alpha_1 \dots \alpha_n}(t) \,, \tag{3.8}$$

where

$$\langle 0|O^*_{\alpha_1 \dots \alpha_n}(t) O_{\beta_1 \dots \beta_n}(0)|0\rangle = \delta^4(t) C_{\alpha_1 \dots \alpha_n ; \beta_1 \dots \beta_n} \,, \tag{3.9}$$

and $C_{\alpha_1 \dots \alpha_n ; \beta_1 \dots \beta_n}$ is a numerical tensor built out of the g's.

For example, for $n = 1$ one has $C_{\alpha_1 ; \beta_1} = g_{\alpha_1 \beta_1}$, for $n = 2$ one has

$$C_{\alpha_1 \beta_1 \alpha_2 \beta_2} = g_{\alpha_1 \alpha_2} g_{\beta_1 \beta_2} - \tfrac{1}{2}(g_{\alpha_1 \beta_1} g_{\alpha_2 \beta_2} + g_{\alpha_1 \beta_2} g_{\alpha_2 \beta_1}) \,.$$

From (3.8) and (3.9) we get the generalized Wilson expansion for any two local tensor operators

$$O_{\alpha_1 \dots \alpha_n}(x) O_{\beta_1 \dots \beta_m}(y) = \int \mathrm{d}^4 t \langle 0|O_{\alpha_1 \dots \alpha_n}(x) O_{\beta_1 \dots \beta_m}(y) O^*_{\gamma_1 \dots \gamma_j}(t)|0\rangle O^{\gamma_1 \dots \gamma_j}(t) \,, \tag{3.10}$$

where $O^*_{\gamma_1 \dots \gamma_j}(x)$ is the shadow of $O_{\gamma_1 \dots \gamma_j}(x)$ and the right-hand side of eq. (3.10) is the contribution to the operator product of a particular irreducible representation of the conformal algebra.

([17]) S. FERRARA, R. GATTO and A. F. GRILLO: Frascati preprint LNF 71/79 (to appear in *Springer Tracts*); S. FERRARA, R. GATTO, A. F. GRILLO and G. PARISI: *Phys. Lett.*, **38** B, 333 (1972).

Expansion (3.10) holds everywhere in space-time if exact conformal symmetry is assumed. However, in presence of symmetry breaking, it holds at the leading order near the light-cone and reduces to [17]

$$(3.11)\qquad O_{\alpha_1\dots\alpha_n}(x)\,O_{\beta\,\dots\beta_n}(y) \underset{(x-y)^2\to 0}{\widetilde{}} (x^2)^{-\frac{1}{2}(l_n+l_m-l_j+n+m+j)}\,x_{\alpha_1}\dots x_{\alpha_n}x_{\beta_1}\dots x_{\beta_m}x^{\gamma_1}\dots x^{\gamma_j}\cdot \cdot\int_0^1 \mathrm{d}u\, f^j_{nm}(u)\,O_{\gamma_1\dots\gamma_j}(ux)$$

with

$$f^j_{nm}(u) = u^{-\frac{1}{2}(l_n-l_m+l_j+j+m-n)-1}(1-u)^{\frac{1}{2}(l_m-l_n+l_j+j+n-m)-1}\,.$$

The manifest covariance of (3.10) is a consequence of the transformation law (2.7).

Note that the projection operator (3.8) allows one to write down the contribution of a general « skeleton graph » to the general n-point function [14]

$$(3.12)\qquad \langle 0|A_1^{\{J_1\}}(x_1)\dots A_n^{\{J_n\}}(x_n)|0\rangle\,.$$

This is achieved by inserting the appropriate projection operators $\mathscr{P}_{l_n,n}$ and it is completely equivalent to writing a multiple operator expansion like (3.10) for any chosen configuration. This contribution of a skeleton graph to the n-point function (3.12) is therefore of the form

$$(3.13)\qquad \int \mathrm{d}^4t_1\dots \mathrm{d}^4t_{n-3}\langle 0|A_1^{\{J_1\}}(x_1)\,A_2^{\{J_2\}}(x_2)\,O_1^{\{\alpha_1\}}(t_1)|0\rangle\langle 0|O_{1\{\alpha_1\}}(t_1)\,A_3^{\{J_3\}}(x_3)\cdot \cdot O_2^{*\{\alpha_2\}}(t_2)|0\rangle|\dots\langle 0|O_{n-3}^{\{\alpha_{n-3}\}}(t_{n-3})\,A_{n-1}^{\{J_{n-1}\}}(x_{n-1})\,A_n^{\{J_n\}}(x_n)|0\rangle\,.$$

4. – Propagators and vertices.

The covariance of two- and three-point functions under the operation R allows to write them down in a compact form for any (integer) spin.

Note in fact that for spin 1 we can write [17]

$$(4.1)\qquad \langle 0|J_\mu(x)\,J_\nu(y)|0\rangle = c_1[(x-y)^2]^{-l}\,M_{\mu\nu}(x-y)$$

as a consequence of eq. (2.11).

It is easy to verify that the generalization of eq. (4.1) to any spin is

$$(4.2)\qquad \langle 0|O_{\alpha_1\dots\alpha_n}(x)\,O_{\beta_1\dots\beta_n}(y)|0\rangle = c_n[(x-y)^2]^{-l_n}M_{\alpha_1\beta_1}(x-y)\dots M_{\alpha_n\beta_n}(x-y) - (\text{traces})\,.$$

In a similar way one can obtain the general form for the three-point function

$$\langle 0|O_{\alpha_1\dots\alpha_n}(x)\,B(y)\,C(z)|0\rangle\,.$$

In fact, the conformal covariant solution for $n=1$ can be written as

$$(4.3)\qquad \langle 0|O_\alpha(x)\,B(y)\,C(z)|0\rangle = $$

$$= [(y-z)^2]^{-\frac{1}{2}(l_B+l_C+1-l_1)}[(y-x)^2]^{-\frac{1}{2}(l_1+l_B-l_C-1)}[(x-z)^2]^{-\frac{1}{2}(l_1+l_C-l_B-1)}\left[\left(\frac{1}{z-x}\right)_\alpha + \left(\frac{1}{x-y}\right)_\alpha\right]\,.$$

The generalization to higher spin is ($\tau_n = l_n - n$)

$$\langle 0|O_{\alpha_1\ldots\alpha_n}(x)\,B(y)\,C(z)|0\rangle = [(y-z)^2]^{-\frac{1}{2}(l_B+l_C-\tau_n)}[(y-x)^2]^{-\frac{1}{2}(\tau_n+l_B-l_C)}\cdot$$
$$\cdot[(x-y)^2]^{-\frac{1}{2}(\tau_n+l_C-l_B)}\left[\left(\frac{1}{z-x}\right)_{\alpha_1}+\left(\frac{1}{x-y}\right)_{\alpha_1}\right]\cdots\left[\left(\frac{1}{z-x}\right)_{\alpha_n}+\left(\frac{1}{x-y}\right)_{\alpha_n}\right]-(\text{traces})\,. \tag{4.4}$$

The conformal covariance of (4.1) can again be verified using the properties of the matrix $M_{\alpha\beta}(x)$.

Note that the requirement of covariance under R gives the selection rule already derived in ref. [17].

Finally we give the operator-product expansion for two conformal scalars. From (3.10) and (4.4) one has

$$A(x)\,B(y) = \sum_{n,\tau_n}[(x-y)^2]^{-\frac{1}{2}(l_A+l_B-\tau_n^*)}\int \mathrm{d}^4 t[(t-x)^2]^{-\frac{1}{2}(\tau_n^*+l_A-l_B)}\cdot$$
$$\cdot[(t-y)^2]^{-\frac{1}{2}(\tau_n^*+l_B-l_A)}\left[\left(\frac{1}{t-y}\right)_{\alpha_1}+\left(\frac{1}{x-t}\right)_{\alpha_1}\right]\cdots\left[\left(\frac{1}{t-y}\right)_{\alpha_n}+\left(\frac{1}{x-t}\right)_{\alpha_n}\right]O^{\alpha_1\ldots\alpha_n}(t)\,, \tag{4.5}$$

where $\tau_n^* = 4 - l_n - n$, and we recover the expansion in the form given by BONORA, SARTORI and TONIN in ref. [18].

Note that the expansion (4.5), as shown by BONORA in ref. [19], is equivalent to the expansion of ref. [20] obtained by using the isomorphism with the group $O_{4,2}$ on the cone in six-dimension.

([18]) L. BONORA, G. SARTORI and M. TONIN: Padova preprint IFPTH 6/71, to appear in *Nuovo Cimento*, A.
([19]) L. BONORA: *Lett. Nuovo Cimento*, **3**, 548 (1972).
([20]) S. FERRARA, R. GATTO and A. F. GRILLO: *Lett. Nuovo Cimento*, **2**, 1363 (1971).