Laboratori Nazionali di Frascati

LNF-72/44

S. Ferrara and G. Rossi: LIGHT-CONE SINGULARITIES AND LORENTZ POLES.

Estratto da: J. Math. Phys. 13, 499 (1972)

Light-Cone Singularities and Lorentz Poles

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(Received 8 October 1971)

The causal Meyer–Suura structure functions are projected into irreducible representations of the Lorentz group. A clarification of the connection between light-cone singularities and Lorentz poles is obtained: We find that in general a light-cone singularity of the type $1/(-x^2 + i\epsilon x_0)^\alpha$, in the operator product of the hadronic electromagnetic current, is built up from a sequence of Lorentz poles at $\lambda_n = 1 + \alpha - n$ whose residues are polynomials of order n in the virtual photon square mass.

1. INTRODUCTION

In this paper we study the connection between scaling properties (light-cone singularities) and Regge-like behavior (J-plane singularities) of the off-mass-shell Compton amplitude. Recently many authors[1] suggested that presence of light-cone singularities in the commutator of the hadronic electromagnetic current, which are responsible for the scaling properties of the structure functions in the Bjorken limit, could imply the existence of fixed poles in the off-mass-shell forward Compton amplitude. To study this phenomenon, it is necessary to develop the harmonic analysis with respect to the Lorentz group in configuration space, showing explicitly how Lorentz poles contribute to build up light-cone singularities.

In Ref. 2 the Wick-rotated imaginary part of the off-mass-shell forward Compton amplitude has been subjected to an $O(4)$ analysis and the connection between its Lorentz pole content and light-cone singularities has been investigated by performing a Sommerfeld–Watson transform. The authors were able to relate the behavior of the $O(4)$ partial waves at small distances to the scaling properties of the Compton amplitude. These techniques were used to study a wide class of light-cone singularities suggested by ladder models.

In this work we discuss two kinds of expansion of the Compton amplitude; the first one, which is relevant in the Bjorken limit, is given by an integral over all possible light-cone singularities. This expansion is more transparent in momentum space where it appears as an expansion in terms of homogeneous functions of the variables q^2, ν, i.e., over irreducible representations of the group of projective transformations on the complex variables q^2, ν.[3] The second one, relevant in the Regge limit, is obtained by projecting over the irreducible representations of the Lorentz group.

We find in general that an infinite number of Lorentz poles "conspire" to build up a light-cone singularity, more precisely, a term like $1/(-x^2 + i\epsilon x_0)^\alpha$ related to the sequence of Lorentz poles which are located at $\lambda_n = 1 + \alpha - n$ and whose residues are polynomials of order n in the virtual photon mass. The possible nonpolynomiality of the residues should be interpreted as an indication that an infinite sequence of light-cone singularities contribute in Regge limit.

We point out that these poles have nothing to do with conventional Regge poles, i.e., with the behavior of the "structure functions" $f_i(x \cdot p)$ in configuration space for large $x \cdot p$, but have a pure kinematical origin, in the sense that they reflect the presence of a "power type" singularity $(x_+^2)^{-\alpha}$ of the current commutator near the light cone. In particular, if these poles occur at integer points, because of the Regge signature factor, they cannot contribute to the imaginary part of the off-mass-shell Compton amplitude. However they do contribute for nonintegral values, so these singularities could be physically relevant as a manifestation of noncanonical light-cone singularities.

As is well known, such singularities would correspond to renormalization effects of dimension in the operator product of the two currents.

Finally we remark that the techniques we develop may also be useful for studying dynamical situations suggested by some ladder models, in which these kinds of singularities are realized.

Sections 2 and 3 are devoted to studying the properties of the integral transforms we are led to introduce in order to derive the previously mentioned results. In particular the connection of the expansions in momentum and configuration space (related by a Fourier transform) is given. In Sec. 4 the decomposition of a light-cone singularity in terms of Lorentz poles is carried out. The proofs of the main formulas we use are collected in the Appendix.

2. CONFORMAL TRANSFORM

Let us consider the functions[4]

$$V_1(q^2,\nu)=(1/q^2)[W_1(q^2,\nu)+(\nu^2/q^2)W_2(q^2,\nu)], \quad (2.1)$$

$$V_2(q^2,\nu)=-(1/q^2)W_2(q^2,\nu), \quad (2.2)$$

linear combinations of the structure functions $W_1(q^2,\nu)$, $W_2(q^2,\nu)$ defined by the equations

$$W_{\mu\nu}(q,p)=\frac{1}{(2\pi)^4}\int d^4x e^{-iq\cdot x}\langle p|[J_\mu^{\mathrm{el}}(x),J_\nu^{\mathrm{el}}(0)]|p\rangle$$
$$=-\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)W_1(q^2,\nu)+\left(p_\mu-\frac{\nu}{q^2}q_\mu\right)$$
$$\times\left(p_\nu-\frac{\nu}{q^2}q_\nu\right)W_2(q^2,\nu), \quad (2.3)$$

where $J_\mu^{\mathrm{el}}(x)$ is the hadronic electromagnetic current. For the Fourier transforms ($k=1,2$)

$$\begin{aligned}W_k^F(x^2,x\cdot p)&=\int d^4q e^{iq\cdot x}W_k(q^2,\nu),\\ V_k^F(x^2,x\cdot p)&=\int d^4q e^{iq\cdot x}V_k^F(q^2,\nu),\end{aligned} \quad (2.4)$$

the following relations hold:

$$W_1^F(x^2,x\cdot p)=-\Box V_1^F(x^2,x\cdot p)-P_\mu P_\nu\partial^\mu\partial^\nu V_2^F(x^2,x\cdot p),$$
$$W_2'^F(x^2,x\cdot p)=\Box V_2(x^2,x\cdot p). \quad (2.5)$$

Experimental data suggest the following asymptotic behavior:

$$\lim_{\substack{\nu\to\infty\\ \omega=-q^2/2\nu\ \text{fixed}}} V_k(q^2,\nu)\sim\nu^{\alpha_k-2}F_k(\omega) \quad (2.6)$$

with $\alpha_1=1$, $\alpha_2=0$.

A typical contribution $\nu^{\alpha-2}F(\omega)$ (we shall omit the index k from now on) to the structure functions in the Bjorken limit corresponds in the Fourier transform to a term of the type $[x^2]^{-\alpha}f(x\cdot p)$ near the light cone (where we indicate by the symbol $[x^2]^{-\alpha}$ the discontinuity of $1/(-x^2+i0)^\alpha$). The scaling behavior in momentum space has an interesting geometrical interpretation. Let us consider a function $V(q^2,\nu)$ defined on the (four-dimensional) complex affine plane q^2,ν. This space is homogeneous with respect to the group $SL(2,C)$ of projective transformations, i.e., it is equivalent to the quotient space $SL(2,C)/Z$ where Z is the group of matrices of the form $\left(\begin{smallmatrix}1 & z\\ 0 & 1\end{smallmatrix}\right)$. A representation of this group is defined on these functions as follows[5]:

$$T_g V(q^2,\nu)=V(\alpha q^2+\gamma\nu,\beta q^2+\delta\nu), \quad (2.7)$$

where $g=\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ with $\alpha\delta-\beta\gamma=1$.

We observe that the homogeneous functions play a special role in this space, as they form an irreducible subspace for the representation. The theory of harmonic analysis on homogeneous spaces gives the following expansion formula:

$$V(q^2,\nu)=\frac{1}{(2\pi)^2 i}\sum_{n=-\infty}^{+\infty}\int_{-i\infty}^{i\infty}d\alpha\, V(q^2,\nu;n,\alpha), \quad (2.8)$$

where $n_1=(n+\alpha)/2$, $n_2=(-n+\alpha)/2$, and $V(q^2,\nu;n,\alpha)$ is the Mellin transform of $V(q^2,\nu)$ defined as follows:

$$V(q^2,\nu;n,\alpha)=\tfrac{1}{2}\int d\eta d\bar\eta\,\eta^{-n_1}\bar\eta^{-n_2}V(\eta q^2,\eta\nu). \quad (2.9)$$

Equation (2.8) is only valid for L^2 functions, while Eq. (2.9) can be used to analytically continue the Mellin transform in α so that, for non-square-summable functions the expansion formula reads

$$V(q^2,\nu)=\frac{1}{(2\pi)^2 i}\sum_{n=-\infty}^{+\infty}\int_C d\alpha\, V(q^2,\nu;n,\alpha), \quad (2.10)$$

where C is a suitable path in the complex α plane.

If we introduce the Fourier transform as

$$V^F(x^2,u)=\int d^4q e^{iq\cdot x}V(q^2,\nu), \qquad u=x\cdot p, \quad (2.11)$$

then from the previous analysis we obtain in configuration space the expansion formula

$$V^F(x^2,u)=\frac{1}{(2\pi)^2 i}\sum_n\int_C d\alpha\, V^F(x^2,u;n,\alpha), \quad (2.12)$$

where

$$V^F(x^2,u)=\tfrac{1}{2}\int d\eta d\bar\eta\,\eta^{n_1}\bar\eta^{n_2-2}V^F(\eta x^2,u). \quad (2.13)$$

If we perform the sum over n, we get

$$V^F(x^2,u)=\frac{1}{2\pi i}\int_C d\alpha\,\hat V^F(x^2,u;\alpha), \quad (2.14)$$

with the definition

$$\hat V^F(x^2,u;\alpha)=\int_0^\infty dt\,t^{\alpha-1}V^F(tx^2,u). \quad (2.15)$$

Using the causality of the Fourier-transformed structure functions, we can write

$$V^F(x^2,u)=\frac{1}{2\pi i}\int_C d\alpha[x^2]^{-\alpha}f(u;\alpha), \quad (2.16)$$

where we have introduced the "conformal transform" of $V^F(x^2,u)$:

$$f(u;\alpha)=-\frac{1}{2i\sin\pi\alpha}\int_0^\infty d\sigma\,\sigma^{\alpha-1}V^F(\sigma,u) \quad (2.17)$$

and

$$[x^2]^\alpha=\mathrm{disc}\,[1/(-x^2+i0)^\alpha]=-2i\sin\pi\alpha(x_+^2)^{-\alpha}.$$

We have called the transform defined by Eq. (2.17) "conformal transform" to emphasize that a term like $[x^2]^{-\alpha}f(x\cdot p;\alpha)$ is related in the corresponding light-cone operatorial expansions to an infinite set of tensor operators $O_{\alpha_1\cdots\alpha_n}(0)$ classified according to a ladder of irreducible representations of the conformal algebra whose spectrum is given by the eigenvalues of the nonvanishing Casimir operator:

$$2M_{\mu\nu}M^{\mu\nu}+2P_\lambda\cdot K^\lambda-2D^2+8iD$$
$$=4n(n-\alpha-1)+2\alpha^2-8,$$

and $M_{\mu\nu}, P_\lambda, K_\lambda, D$ are the generators of the conformal algebra.[6]

In momentum space we can write the expansion

$$V(q^2, \nu) = \frac{1}{2\pi i} \int_C d\alpha\, \nu^{\alpha-2} F(\omega; \alpha), \qquad (2.18)$$

where the scaling function is expressed in terms of the conformal transform $f(u; \alpha)$ as

$$F(\omega, \alpha) = \frac{1}{(2\pi)^2} 2^{-\alpha} e^{-i\pi\alpha/2} \frac{\sin\pi\alpha}{\Gamma(\alpha)} \int_0^\infty du\, e^{i\omega u} u^{1-\alpha} f(u; \alpha). \qquad (2.19)$$

To obtain the complete diagonalization of the expansions (2.16), (2.18), we define the Mellin transform with respect to the u variable of $f(u; \alpha)$;

$$f(\alpha, \tau) = \int_0^\infty du\, u^{\tau-1} f(u; \alpha), \qquad (2.20)$$

so in configuration space we have

$$V(x^2, u) = \frac{1}{(2\pi i)^2} \int_C d\alpha \int_{C-i\infty}^{C+i\infty} d\tau [x^2]^{-\alpha} u^{-\tau} f(\alpha, \tau), \qquad (2.21)$$

and in momentum space

$$V(q^2, \nu) = \frac{1}{(2\pi)^4} \int_C d\alpha \int_{C-i\infty}^{C+i\infty} d\tau\, 2^{2-2\alpha} e^{-i\pi\alpha/2} e^{-i\pi\tau/2} \times \frac{\sin\pi\alpha}{\Gamma(\alpha)} \Gamma(2-\alpha-\tau)(-q^2)^{\alpha-2} \omega^\tau f(\alpha, \tau), \qquad (2.22)$$

where we have used the relation

$$F(\omega, \alpha) = \frac{1}{(2\pi)^3} 2^{-\alpha} e^{-i\pi\alpha/2} \frac{\sin\pi\alpha}{\Gamma(\alpha)} \omega^{\alpha-2} \times \int_{C-i\infty}^{C+i\infty} d\tau e^{-i\pi\tau/2} \Gamma(2-\alpha-\tau) \omega^\tau f(\alpha, \tau). \qquad (2.23)$$

3. LORENTZ AND WEYL TRANSFORMS

In this section we will expand the amplitude in configuration space directly into irreducible representations of the Lorentz group. The complete diagonalization will be obtained in this case by means of the Weyl transform which is defined as the product of the Lorentz and Mellin transform in the x^2 variable. These transforms obviously commute. We start by projecting out the dependence of $\tilde{V}^F(x^2, \cosh\zeta_x) = V^F(x^2, x\cdot p)$ on $\cosh\zeta_x = (x\cdot p)/\sqrt{x^2}$ by performing the usual Lorentz transform[7]

$$\int_0^\infty \tilde{V}^F(x^2, \cosh\zeta_x) \mathfrak{D}_\lambda(\cosh\zeta_x) \sinh^2\zeta_x\, d\zeta_x = \tilde{V}_\lambda^F(x^2) \qquad (3.1)$$

as defined by Eq. (A1).

The Plancherel theorem gives

$$\tilde{V}^F(x^2, \cosh\zeta_x) = \frac{i}{\pi} \int_{-i\infty}^{i\infty} d\lambda \lambda^2 \tilde{V}_\lambda^F(x^2) \mathfrak{D}_{-\lambda}(\cosh\zeta_x), \qquad (3.2)$$

where the path has to be suitably shifted for non-L^2 functions. Possible behaviors of the Lorentz transform $\tilde{V}_\lambda^F(x^2)$ were studied in Ref. 2 in connection with simple structures suggested by ladder models. The corresponding expansion in momentum space is obtained, by computing the Lorentz transform of the Fourier kernel e^{-iqx} [see Eq. (A5)], by means of the formula

$$\tilde{V}_\lambda(q^2) = \int dR\, R^3 \frac{K_\lambda(R\sqrt{-q^2})}{R\sqrt{-q^2}} \tilde{V}_\lambda^F(R^2), \qquad (3.3)$$

where $K_\lambda(x)$ is a modified Bessel function of third kind (Hankel function) and $R = \sqrt{x^2}$. The inversion formula reads

$$V(q^2, \nu) = 4\pi i \int_C d\lambda \lambda^2 \tilde{V}_\lambda(q^2) \mathfrak{A}_\lambda(\nu, q^2), \qquad (3.4)$$

where $\mathfrak{A}_\lambda(\nu, q^2)$ is a second kind matrix element on the Lorentz group defined by Eq. (A7). We note that the partial wave $\tilde{V}_\lambda(q^2)$ defined by Eq. (3.3) could in principle have λ singularities originated by the Hankel function. This phenomenon is more transparent if we perform the complete diagonalization by means of the Mellin transform in the variable x^2. We define the Weyl transform as

$$\tilde{V}_{\lambda\rho}^F = \int\int \tilde{V}^F(x^2, \cosh\zeta_x) \mathfrak{D}_\lambda(\cosh\zeta_x) \times \sinh^2\zeta_x\, d\zeta_x (x^2)^{\rho-1} dx^2 \qquad (3.5)$$

according to Eq. (A9). We call it a Weyl transform since it performs the diagonalization with respect to irreducible representations of the Weyl group. The inversion formula is given by

$$\tilde{V}^F(x^2, \cosh\zeta_x) = \frac{1}{2\pi^2} \int d\lambda \lambda^2 \int d\rho (x^2)^{-\rho} \mathfrak{D}_\lambda(\cosh\zeta_x) \tilde{V}_{\lambda\rho}^F \qquad (3.6)$$

and in momentum space, via Eq. (A12), we have

$$V(q^2, \nu) = \frac{4}{\pi} \int d\lambda \int d\rho\, \lambda^2 2^{-2\rho} \Gamma(\tfrac{1}{2}\lambda - \rho + \tfrac{3}{2}) \times \Gamma(-\tfrac{1}{2} - \rho + \tfrac{3}{2})(-q^2)^{\rho-2} \mathfrak{A}_\lambda(\nu, q^2) \tilde{V}_{\lambda\rho}^F. \qquad (3.7)$$

We observe that the paths of integration in the λ and ρ planes are along the imaginary axis for L^2 functions and they have to be suitably shifted for non-L^2 functions.

The Weyl transform is simply connected to the conformal transform introduced in the previous section. To see this, we start by rewriting Eq. (2.21) as

$$V(x^2, u) = \frac{2}{(2\pi i)^2} \int\int d\alpha d\rho [x^2]^{-\rho} (\cosh\zeta_x)^{2\alpha-2\rho} f(\alpha, 2\rho - 2\alpha) \qquad (3.8)$$

after the change of variable $\tau = 2\rho - 2\alpha$. This connection clarifies the kinematical origin of the Lorentz-pole content of a light-cone contribution; this will be shown explicitly in the next section.

4. DECOMPOSITION OF A LIGHT-CONE SINGULARITY INTO LORENTZ POLE CONTRIBUTIONS

In this section we want to investigate the connection between the two integral representations for the causal structure functions $V^F(x^2, u)$, (2.21) and (3.6) [and their related momentum space versions (2.23) and (3.7)]. If we remember the structure of the Eq. (3.8), we see that the transformation function which relates the two expansions is nothing but the Lorentz transform of the power $(\cosh\zeta_x)^{2\rho-2\alpha}$. Its Lorentz transform is given by Eq. (A13). So we have in terms of irreducible Lorentz representations,

$$\tilde{V}^F(x^2, \cosh\zeta_x) = \frac{-1}{(2\pi i)^3} \int\int\int d\alpha\, d\rho d\lambda\, \lambda^2 \frac{2^{2\rho-2\alpha-1}}{\Gamma(2\rho - 2\alpha)} \times f(\alpha, 2\rho - 2\alpha) \Gamma(\tfrac{1}{2}\lambda - \tfrac{1}{2} + \rho - \alpha) \times \Gamma(-\tfrac{1}{2}\lambda - \tfrac{1}{2} + \rho - \alpha)[x^2]^{-\rho} \mathfrak{D}_\lambda(\cosh\zeta_x) \qquad (4.1)$$

and in momentum space, by means of Fourier transform,

$$V(q^2,\nu)=\frac{2}{(2\pi)^2 i}\iiint d\alpha d\rho d\lambda\lambda^2\frac{2^{-2\alpha}}{\Gamma(2\rho-2\alpha)}$$
$$\times f(\alpha,2\rho-2\alpha)(-2i\sin\pi\alpha)$$
$$\times\Gamma(\tfrac{1}{2}\lambda-\tfrac{1}{2}+\rho-\alpha)\Gamma(-\tfrac{1}{2}\lambda-\tfrac{1}{2}+\rho-\alpha)$$
$$\times\Gamma(\tfrac{1}{2}\lambda-\rho+\tfrac{3}{2})\Gamma(-\tfrac{1}{2}\lambda-\rho+\tfrac{3}{2})(-q^2)^{\rho-2}\,\mathcal{Q}_\lambda(\nu,q^2) \quad (4.2)$$

Comparing Eqs. (3.7) and (4.2), we have the connection between the two previously introduced conformal and Weyl transforms:

$$\tilde{V}^F_{\lambda\rho}=\frac{1}{8\pi i}\int d\alpha\frac{2^{2\rho-2\alpha}}{\Gamma(2\rho-2\alpha)}\Gamma(\tfrac{1}{2}\lambda-\tfrac{1}{2}+\rho-\alpha)$$
$$\times\Gamma(-\tfrac{1}{2}\lambda-\tfrac{1}{2}+\rho-\alpha)f(\alpha,2\rho-2\alpha)(-2i\sin\pi\alpha). \quad (4.3)$$

This result clearly means that a Weyl contribution is in general built up from an infinite sequence of light-cone singularities. In order to study the matching between light-cone singularities and Lorentz poles, we change the order of integration in the integral representation (4.2) and perform the λ-integration explicitly by means of Cauchy's theorem. We get

$$V(q^2,\nu)=\sum_{n=0}^{\infty}\frac{1}{\pi}\iint d\alpha d\rho\frac{(-1)^{n-1}}{(n-1)!}\frac{2^{-2\alpha}}{\Gamma(2\rho-2\alpha)}$$
$$\times(-2i\sin\pi\alpha)f(\alpha,2\rho-2\alpha)\Gamma(2\rho-2\alpha-1+n)$$
$$\times\Gamma(-\alpha+1+n)\Gamma(\alpha-2\rho+2-n)$$
$$\times(2\rho-2\alpha-1+2n)^2(-q^2)^{\rho-2}\mathcal{Q}_{2\rho-2\alpha-1+2n}(\nu,q^2)$$
$$+\sum_{n=0}^{\infty}\frac{1}{\pi}\iint d\alpha d\rho\frac{(-1)^{n-1}}{(n-1)!}\frac{2^{-2\alpha}}{\Gamma(2\rho-2\alpha)}(-2i\sin\pi\alpha)$$
$$\times f(\alpha,2\rho-2\alpha)\Gamma(-\alpha+1+n)$$
$$\times\Gamma(-\alpha+2\rho-2-n)\Gamma(-2\rho+3+n)$$
$$\times(-2\rho+3+2n)^2(-q^2)^{\rho-2}\mathcal{Q}_{-2\rho+3+2n}(\nu,q^2), \quad (4.4)$$

where we have taken the contribution of the Lorentz poles at $\lambda=2\rho-2\alpha-1+2n$, $\lambda=-2\rho+3+2n$ and closed the integration path in the right half-plane, where the functions $\mathcal{Q}_\lambda(\nu,q^2)$ go to zero.

To see the Lorentz pole content of a light-cone contribution, we assume that the ρ integration can be performed by an appropriate deformation of the integration path in such a way that the Mellin transform is analytic in the corresponding region of the ρ plane. This corresponds, in Regge pole language, to considering the Compton amplitude with true Regge poles subtracted.[1] Note that a conventional Regge pole would correspond to a pole in the Mellin transform and would give rise to a different behavior in q^2 of the residue function.

Performing the integral, we get

$$V(q^2,\nu)=\sum_{n,m=0}^{\infty}4\int d\alpha\frac{(-1)^{n-1}}{(n-1)!}\frac{(-1)^{m-1}}{(m-1)!}$$
$$\times\frac{2^{-2\alpha}\sin\pi\alpha}{\Gamma(-\alpha+2-n+m)}f(\alpha,-\alpha+2-n+m)$$
$$\times\Gamma(-\alpha+1+n)\Gamma(-\alpha+1+m)$$
$$\times(-\alpha+1+n+m)^2(-q^2)^{[(\alpha+2-n+m)/2]-2}$$
$$\times\mathcal{Q}_{-\alpha+1+n+m}(\nu,q^2)+\sum_{n,m=0}^{\infty}4\int d\alpha\frac{(-1)^{n-1}}{(n-1)!}$$
$$\times\frac{(-1)^{m-1}}{(m-1)!}\frac{2^{-2\alpha}\sin\pi\alpha}{\Gamma(-\alpha+2+n-m)}$$
$$\times f(\alpha,-\alpha+2+n-m)\Gamma(-\alpha+1+n)$$
$$\times\Gamma(-\alpha+1+m)(-\alpha+1+n+m)^2$$
$$\times(-q^2)^{[(\alpha+2+n-m)/2]-2}\mathcal{Q}_{-\alpha+1+n+m}(\nu,q^2), \quad (4.5)$$

where we have taken the contributions of the poles at $\rho=(-\alpha+2-n+m)/2$, $\rho=(2+\alpha+n-m)/2$ in the first and second integral of (4.5), respectively. The integral has been closed in such a way that the background goes to zero by moving the integration path at infinity.

Formula (4.5) can be rewritten as

$$V(q^2,\nu)=\sum_{n,m=0}^{\infty}4\int d\alpha\Bigg(\frac{f(\alpha,-\alpha+2-n+m)}{\Gamma(-\alpha+2-n+m)}$$
$$\times(-q^2)^{[(\alpha+2-n+m)/2]-2}+n\rightleftarrows m\Bigg)\frac{(-1)^{n-1}}{(n-1)!}$$
$$\times\frac{(-1)^{m-1}}{(m-1)!}2^{-2\alpha}\sin\pi\alpha\,\Gamma(-\alpha+1+m)$$
$$\times\Gamma(-\alpha+1+m)(-\alpha+1+n+m)^2$$
$$\times\mathcal{Q}_{-\alpha+1+n+m}(\nu,q^2). \quad (4.6)$$

To see the behavior of a light-cone singularity, we pick up a contribution to the integrand in (4.6). In the Regge limit we have

$$\underset{\nu\to\infty}{V_\alpha(q^2,\nu)}=\sum_{n,m=0}^{\infty}-4\Bigg(\frac{f(\alpha,-\alpha+2-n+m)}{\Gamma(-\alpha+2-n+m)}$$
$$\times(-q^2)^m+n\rightleftarrows m\Bigg)\frac{(-1)^{n-1}}{(n-1)!}\frac{(-1)^{m-1}}{(m-1)!}$$
$$\times 2^{-2\alpha}\sin\pi\alpha\,\Gamma(-\alpha+1+n)$$
$$\times\Gamma(-\alpha+1+m)(-\alpha+1+n+m)$$
$$\times e^{i\pi(2-\alpha+n+m)/2}\,\nu^{\alpha-2-n-m}, \quad (4.7)$$

where we have used the asymptotic behavior of the functions $\mathcal{Q}_\lambda(\nu,q^2)$ defined in Eq. (A7). In particular we observe that the Γ functions exactly cancel each other for $\alpha=$ negative integer, corresponding to the case of a derivative of a δ singularity on the light-cone. In particular the leading pole corresponding to the α-light-cone singularity goes as

$$\underset{\nu\to\infty}{V_\alpha(q^2,\nu)}\sim\text{const}\cdot\nu^{\alpha-2} \quad (4.8)$$

with residue independent of q^2. In general the residue of the nth nonleading poles is a polynomial in q^2 of order n. These results establish that a light-cone singularity $(x_+^2)^{-\alpha}$ in the current commutator corresponds to a sequence of poles at $J_n=\alpha-n$ whose residues are polynomial of order n in the photon square mass.

APPENDIX

In this appendix we recall some formula which we need in the text. We start by recalling that the causal structure function $V^F(x^2, x\cdot p) = \tilde{V}^F(x^2, \cosh\zeta_x)$ ($\cos\zeta_x = x\cdot p/\sqrt{x^2}$) can be considered, for fixed x^2, as a bicovariant function[8] defined over $SL(2, C)$ (the universal covering group of the Lorentz group). This means that it is a function $\tilde{V}(x^2, a)$, $a \in SL(2, C)$, which satisfies the covariance relation $\tilde{V}(x^2, a) = \tilde{V}(x^2, h_1 a h_2)$ for $h_1, h_2 \in SU(2)$. Its Lorentz transform is given by the formula

$$\int_{SL(2,C)} \tilde{V}^F(x^2, a)\mathfrak{D}^{0\lambda}_{0000}(a)d^6a = 4\pi^3 \int_0^\infty \tilde{V}^F(x^2, \cosh\zeta_x) \times \mathfrak{D}_\lambda(\cosh\zeta_x)\sinh^2\zeta_x d\zeta_x = 4\pi^3 \tilde{V}^F_\lambda(x^2), \quad \text{(A1)}$$

where

$$\mathfrak{D}_\lambda(\cosh\zeta_x) = d^{0\lambda}_{000}(\cosh\zeta_x) = (\sinh\lambda\zeta_x)/(\lambda \sinh\zeta_x) \quad \text{(A2)}$$

is a matrix element of an irreducible representation of the type $(0, \lambda)$. This function is called an elementary spherical harmonic of $SL(2, C)$. The Plancherel theorem gives

$$\tilde{V}^F(x^2, \cosh\zeta_x) = \frac{i}{\pi}\int_{-i\infty}^{i\infty} d\lambda\lambda^2 \tilde{V}^F_\lambda(x^2)\mathfrak{D}_{\lambda}(\cosh\zeta_x) \quad \text{(A3)}$$

for functions L^2 over $SL(2, C)$. For non-L^2 functions the integration path must be suitably shifted.

Computation of the Lorentz transform of the inverse Fourier transform

$$V(q^2, \nu) = \int e^{-iq\cdot x} V^F(x^2, x\cdot p)d^4x \quad \text{(A4)}$$

requires the knowledge of the Lorentz transform of the Fourier kernel $e^{-iq\cdot x}$. This has been evaluated in Ref. 9, and we get

$$\int e^{-iq\cdot x}\mathfrak{D}^{0\lambda}_{0000}(a)d^6a = 2\pi^2(1/R\sqrt{-q^2})K_\lambda(R\sqrt{-q^2})\mathfrak{D}_\lambda(\nu, q^2), \quad \text{(A5)}$$

where $K_\lambda(x)$ is a modified Bessel function of third kind[10] and

$$\mathfrak{D}_\lambda(\nu, q^2) = d^{0\lambda}_{000}(\nu/\sqrt{q^2}) \quad \text{(A6)}$$

is a matrix element of a $(0, \lambda)$ representation of $SL(2, C)$ continued to imaginary values of $\cosh\zeta_q$ by means of the formulas

$$\mathfrak{D}_\lambda(\nu, q^2) = \mathfrak{A}_\lambda(\nu, q^2) - \mathfrak{A}_{-\lambda}(\nu, q^2),$$

$$\mathfrak{A}_{-\lambda}(\nu, q^2) = \frac{1}{\lambda}\frac{1}{\sqrt{\nu^2 - q^2}}(\nu + \sqrt{\nu^2 - q^2})^\lambda \times (-q^2)^{(1-\lambda)/2} e^{i\pi(1-\lambda)/2}. \quad \text{(A7)}$$

At this point we have to make a remark: In principle the projection of the Fourier kernel $e^{-iq\cdot x}$, which acts from a homogeneous space of the kind $SL(2, C)/SU(2)$ ($x^2 > 0$) to one of the kind $SL(2, C)/SU(1,1)$ ($q^2 < 0$) receives contributions also from irreducible representations of the type $(M, 0)$. Nevertheless, as explained in Ref. 11, the Plancherel measure in the inversion formula in momentum space has a support $\Omega_{x,q}$ which is the intersection of the supports Ω_x, Ω_q of the Plancherel measures on the two different homogeneous spaces so

$$\Omega_{x,q} = \Omega_x \cap \Omega_q = (0, \lambda) \cap [(0, \lambda) + (M, 0)] = (0, \lambda).$$

We then obtain the Lorentz expansion in q-space in the form

$$V(q^2, \nu) = 2\pi i \int d\lambda\lambda^2 \tilde{V}_\lambda(q^2)\mathfrak{D}_\lambda(\nu, q^2), \quad \text{(A8)}$$

where

$$\tilde{V}_\lambda(q^2) = \int dR\, R^3 \{[K_\lambda(R\sqrt{-q^2})]/(R\sqrt{-q^2})\}\tilde{V}^F_\lambda(R^2)$$

and we have used the formula

$$\theta(x^2)\theta(x^0)d^4x = (1/\pi)R^3 dR d^3X,$$

where d^3X is the invariant measure over $SL(2, C)/SU(2)$ and is defined by the formula

$$d^3u d^3X = d^6a,$$

d^3u, d^6a being the Haar measures over $SU(2)$, $SL(2, C)$, respectively.

We observe that, to obtain complete diagonalization of the Lorentz expansion, we have to perform the Mellin transform in the variable $R = \sqrt{x^2}$. If we define the Weyl transform as

$$4\pi^3 \tilde{V}^F_{\lambda\rho} = \int \tilde{V}^F(x^2, \cosh\zeta_x)\mathfrak{D}_\lambda(a)(x^2)^{\rho-1}d^6a\, dx^2, \quad \text{(A9)}$$

we get

$$V^F(x^2, x\cdot p) = \frac{1}{2\pi^2}\int_{-i\infty}^{i\infty} d\lambda\lambda^2 \times \int_{C-i\infty}^{C+i\infty} d\rho (x^2)^{-\rho}\mathfrak{D}_\lambda(\cosh\zeta_x)\tilde{V}^F_{\lambda\rho} \quad \text{(A10)}$$

and the corresponding expansion in momentum space is

$$V(q^2, \nu) = \frac{2}{\pi}\int_{-i\infty}^{i\infty} d\lambda \int_{C-i\infty}^{C+i\infty} d\rho\lambda^2 \tilde{V}^F_{\lambda\rho}2^{-2\rho}(-q^2)^{\rho-2} \times \Gamma(\tfrac{1}{2}\lambda - \rho + \tfrac{3}{2})\Gamma(-\tfrac{1}{2}\lambda - \rho + \tfrac{3}{2})\mathfrak{D}_\lambda(\nu, q^2). \quad \text{(A11)}$$

To derive the last equation, we performed the Mellin transform of the function[12] $K_\lambda(x)$

$$\int \frac{dx}{x} x^{-2\rho+3}K_\lambda(x) = 2^{-2\rho+1}\Gamma(\tfrac{1}{2}\lambda - \rho + \tfrac{3}{2})\Gamma(-\tfrac{1}{2} - \rho + \tfrac{3}{2}). \quad \text{(A12)}$$

Finally we perform the Weyl transform of a light-cone singularity contribution. This is necessary to relate the expansions introduced in Secs. 2 and 3. We get[13]

$$\int (1/x^2)^\alpha f(x\cdot p, \alpha)\mathfrak{D}_\lambda(a)(x^2)^{\rho-1}d^6a\, dx^2 = 2\int dx\, x^{-2\alpha+2\rho-1}f(x, \alpha)\int(\cosh\zeta_x)^{2\alpha-2\rho}\mathfrak{D}_\lambda(a)d^6a$$
$$= \pi^3 f(\alpha, -2\alpha + 2\rho)[2^{2\rho-2\alpha}/\Gamma(2\rho - 2\alpha)] \times \Gamma(\tfrac{1}{2}\lambda - \tfrac{1}{2} + \rho - \alpha)\Gamma(-\tfrac{1}{2}\lambda - \tfrac{1}{2} + \rho - \alpha). \quad \text{(A13)}$$

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