

LNF-72/33
19 Aprile 1972

A. Turrin: STABILITY LIMITS OF THE PHASE OSCILLATIONS
IN A MICROTRON. -

LNF-72/33
19 Aprile 1972

A. Turrin: STABILITY LIMITS OF THE PHASE OSCILLATIONS
IN A MICROTRON. -

(Submitted to Nuclear Instr. and Meth. for publication)

ABSTRACT. --

The finite-difference equations existing in the literature for phase oscillations in a microtron are discussed first. It turns out that these equations are incorrect.

A technique is then developed to analyse the effect of computing the stability limits with the aid of these equations. It is shown that the resulting phase stable regions really constitute a wrong result.

Finally, a new treatment of the phase oscillations in a microtron is given. The stability limits are derived from the general theory of phase oscillations.

The results are:

- i) The range of equilibrium phase angles extends from 0° (corresponding to a peak voltage across the gap $V' = m_0 c^2 / e$) up to 90° (corresponding to very high peak voltages).
- ii) The bucket area is an increasing function of the peak voltage.
- iii) At high peak voltages stable oscillations are connected with large radial excursions from the synchronous orbit.

2.

1. -- ANALYSIS OF THE FINITE-DIFFERENCE PHASE EQUATIONS
EXISTING IN THE LITERATURE. -

The equations as given by C. Henderson, F. F. Heymann and R. E. Jennings⁽¹⁾ and as used currently to day are:

$$(1) \quad \Delta \varphi_{h+1} = \Delta \varphi_h + 2 \pi \nu \Delta \varepsilon_h$$

$$(2) \quad \Delta \varepsilon_{h+1} = \Delta \varepsilon_h + \frac{\cos(\varphi_s + \Delta \varphi_{h+1})}{\cos \varphi_s} - 1 .$$

Here $\varphi_s (> 0)$ is the equilibrium phase angle ($\cos \varphi_s = m_0 c^2 / (eV')$); $\Delta \varphi_h = \varphi_h - \varphi_s$ is the phase shift from the synchronous particle in going from the (h-1)th up to the hth traversal of the accelerating gap (V' is the peak voltage), and $\Delta \varepsilon_h$ denotes the energy deviation (in rest masses $m_0 c^2$ of the electron) from the energy of the synchronous particle measured at the hth exit of the gap. ν is an integer number given by the ratio $\Delta T_s / T_0$, where $\Delta T_s = \text{constant}$ is the increment of revolution period of the synchronous particle at each hth revolution and where T_0 is the period of the accelerating voltage.

Let us start from the general theory of phase oscillations expressed in differential terms, in order to try to reobtain equations like (1) and (2) from energy-gain-per-turn considerations.

The differential equations which are valid for any kind of constant gradient accelerator are given by D. Bohm and L. Foldy⁽²⁾ (B. and F.) and may be rewritten as follows:

$$(3) \quad \frac{d}{dt} \Delta \varphi = \nu h \omega_s \frac{\Delta E}{E_s}$$

$$(4) \quad \frac{d}{dt} \left(\frac{\Delta E}{\omega_s} \right) = \frac{eV'}{2\pi} (\cos \varphi - \cos \varphi_s).$$

Here h indicates we are referring to the h th orbit.

One finds equation (3) by taking equation (15) of B. and F., inserting in it the harmonic number νh (obviously, h acts as a counter of subsequent orbits), and putting the field index value equal to zero. Equation (4) may be written by taking the equation immediately before eq. (17) of B. and F., replacing $\sin\varphi$ by $\cos\varphi$, and neglecting the radiation loss term.

For the microtron, let us replace $E_s = h m_0 c^2$ in (3) and $eV = m_0 c^2 / \cos\varphi_s$ in (4).

One gets

$$(5) \quad \frac{d}{dt} \Delta\varphi = \nu \omega_s \Delta\varepsilon$$

$$(6) \quad \frac{d}{dt} \left(\frac{\Delta\varepsilon}{\omega_s} \right) = \frac{1}{2\pi \cos\varphi_s} (\cos\varphi - \cos\varphi_s).$$

Now, the only way to find equations like (1) and (2) is to replace $d/dt(\Delta\varepsilon/\omega_s)$ in (6) by $\frac{1}{\omega_s} \frac{d}{dt} \Delta\varepsilon$ and then to integrate (5) and (6) over one whole orbit. But the microtron is a machine in which the synchronous particle angular velocity ω_s decreases at every traversal of the accelerating gap. Therefore, putting $1/\omega_s$ before the d/dt operator in eq. (6) and then integrating over one complete revolution is a mistake. In fact, integrating correctly over the $(h+1)$ th orbit, one obtains

$$\frac{\Delta\varepsilon_{h+1}}{\omega_{s_{h+1}}} - \frac{\Delta\varepsilon_h}{\omega_{s_h}} = \frac{1}{\omega_{s_{h+1}}} \left(\frac{\cos(\varphi_s + \Delta\varphi_{h+1})}{\cos\varphi_s} - 1 \right)$$

instead of (2). Consequently, one cannot make calculations with equations like (1) and (2) without having errors present.

In the following section we shall accept the errors expected when using equations like (1) and (2). A straight-forward procedure that gives the stability limits as determined by (1) and (2) will be developed.

4.

2. -- THE STABLE, THE UNSTABLE FIXED-POINT AND THE SEPARATRIX -

First of all, we shall make three statements about the non-linear transformation given by (1) and (2):

i) In the phase plane $\Delta\varphi, \Delta\varepsilon$ (see fig. 1) there exist two points with the property that they remain unchanged during step-by-step iterations made by (1) and (2). These are the points

$$(7) \quad \begin{aligned} \Delta\varepsilon &= 0 \\ \Delta\varphi &= 0 \end{aligned} \quad \text{at} \quad \varphi_f = \pm \varphi_s \quad (\varphi_s > 0).$$

These points are called fixed points.

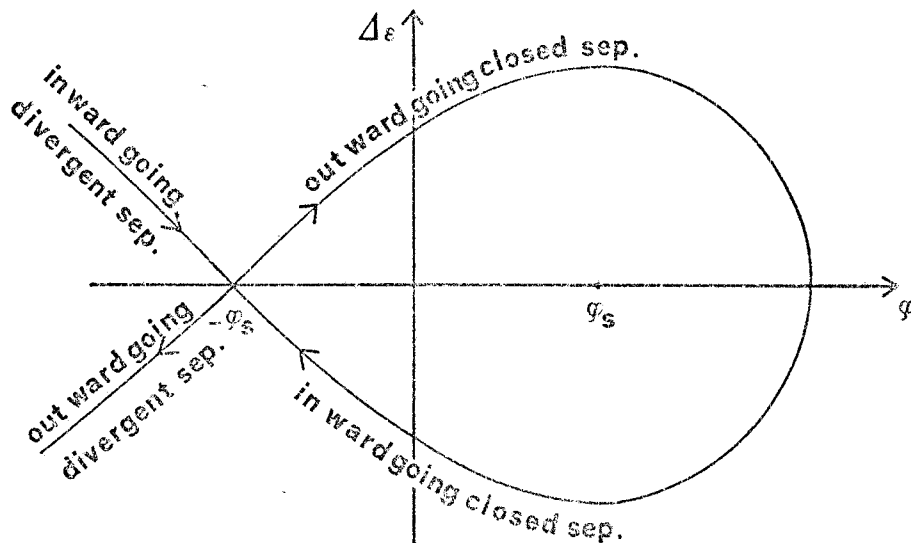


FIG. 1 - Schematic phase-plane diagram. The separatrix, which passes twice through the unstable fixed-point, encloses the stable region.

ii) The local properties of the fixed-points are deduced from the transfer matrix for the linearized motion in their neighbourhood. This matrix is the Jacobian matrix of (1) and (2) calculated at the fixed-points

$$\varphi_f = \pm \varphi_s.$$

$$(8) \quad J_f = \begin{vmatrix} \left(\frac{\partial \Delta \varphi_{h+1}}{\partial \Delta \varphi_h}\right)_f & \left(\frac{\partial \Delta \varphi_{h+1}}{\partial \Delta \varepsilon_h}\right)_f \\ \left(\frac{\partial \Delta \varepsilon_{h+1}}{\partial \Delta \varphi_h}\right)_f & \left(\frac{\partial \Delta \varepsilon_{h+1}}{\partial \Delta \varepsilon_h}\right)_f \end{vmatrix} = \begin{vmatrix} 1 & 2\pi\nu \\ -\operatorname{tg} \varphi_f & 1 - 2\pi\nu \operatorname{tg} \varphi_f \end{vmatrix}.$$

The value of the half-trace of J_f determines whether the motion in the neighbourhood of the corresponding fixed-point is stable ($\left|\frac{1}{2}\operatorname{tr} J_f\right| < 1$) or unstable ($\left|\frac{1}{2}\operatorname{tr} J_f\right| > 1$).

We have

$$(9) \quad \operatorname{tr} J_f = \frac{1}{2}\operatorname{tr} J_f = 1 - \pi\nu \operatorname{tg} \varphi_f,$$

and this shows that the point at $\varphi_f = +\varphi_s$ is stable, and the point at $\varphi_f = -\varphi_s$ is unstable.

iii) There exists a closed curve which separates the stable from the unstable phase-plane region. This curve is called separatrix.

The separatrix crosses itself at the unstable fixed-point.

The slopes of the separatrix at the unstable fixed-point are given by the eigenvectors of the Jacobian matrix (8) calculated for the unstable fixed-point. These slopes are given by

$$(10) \quad \frac{1}{\left(\frac{\partial \Delta \varphi_{h+1}}{\partial \Delta \varepsilon_h}\right)_f} \left[\frac{1}{2} \left[\left(\frac{\partial \Delta \varepsilon_{h+1}}{\partial \Delta \varepsilon_h}\right)_f - \left(\frac{\partial \Delta \varphi_{h+1}}{\partial \Delta \varphi_h}\right)_f \right] + \left[\left(\frac{1}{2}\operatorname{tr} J_f\right)^2 - 1 \right]^{1/2} \right] = \\ = \frac{1}{2} \operatorname{tg} \varphi_s + \frac{1}{2\pi\nu} \left[(1 + \pi\nu \operatorname{tg} \varphi_s)^2 - 1 \right]^{1/2}.$$

Knowing the coordinates of the unstable fixed-point and the slopes of the separatrix at this point, one can make a digital computation of the separatrix. This is done by following with the aid of eqs. (1) and (2) successive shadows of many representative points that are distributed on the outward-going eigenvector (sign plus in (10)) in the neighbourhood of the unstable fixed-point; and, conversely, by following those that are distributed on the inward-going eigenvector (sign minus in

6.

(10)) in the same neighbourhood with the aid of the inverse transformation of (1) and (2), i. e.

$$(11) \quad \Delta \varphi_h = \Delta \varphi_{h+1} - 2\pi\nu \Delta \varepsilon_h$$

$$(12) \quad \Delta \varepsilon_h = \Delta \varepsilon_{h+1} - \frac{\cos(\varphi_s + \Delta \varphi_{h+1})}{\cos \varphi_s} + 1.$$

This will be done in the following section.

3. -- DIGITAL CALCULATION OF THE SEPARATRIX BY MEANS OF EQUATIONS (1) AND (2) --

Using the procedure outlined in the previous section, we have calculated the separatrix, limiting ourselves to only one typical case already considered in reference⁽¹⁾, i. e. for $\varphi_s = 21.79$ degrees, which corresponds to a peak voltage across the gap $V' = 550$ kV. ($\nu = 1$).

The results, which are shown in fig. 2, are seen to be in quite agreement with the result of the digital computations reported in reference⁽¹⁾ (see fig. 4, curve C of the quoted reference, $V' = .938V_o$, i. e. $V_o = 560$ kV).

However, it is the shape of the separatrix that appears physically unacceptable. Liouville's theorem states that the representative points move in a phase-plane like an incompressible liquid. Thus, an inspection of fig. 3 readily reveals one fact: the matching between the outward and the inward-going separatrix at the right hand of φ_s fails, and after the occurrence of this failure we get a nonsensical motion. Mismatch occurs essentially because these two branches of the separatrix are not found to be symmetrical with each other about the φ -axis. This is another proof that finite-difference equations like (1) and (2) work incorrectly.

So, it is necessary to reformulate the whole problem, as will be done in the following section.

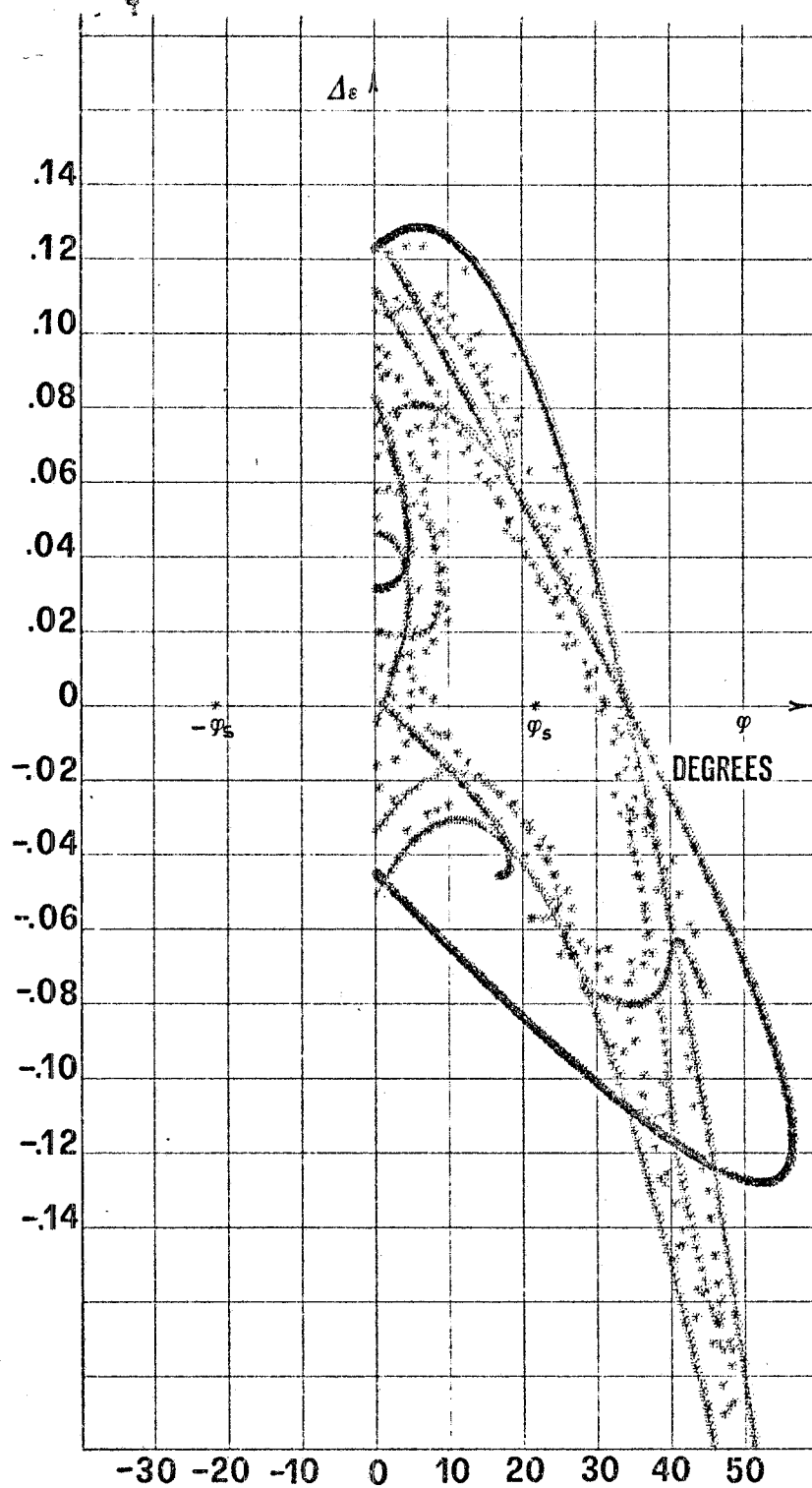


FIG. 2 - Successive positions of many representative points which started from the ingoing and outgoing closed separatrix in the neighbourhood of the unstable fixed point. $V^1 = 550$ kV ($V_0 = 560$ kV). The clean region around φ_s is the phase stable region obtained by the Authors of ref. (1).

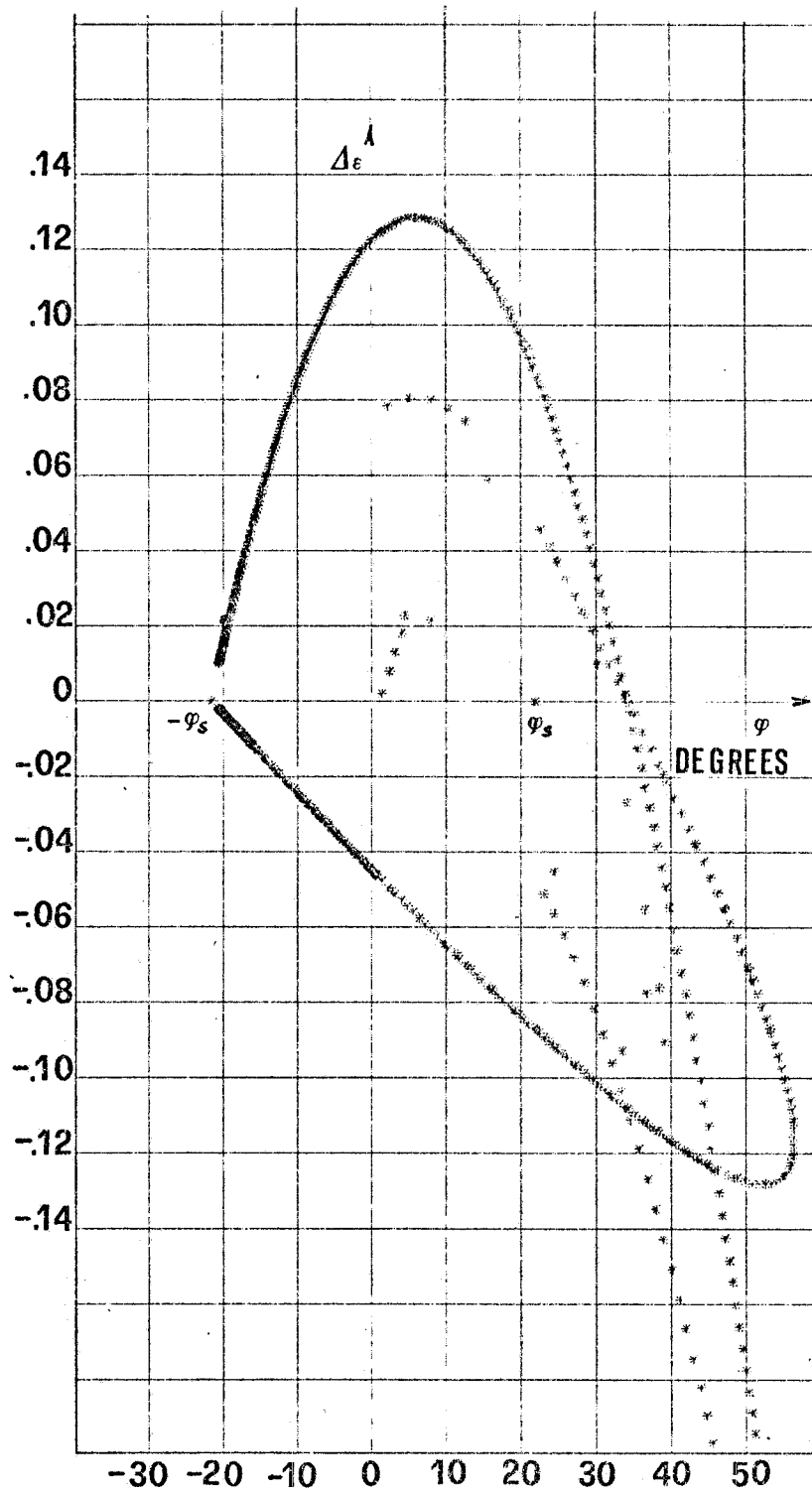


FIG. 3 - The same situation as described in Fig. 2. Only a few shadows of many starting points are plotted to demonstrate the failure of the matching between the outward and inward going closed separatrix.

4. -- CANONICAL FORM OF THE PHASE EQUATIONS AND THE SEPARATRIX -

For a fixed field accelerator, it is convenient to formulate the problem in terms of the pair of conjugate variables $(\frac{\Delta E}{\omega_s})$, $\Delta \varphi$ (as has been already done by K.R. Symon and A. M. Sessler⁽³⁾), so that eqs. (5) and (6) become:

$$(13) \quad \frac{d}{dt} \Delta \varphi = \nu \omega_s^2 \left(\frac{\Delta \varepsilon}{\omega_s} \right)$$

$$(14) \quad \frac{d}{dt} \left(\frac{\Delta \varepsilon}{\omega_s} \right) = \frac{1}{2\pi \cos \varphi_s} (\cos \varphi - \cos \varphi_s).$$

The Hamiltonian function H from which (13) and (14) are derivable as canonical equations

$$\left(\frac{d}{dt} \Delta \varphi = \frac{\partial H}{\partial \left(\frac{\Delta \varepsilon}{\omega_s} \right)} ; \quad \frac{d}{dt} \left(\frac{\Delta \varepsilon}{\omega_s} \right) = - \frac{\partial H}{\partial \Delta \varphi_s} \right)$$

is given below:

$$(15) \quad H = \frac{1}{2} \nu \omega_s^2 \left(\frac{\Delta \varepsilon}{\omega_s} \right)^2 - \frac{1}{2\pi \cos \varphi_s} (\sin \varphi - \varphi \cos \varphi_s) \cong \\ \cong \frac{1}{2} \nu (\Delta \varepsilon)^2 - \frac{1}{2\pi \cos \varphi_s} (\sin \varphi - \varphi \cos \varphi_s).$$

To find the fixed-point we set $\frac{d}{dt} \Delta \varphi = 0$ and $\frac{d}{dt} \left(\frac{\Delta \varepsilon}{\omega_s} \right) = 0$ in eq. (13) and (14). This leads to the two points given by (7).

The separatrix is given by the equation $H = H(\Delta \varepsilon = 0, \varphi = -\varphi_s)$, which can be written as

$$(16) \quad \Delta \varepsilon = \pm \left[\frac{1}{\pi \nu \cos \varphi_s} (\sin \varphi + \sin \varphi_s - (\varphi + \varphi_s) \cos \varphi_s) \right]^{1/2}.$$

In fig 4 the separatrices are plotted for five values of V' (see fig. 4 of reference⁽¹⁾ for comparison). $\nu = 1$.

It turns out from (16) that the stable area is an increasing function of the peak voltage.

The half-height of the bucket, i. e. the maximum stable energy excursion, is found by putting $\varphi = \varphi_s$ in the separatrix equation (16). One

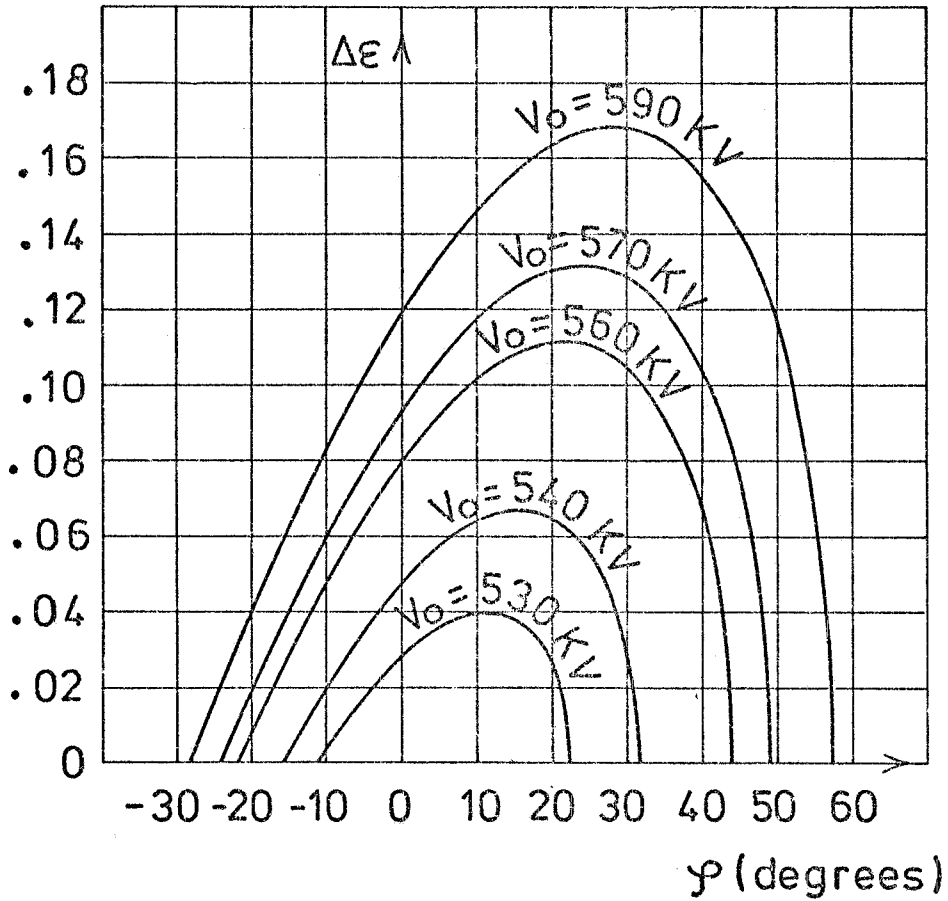


FIG. 4 - Region of phase stability in microtrons for various peak voltages. These curves are symmetrical about the φ -axis. The bucket area is an increasing function of the peak volt.

gets:

$$(17) \quad \Delta \varepsilon_{\max} = \left[\frac{2}{\pi \nu} (\operatorname{tg} \varphi_s - \varphi_s) \right]^{1/2} .$$

At the h th orbit

$$(18) \quad \frac{\Delta E_{\max}}{E_s} = \frac{\Delta R_{\max}}{R_s} = \frac{\Delta \varepsilon_{\max}}{h} .$$

5. -- CONCLUSION -

The conclusion can be drawn that in principle there is nothing in phase oscillations dynamics for the microtron that restricts operation to low peak voltages. The only real restriction that comes out is the fact that for large peak voltage values particles must perform large stable excursions from the synchronous orbit, so that they strike the walls of the vacuum chamber or the cavity anyway.

ACKNOWLEDGEMENTS. -

It is a pleasure to thank Prof. I. F. Quercia and the Electron Synchrotron Staff for their kind interest in the progress of the work.

The techniques applied above are extracted from two papers of Prof. H. G. Hereward. These are:

- i) "The possibility of resonant extraction from the C. P. S. ",
CERN AR/Int. GS/61-5, Dec. 7th, 1961.
- ii) "What are the equations for the phase oscillations in a synchrotron?"
CERN 66-6, Proton Synchrotron Machine Division, Feb. 10th, 1966.

REFERENCES. -

- (1) - C. Henderson, F. F. Heymann and R. E. Jennings, Proc. Phys. Soc. (London) 66 B, 41 (1953).
- (2) - D. Bohm and L. Foldy, Phys. Rev. 70, 249 (1946).
- (3) - K. R. Symon and A. M. Sessler, Proc. CERN Symp. on High Energy Accelerators, page 44 (1956).