COMITATO NAZIONALE PER L'ENERGIA NUCLEARE
Laboratori Nazionali di Frascati

LNF-72/38
11 Maggio 1972

S. Ferrara, A. F. Grillo and G. Parisi: CONFORMAL SYMMETRY AT LIGHT-LIKE DISTANCES AND ASYMPTOTIC BEHAVIOUR OF ELECTROMAGNETIC FORM FACTORS. -

Servizio Documentazione
dei Laboratori Nazionali di Frascati del CNEN
Casella Postale 70 - Frascati (Roma)

Laboratori Nazionali di Frascati del CNEN
Servizio Documentazione

LNF-72/38
11 Maggio 1972

S. Ferrara, A. F. Grillo and G. Parisi[x]: CONFORMAL SYMMETRY AT LIGHT-LIKE DISTANCES AND ASYMPTOTIC BEHAVIOUR OF ELECTROMAGNETIC FORM FACTORS. -

ABSTRACT. -

Using conformal symmetry for the operator product of the hadronic electromagnetic current and of the pion interpolating field, at light-like distances, we connect the behaviour of the structure function in the scaling limit to the on-mass shell asymptotic behaviour of the electromagnetic pion form factor. Our results are based on the analysis of dimension spectrum of local operators appearing in the operator product expansion and on the assumption of precocious scaling. As an application of our results, predictions on the asymptotic behaviour of the e.m. pion form factor as well as for the threshold behaviour of the structure function for the singly inclusive π-production in e^+e^- annihilation, are given.

1. - INTRODUCTION. -

It has been recently suggested[1] that there is a simple relation between the asymptotic behaviour of the on mass-shell form factor and the dimension of the interpolating field.

This result seems at first sight to be wrong: the interpolating field[2] can be chosen arbitrarily, provided it has a non-zero matrix element between the vacuum and one-particle states, and on-shell quantities cannot depend on its dimension.

(x) - This research has been sponsored in part by CNR-GNSM.

In this paper we present arguments suggesting that the asymptotic behaviour of the on-shell form factor depends on the dimension spectrum of the operators of the theory. We find that the old naive result is incorrect for too high values of the dimension.

If we assume that the spectrum of the dimensions is the one suggested by naive quark-model(1, 3) we find, as an example, a definite prediction for the asymptotic e.m. pion form factor.

To derive these results we assume that the Wilson(4) expansion for the product of two operators on the light-cone(5) is conformal(6) covariant and moreover that the structure functions for fields of low dimensions satisfy precocious scalin(3).

From the first assumption we derive a treshold theorem for the behaviour of the structure function near $\omega = 1$.

The second assumption enables us to saturate some mass dispersion relations with only low mass states and derive a relation between the asymptotic behaviour of the form factor and the treshold behaviour $\omega \to 1$, as suggested by Brandt and Preparata(7). The arbitrariness of the interpolating field does not affect our results. We are making the assumption of precocius scaling only for low dimension fields, not far arbitrary fields. This seems reasonable as far as high dimension fields excite many more states from the vacuum than low dimension fields, so that the asymptotic regime starts at a much higher energy and one cannot saturate a mass dispersion relation with only a few low mass states.

In section 2 we derive the general form of the off-mass shell form factor using conformal invariance on the light-cone.

In section 3 we relate the $\omega \simeq 1$ limit to the asymptotic behaviour of e.m. form factors.

Finally some formulas needed in the text are collected in the appendix.

2. - CONFORMAL ALGEBRA ON THE LIGHT-CONE AND OFF-MASS SHELL FORM FACTORS: TRESHOLD THEOREM. -

Let us consider the off-mass shell three-point structure function related to the electromagnetic vertex of the π-field,

$$(2.1) \qquad W_{\mu}(p, q) = \mathrm{Im} \int d^4x \, e^{iq \cdot x} \langle 0 | T(J_{\mu}(x) \phi(0)) | p \rangle$$

where $|p\rangle$ is a one-particle state ($p^2 = m_\pi^2$) and ϕ a local interpolating field, i.e. $\langle 0|\phi(0)|p\rangle \neq 0$.

We study now the structure function $W_\mu(p,q)$ in the scaling region which corresponds to the light-cone limit of the matrix element

(2.2)
$$\lim_{x^2 \to 0} \langle 0|T(J_\mu(x)\phi(0))|p\rangle$$

This matrix-element can be evaluated using a conformal covariant operator product expansion(6)

(2.3)
$$T(J_\mu(x)\phi(0)) \underset{x^2\to 0}{\sim} (\partial_\mu \partial^\lambda - g_\mu^\lambda \Box) \sum_n \left(\frac{1}{-x^2+i\varepsilon}\right)^{1/2(1-\tau_n)} \cdot x^{\alpha_1} \ldots x^{\alpha_{n-1}} \int_0^1 du\, f_n(u)\, O^n_{\lambda\alpha_1\ldots\alpha_{n-1}}(ux)$$

where

(2.4)
$$f_n(u) = c_n\, u^{\frac{d_n-1}{2}} \cdot (1-u)^{\frac{d_n-1^x}{2}}$$

and $\tau_n = l_n - n$, $d_n = l_n + n$, $l^x = 4 - l$.

Note that eq. (2.3) takes automatically account of current conservation.

$O^n_{\alpha_1\ldots\alpha_n}$ are a set of local tensor operators of given dimension l_n and irreducible under the conformal algebra i.e. (6)

(2.5)
$$\left[O^n_{\alpha_1\ldots\alpha_n}(0),\ K_\lambda\right] = 0$$

The advantage of the conformal covariant prescription is that the contributions of higher order tensors such as

$$\partial_{\alpha_{n+1}} \ldots \partial_{\alpha_m} O_{\alpha_1\ldots\alpha_n}(x)$$

is given from eq. (2.3) in terms of tensors of the type (2.5).

Inserting eq. (2.3) in (2.1) we get, after some computations shown in the Appendix

$$\lim_{\substack{-q^2 \to \infty;\ \nu = p\cdot q \to \infty \\ \omega = 2m_\pi \nu/-q^2 \text{ fixed}}} W_\mu(p,q) = p_\mu W_2(q^2,\nu) + q_\mu W_1(q^2,\nu) =$$

(2.6)

$$= (p_\mu q^2 - p\cdot q\, q_\mu) \sum_n C_n(q^2)^{1/2(1-\tau_n)-2} (\omega)^{1/2(1+\tau_n)} (\omega-1)^{1-2} \cdot$$

$$\cdot\ \theta(\omega-1)\, {}_2F_1\left(\frac{1-d_n}{2}, \frac{d_n - 1^x}{2} + 1; 1-1; 1-\frac{1}{\omega}\right)$$

If we consider now the treshold behaviour $\omega \to 1$, we get

(2.7) $$\lim_{\omega \to 1} W_\mu(p,q) \sim (p_\mu q^2 - p\cdot q\, q_\mu)(\omega-1)^{1-2} \sum_n C_n(q^2)^{1/2(1-\tau_n)-2}$$

as

$$\lim_{\omega \to 1} {}_2F_1\left(\frac{1-d_n}{2}, \frac{d_n - 1^x}{2} + 1; 1-1; 1-\frac{1}{\omega}\right) = 1$$

We note that the treshold behaviour of the structure function depends only by the dimension of the field $\phi(x)$.

3. - ASYMPTOTIC BEHAVIOUR OF ELECTROMAGNETIC FORM FACTORS. -

In this section we connect, using the hypothesis of precocious scaling, the treshold behaviour of the structure function to the asymptotic behaviour of the on-shell electromagnetic form factor.

Let us consider the treshold limit $\omega \to 1$ but $s = q^2(1-\omega) \to \infty$, then we get

(3.1) $$W_2(q^2, s) \underset{\omega \to 1}{\sim} s^{1-2} \sum_n C_n(q^2)^{1-\frac{1}{2}(1+\tau_n)}$$

so the electromagnetic form factor is given by

$$(3.2) \qquad \underset{q^2 \to \infty,\ \omega \to 1}{F}(q^2) = \mathrm{Im}\, G^{-1}(s)\, W_2(q^2, s) \sim \sum_n (q^2)^{1-\frac{1}{2}(l+\tau_n)}$$

where the imaginary part of the inverse propagator $G^{-1}(s)$ of the virtual π-meson exactly cancels the s-dependence in eq. (3.1). If we now assume that eq. (3.2) still holds near $s \sim m_\pi^2$ we get that eq. (3.2) defines the asymptotic behaviour of the on-shell e.m. form factor.

This result may be obtained, in a more refined form, using a mass-dispersion relation approach as pointed out by Brandt and Preparata[7].

From eq. (3.2) we see that the leading term is obtained by the contribution of the ϕ field itself only if $\tau_n > 1$. However, as the twist is not less than two we get that for $1 \leq l \leq 2$ the form factor decrease as

$$(3.3) \qquad \underset{q^2 \to \infty}{F_\pi}(q^2) \sim \left(\frac{1}{q^2}\right)^{l-1}$$

However if one assumes that the dimension spectrum is the same of the canonical quark model one finds $l = 3/2 + 3/2 = 3$ and $\tau_n = 2$ for the minimum twist (e.g. the Axial Current) so that $F_\pi(q^2)$ becomes

$$(3.4) \qquad \underset{q^2 \to \infty}{F_\pi}(q^2) \sim (q^2)^{-1/2} = (q^2)^{-3/2}$$

A consistent check of this approach can be given in the framework of the renormalization group.

Gell'Mann and Zachariasen[8] found that, if $\phi(x)$ is the fundamental field, then

$$(3.5) \qquad \lim_{q^2 \to \infty} \frac{q^2}{F(q^2)G(q^2)} = 1$$

where $F(q^2)$ and $G(q^2)$ are respectively the on-shell form factor and the propagator of the field, and we see that (3.5) is nothing but eq. (3.3).

However we agree with this result only for $1 \leq l \leq 2$. This is clear from the fact that eq. (3.5) is derived from a perturbative approach where $l-1$ is small.

As a final remark we note that the asymptotic behaviour of the π-electromagnetic form factor can be related, using duality[9] arguments, (or also the parton model[10]) to the treshold behaviour of the structure function $\overline{F}_2(\omega)$ for the inclusive process $e^+e^- \to \pi$ + anything.

In fact we get

$$\overline{F}_2(\omega) \underset{\omega \to 1}{\sim} (\omega-1)^{1+\tau_n-3} \tag{3.6}$$

In particular, in the canonical quark model eq. (3.6) becomes

$$\overline{F}_2(\omega) \underset{\omega \to 1}{\sim} (\omega-1)^{2} \tag{3.7}$$

ACKNOWLEDGEMENTS. -

We are indebted with R. Brandt, C. Orzalesi and E. Etim for discussions.

APPENDIX. -

Let us consider the T operator product expansion of a conserved vector current with a scalar field on the light-cone. Current conservation together with the general form of the operator product expansion of any two tensors on the light-cone, derived in ref. (11) gives the unique solution

$$T(J_\mu(x)\phi(0)) \underset{x^2 \to 0}{\sim} \sum_n (\partial_\mu \partial^\lambda - g_\mu^\lambda \Box)\left(\frac{1}{-x^2+i\varepsilon}\right)^{1/2(1-\tau_n)} \cdot$$

$$(A.1) \qquad \cdot\, x^{\alpha_1} \dots x^{\alpha_{n-1}} \int_0^1 du\, f_n(u)\, O^n_{\lambda\alpha_1 \dots \alpha_{n-1}}(ux)$$

when

$$f_n(u) = C_n\, u^{\frac{d_n - l^x}{2}} (1-u)^{\frac{d_n - l^x}{2}}$$

and

$$\tau_n = l_n - n, \quad d_n = l_n + n, \quad l^x = 4 - l.$$

The structure function in configuration space turns out to be

$$\langle 0 | T(J_\mu(x)\phi(0)) | p \rangle \underset{x^2 \to 0}{\sim} (p \cdot \partial\, \partial_\mu - p_\mu \Box) \sum_n \left(\frac{1}{-x^2+i\varepsilon}\right)^{1/2(1-\tau_n)} \cdot$$

$$(A.2) \qquad \cdot\, (x p)^{n-1} \int_0^1 du\, f_n(u)\, e^{iux \cdot p}$$

As eq. (A.2) is Lorentz-covariant we can put $p = (\underline{t}, 0)$ and write eq. (A.2) as

$$\langle 0 | T(J_\mu(x)\phi(0)) | p \rangle \sim \sum_n t^{n-1} \frac{\partial}{\partial t^{n-1}} (p \cdot \partial\, \partial_\mu - p_\mu \Box) \cdot$$

$$(A.3) \qquad \cdot \left(\frac{1}{-x^2+i\varepsilon}\right)^{1/2(1-\tau_n)} \int_0^1 du\, f_n(u)\, u^{1-n}\, e^{iux \cdot p}$$

From eq. (A.3) we can easily compute the Fourier transform and the structure function in q-space, in fact we get (apart some inessential multiplicative factors) in the limit $-q^2\to\infty$, $v = p\cdot q\to\infty$ but $\omega = -2m_\pi \nu/q^2$ fixed $(1\leq\omega\leq\infty)$

$$W_\mu(p,q) = \mathrm{Im}\int d^4x\, e^{iqx}\langle 0|T(J_\mu(x)\phi(0))|p\rangle \underset{x^2\to 0}{=}$$

$$= (p\cdot q\, q_\mu - p_\mu q^2)\sum_n t^{n-1}\frac{\partial^{n-1}}{\partial t^{n-1}}\int_0^1 du\, f_n(u)\, u^{1-n}\,\mathrm{disc}\left[\frac{1}{(q+up)^2}\right]^{2-\frac{1}{2}(1-\tau_n)} =$$

$$= (p\cdot q\, q_\mu - p_\mu q^2)\sum_n C_n (q^2)^{\frac{1}{2}(1-\tau_n)}(\omega)^{\frac{1}{2}(1-\tau_n)-2}\,.$$

$$\cdot\int_0^1 du\, u^{\frac{d_n-1}{2}}(1-u)^{\frac{d_n-1^x}{2}}\,\mathrm{disc}\left(\frac{1}{\omega}-u\right)^{\frac{1}{2}(1-d_n)-1} =$$

(A.4)

$$= (p\cdot q\, q_\mu - p_\mu q^2)\sum_n C_n (q^2)^{\frac{1}{2}(1-\tau_n)-2}(\omega)^{\frac{1}{2}(1-\tau_n)-2}\,\theta(\omega-1)\,\cdot$$

$$\cdot\int_0^x du\,(1-u)^{\frac{d_n-1}{2}}\,u^{\frac{d_n-1^x}{2}}\,(x-u)^{\frac{1}{2}(1-d_n)-1} = \qquad \left(x = 1-\frac{1}{\omega}\right)$$

$$= (p_\mu q^2 - p\cdot q\, q_\mu)(\omega-1)^{1-2}\sum_n C_n (q^2)^{\frac{1}{2}(1-\tau_n)-2}(\omega)^{-\frac{1}{2}(1+\tau_n)}\,\theta(\omega-1)\,\cdot$$

$$\cdot\, {}_2F_1\left(\frac{1-d_n}{2}, \frac{d_n-1^x}{2}+1; 1-1; 1-\frac{1}{\omega}\right).$$

whene the last integral has been computed as it is a Riemann-Liouville fractional integral(12).

REFERENCES. -

(1) - A. A. Migdal, Phys. Lett. 37 B, 98 (1971).
(2) - H. Araky and R. Haag, Comm. Math. Phys. 5, 77 (1967).
(3) - C. A. Orzalesi and P. D. Raskin, Lett. Nuovo Cimento 1, 331 (1971).
(4) - K. Wilson, Phys. Rev. 179, 1499 (1969).
(5) - R. Brandt and G. Preparata, Nuclear Phys. 27 B, 541 (1971); Y. Frishman, Phys. Rev. Letters 25, 966 (1970).
(6) - S. Ferrara, R. Gatto and A. F. Grillo, Phys. Lett. 36B, 124 (1971); Nuclear Phys. B 34, 349 (1971).
(7) - See ref. (5) and also G. Preparata, Rivista Nuovo Cimento, 1, 423 (1971).
(8) - M. Gell'Mann and Zachariasen, Phys. Rev. 123, 1065 (1961).
(9) - E. Bloom and F. Gilman, Phys. Rev. Letters 25, 1140 (1970).
(10) - S. Drell and T. M. Yan, Phys. Rev. Letters 24, 181 (1970).
(11) - S. Ferrara, R. Gatto and A. F. Grillo, Frascati preprint LNF-71/70 (1971); Phys. Rev., to be published.
(12) - Bateman manuscript Project, Tables of Integral Transform, Vol. 2.