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G. Parisi and F. Zirilli: ANOMALOUS DIMENSIONS IN ONE DIMENSIONAL QUANTUM FIELD THEORY. -

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G. Parisi and F. Zirilli[o]: ANOMALOUS DIMENSIONS IN ONE DIMENSIONAL QUANTUM FIELD THEORY[x]. -

ABSTRACT. -

We study the short distances behaviour of the Green's functions of two operators in a soluble one dimensional model of quantum field theory with adimensional coupling constant. Not integer power behaviour comes out and the leading terms of the Wilson's expansion of two operators at short distances are determined.

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I. - INTRODUCTION. -

The behaviour of Green's functions at short distances is one of the most interesting and controversial problems in quantum field theory[1].

The first fundamental step towards a deeper understanding of the problem was done by K. Wilson[2]. He suggested that the product of two operators satisfies the following asymptotic expansion at short distances:

$$\phi(x)\,\phi(0) \simeq \Sigma_n\, \xi_n(x)\, O_n(0) \tag{1}$$

where the functions $\xi_n(x)$ become singulars when x goes to zero.

In free field theory these functions are integer powers, in perturbation theory this simple behaviour is destroyed by the appearance of terms of the form $g\log(x^2)$.

K. Wilson suggested that the logarithms may sum themselves to a power which may not be integer: this happens in the soluble two dimensional Thirring model[3]. The exponent of the power is dependent from the renormalized coupling constant.

In this work we prove that similar patologies arise also in a very simple one dimensional model of quantum field theory.

We study the Lagrangian:

$$= \int dt \left\{ \frac{1}{2}\left[\dot{\phi}(t)\right]^2 - \frac{m^2}{2}\left[\phi(t)\right]^2 - g\left[\phi(t)\right]^{-2} \right\} \tag{2}$$

This is the only possible one with adimensional coupling constant: the lagrangian density must have dimension 1, the field ϕ has therefore dimension -1/2, so that the only possible scale invariant interaction is $g\phi^{-2}$.

This model can be solved using the equivalence between one dimensional quantum field theory and not relativistic quantum mechanics. The analogous quantum mechanical problem is the quantal oscillator: an harmonic oscillator with centrifugal potential g/x^2; its solution is known from the early days of quantum mechanics[4].

We find that for $g > -1/8$ the Wilson expansion for the product of two fields ϕ is:

$$(3) \qquad \phi(t)\,\phi(0) \simeq \phi^2(0) + t\,\dot\phi(0)\,\phi(0) + \frac{t^2}{2}\,\ddot\phi(0)\,\phi(0) + t^{a+2}\,\phi(0)^{-2a-1}\,\delta\left[\phi(0)\right]$$

where $a = \frac{1}{2}\sqrt{1+8g}$ and all the neglected terms in the Wilson's expansion are of the type $t^{\beta_i}\,O_i(0)$ where β_i is equal or to n or to $n+a+2$ with integer n.

We note that $\phi^{-2a-1}\,\delta[\phi]$ is not an operator but a not bounded sesquilinear form, which is defined on any finite linear combination of the energy eigenvectors.

This result is interesting because it shows that anomalous dimensions in the short distance behaviour are a very common phenomenum, which is present not only in relativistic quantum field theory, but also in the old not relativistic quantum mechanics.

In section II we rederive the analogous between one dimensional quantum field theory and not relativistic quantum mechanics. We write the solution of the quantal oscillator and we use it to compute the Wightman functions for the quantum field problem.

In section III we study the behaviour of the two point Wightman function at short distances and we find anomalous dimensions. We extend our study to the two point correlation function between two arbitrary energy eigenstates and finally arrive to the Wilson expansion (3).

In section IV we shortly discuss our results and make an interesting but unproven conjecture.

II. - SOLUTION OF THE MODEL. -

The Lagrangian of our one dimensional problem is (2) .

One can easily find the associated Eulero Lagrange equation:

$$\ddot{\phi}(t) = m^2 \phi(t) - 2 g \left[\phi(t)\right]^{-3} \tag{4}$$

and the Hamiltonian:

$$H = \frac{1}{2}\left[\pi(t)\right]^{-2} + \frac{1}{2} m^2 \left[\phi(t)\right]^2 + g\left[\phi(t)\right]^{-2} \tag{5}$$

ϕ and π satisfies the canonical commutation relations:

$$\left[\phi(t), \pi(t)\right] = -i \tag{6}$$

In order to find the eigenvectors of the Hamiltonian we use the standard rappresentation:

$$\phi(0) \longrightarrow x; \qquad \pi(0) \longrightarrow i \frac{d}{dx} \tag{7}$$

This trick reduces the problem to find the eigensolutions of the following differential equation:

$$\left[-\frac{d^2}{2dx^2} + \frac{1}{2} m^2 x^2 + \frac{g}{x^2}\right] \psi(x) = E\,\psi(x) \tag{8}$$

Eq. (8) is the Schrodinger equation for the quantual oscillator. The eigenfunctions and the eigenvectors of eq. (8) are:

$$E_n = m\left[2n+a+1\right] \; ;$$

(9)

$$\psi_n(x) = (4m)^{1/4}\left[\frac{\Gamma(n+1)}{\Gamma(a+n+1)}\right]^{-1/2}\left[m x^2\right]^{\frac{2a+1}{4}} \exp - \frac{m x^2}{2} \; L_n^a\left[m x^2\right]$$

where $a = \frac{1}{2}\sqrt{1+8g}$, n is a non negative integer and $L_n^a(x)$ are the Laguerre polinomials[(5)].

If $g \lesssim -1/8$ the spectrum of the hamiltonian is no more lower bounded: there exist no ground state and the physical meaning of the problem is lost.

The Wightman functions of the theory are:

(10) $$\langle 0|\, \phi(t_1) \ldots\ldots \phi(t_n)|0\rangle = \langle \psi_o|\, x(t_1) \ldots\ldots x(t_n)|\, \psi_o\rangle$$

where ψ_o is the ground state of (8) and x(t) is the position operator at time t in the Heisemberg rapresentation. We note that x(t) satisfies the same equation of motion (4) of $\phi(t)$.

If we define:

(11) $$x_{nm} = \langle \psi_n|x|\psi_m\rangle$$

we have from (10)

(12) $$\langle 0|\, \phi(t)\, \phi(0)|0\rangle = \sum_n e^{it E_n}|x_{on}|^2$$

Similar expression can easily derived for general N point Wightman functions.

III. - THE WILSON EXPANSION. -

In this section we compute the two point Wightman function and we study its behaviour at small t.

The formula for x_{on} is

$$x_{on} = \frac{-1}{2(\pi m)^{1/2}} \left[\Gamma(a+n+1)\,\Gamma(a+1)\,\Gamma(n+1)\right]^{-1/2} \Gamma\left(a+\frac{3}{2}\right)\Gamma\left(n-\frac{1}{2}\right) \tag{13}$$

In the limit $n \to \infty$ we find:

$$x_{on} \simeq \frac{-1}{2(\pi m)^{1/2}} \Gamma\left(a+\frac{3}{2}\right) \Gamma^{-1/2}(a+1)\, n^{-\left(\frac{a}{2}+\frac{3}{2}\right)} \tag{14}$$

This asymptotic behaviour of x_{on} implies that in the small t region:

$$\langle 0|\, \phi(t)\, \phi(0)\, |0\rangle = \frac{1}{m}\left[C_0 + C_1\cdot(mt) + C_2\cdot(mt)^2 + C_3\cdot(mt)^{a+2} + \ldots\ldots\right] \tag{15}$$

where, C_0, C_1, C_2, C_3 are g dependent constants and the neglected terms have higher power in t . It is interesting to observed that the power of the forth term is not integer and it is a continuous function of the coupling constant.

We now look for the two point correlation function between two arbitrary energy eigenstates s and r.

We need only to compute the asymptotic behaviour for large n of x_{sn}.

If we decompose

(16) $$L_s^a(x^2 m) = \sum_o^s {}_k\, b_k^s (x^2 m)^k$$

we find, using eq. (9), that:

(17) $$x_{sn} = \frac{1}{m^{1/2}} \left[\frac{\Gamma(s+1) \quad \Gamma(n+1)}{\Gamma(a+s+1)\,\Gamma(n+1+a)}\right]^{1/2} \sum_o^s {}_k \left\{ b_k^s \frac{\Gamma(a+k+\frac{3}{2})\,\Gamma(n-k-\frac{1}{2})}{\Gamma(n+1)\,\Gamma(-k-\frac{1}{2})} \right\} \simeq$$

$$\simeq \frac{1}{m^{1/2}} \Gamma^{1/2}(s+1)\,\Gamma^{-1/2}(a+s+1) \sum_o^s {}_k \left\{ \Gamma(s+k+\frac{3}{2})\, b_k^s\, n^{-a/2-3/2-k} \right\}$$

The terms proportional to b_k^s with $k \neq 0$ go faster to zero. The final result for small t is

(18) $$\langle r|x(t)\,x(0)|s\rangle = \frac{1}{m}\left[D_o^{r,s} + D_1^{r,s}\cdot(mt) + D_2^{rs}\cdot(mt)^2 + \right.$$

$$\left. + D_3 \left[\frac{\Gamma(s+1)\,\Gamma(r+1)}{\Gamma(a+s+1)\,\Gamma(a+r+1)}\right]^{1/2} b_o^s\, b_o^r\, (mt)^{2+a}\right.$$

Where $D_o^{r,s}$, $D_1^{r,s}$, $D_2^{r,s}$ are some constants dependent from g, r and s, but D_3 if function of only g. b_o^s can also defined as:

(19) $$b_o^s = \lim_{x \to 0} \frac{1}{(4m)^{1/4}} \frac{\Gamma(a+s+1)}{\Gamma(s+1)} (m x^2)^{-\frac{2a+1}{4}} \psi_s(x)$$

so that (18) is equivalent to:

(20) $$\langle r|x(t)\,x(0)|s\rangle \simeq \langle r|x^2(0)|s\rangle + t\,\langle r|\dot{x}(0)\,x(0)|s\rangle + \frac{t^2}{2}\langle r|\ddot{x}(0)\,x(0)|s\rangle +$$

$$+ t^{a+2} \frac{D_3}{2} \int \psi_2^+(x)\, \psi_s(x)\, x^{-2a-1}\, \delta(x) + \ldots$$

The coefficient of t^{a+2} can also interpreted as the mean value of the sesquilinear form $x^{-2a-1}\delta(x)$ between the states.

Eq. (20) can be rewritten in a operatiorial form and in this way we find the Wilson expansion (3) for the product of two fields.

We note that the operational form of the short distance singularities is mass indǐependent and the index of the power depends only from the adimensional coupling constant.

One can investigate the general form of the neglected terms, computing the exact two point correlation function. This can be done inserting in (18) the exact expression (17) for x_{rn}, not its asymptotic expansion.

One finds

$$\langle r|x(t)x(0)|s\rangle = \frac{e^{-irtm}}{m}\left[\frac{\Gamma(s+1)\Gamma(r+1)}{\Gamma(a+s+1)\Gamma(a+r+1)}\right]^{1/2}\frac{1}{\Gamma(1+a)}\sum_{o}^{r}{}_{k}\sum_{o}^{s}{}_{k'}\, b^{s}_{k'}\, b^{r}_{k} \tag{21}$$

$$\Gamma(a+k+\frac{3}{2})\Gamma(a+k'+\frac{3}{2})F(-k-\frac{1}{2}, -k'-\frac{1}{2}; 1+a; e^{im\cdot t})$$

Using the well known decomposition of the hypergeometric function[5] we arrive to:

$$\langle r|x(t)x(0)|s\rangle = \frac{e^{-irtm}}{m}\left[\frac{\Gamma(s+1)\Gamma(r+1)}{\Gamma(a+s+1)\Gamma(a+r+1)}\right]\sum_{o}^{r}{}_{k}\sum_{o}^{s}{}_{k'}\, b^{s}_{k'}\, b^{r}_{k}$$

$$\Big\{\Gamma(a+k+k'+2)F(-k-\frac{1}{2}, k'-\frac{1}{2}; -a-k-k'-1; 1-e^{imt})$$

$$+\left[1-e^{imt}\right]^{2+a+k+k'}\frac{\Gamma(-a-k-k'-2)\Gamma(a+k+\frac{3}{2})\Gamma(a+k'+\frac{3}{2})}{\Gamma(-k-\frac{1}{2})\Gamma(-k'-\frac{1}{2})}\times \tag{22}$$

$$\times\ F(a+k+\frac{3}{2}, a+k'+\frac{3}{2}; a+k+k'+3; 1-e^{imt})\Big\}$$

The first term generate short distances singularities with integer powers and the second term contains only powers of the form n+a+2. (The hypergeometric function is regular at the origine).

IV. - CONCLUSIONS. -

The results of our study one that in a one dimensional model of quantum field theory with adimensional coupling constant the fundamental field doesn't change dimension, but operators with anomalous dimension appear in the Wilson expansion of the product of two fields. The dimension of these operators is coupling constant dependent.

We find also a very simple expression for the leading anomalous term of the Wilson expansion.

In this line of work the next step is to study other models with singular potential and with dimensional coupling constant. Our feeling is that in this model too there are not integer powers in the short distances Wilson expansion, but the anomalous dimensions should not be coupling constant dependent. It would be very interesting to verify this conjecture.

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