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G. Parisi : CONFORMAL INVARIANCE IN PERTURBATION THEORY. -

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ABSTRACT. -

Asymptotic conformal invariance is proved to be true at all orders in perturbation theory. The correct Ward Identities for broken conformal invariance are derived : they are the extension of the Callan Symanzik equation from scale to conformal transformations.

There exist formal proofs that a quantum field theory with dimensionless coupling costant and without derivative coupling is asymptotically invariant under the conformal group[1]. Unfortunately, in perturbation theory ultraviolet divergences destroy the canonical commutation relations, the proofs are no more valid and such a powerful asymptotic symmetry is lost[2].

Callan[3] and Symanzik [4] have however derived, at all perturbation orders, an equation (C. S. eq.) which replaces the wrong naive Ward identities of scale invariance. From a non trivial analysis[5] of this equation it follows that there exit a function $\beta(g)$ of the renormalized coupling costant g such that, if $\beta(g)$ has a zero at some $g = g_c \neq 0$, asymptotic dilatation invariance is restablished in the deep euclidean region through the phenomenon of renormalization of dimensions: the field aquires an anomalous dimension. Both the value of the anomalous dimension and the form of the leading term of the Green functions in the deep euclidean region are independent from the coupling costant.

A very interesting question is if, under the same hypothesis, an asymptotically scaling invariant field theory is asymptotically invariant under the whole conformal algebra.

Asymptotic conformal invariance has recently been used to derive divergence free equations that fix the value of the anomalous dimention[6], improved light-cone expansions[7], and relations between the dimention of the interpolating field and the asymptotic behaviour of the form factor[8]. Conformal invariance yelds striking constraints on the form of the Green functions at high momenta : The three point function of three operators of given dimensions is completely determined apart from a multiplicative costant[9]. This result implies the existence of light-cone singularities in the product of two operators : the type of the singularity is the one suggested by the Wilson rule[10].

The aim of this letter is to prove that if a renormalizable Lagrangian field theory with dimensionless coupling costant and without derivative couplings is asymptotically scale invariant (i. e. the function $\beta(g)$ has a zero at $g_c \neq 0$), it is also asymptotically conformal invariant. A by product of this analysis is that the zero mass theory renormalized a là Gell-Mann and Low[11] is exactly conformal invariant. In order to prove this statement we must find a generalization of the C. S. eq. to conformal transformations. Let us work for simplicity in the $g\phi^4$ theory. The C. S. eq. can be formally interpreted as the zero momentum Ward identity for a current $D^{\mu}(x)$, whose integrated forth component generates scale transformations, the field ϕ having dimension $1 + \gamma(g)$ and with

(1) $$\partial_\mu D^{\mu}(x) = - \eta(g)\, m^2 : \phi^2(x) : - \beta(g) : \phi^4(x) : \equiv \theta(x)$$

In the configuration space the C.S. eq. has indeed the typical form of a Ward identity:

$$\sum_i \left[x_i^\mu \frac{\partial}{\partial x_i^\mu} - (1+\gamma(g)) \right] \langle 0 | T\left[\phi(x_1) \ldots \phi(x_N) \right] | 0 \rangle =$$

(2)

$$= \int d^4y \, \langle 0 | T\left[\phi(x_1) \ldots \phi(x_N) \, \theta(y) \right] | 0 \rangle .$$

Let us euristically suppose that also at higher orders in pertubation theory there exists a current $K^{\mu\lambda}(x)$ which generate the special conformal transformation and that the naive result

(3) $$\partial_\mu K^{\mu\lambda}(x) = 2x^\lambda \, \partial_\mu D^\mu(x) = 2x^\lambda \, \theta(x)$$

remains true.

If we suppose that the field ϕ transforms irreducibly under a special conformal transformation, being an operator with $K^\lambda = 0$ and of dimention $1+\gamma(g)$, we obtain the following zero momentum Ward identities.

$$\sum_i \left[2x_i^\lambda x_i^\rho \frac{\partial}{\partial x_i^\rho} - x_i^2 \frac{\partial}{\partial x_i^\lambda} - 2(1+\gamma(g))\, x_i^\lambda \right] \cdot$$

(4) $$\cdot \quad 0 | T\left[\phi(x_1) \ldots \phi(x_N) \right] | 0 =$$

$$= 2 \int d^4y \, y^\lambda \langle 0 | T\left[\phi(x_1) \ldots \phi(x_N) \, \theta(y) \right] | 0 \rangle$$

The equation (4) is true in the tree diagram approssimation, but may be destroyed during the renormalization; we shall now show that this does not happen and the relation (4) remains unchanged at each order in perturbation theory.

The main difficulties that can be found in proving eq. (4), are due to the presence in the standard perturbation theory of divergent integrals wich must be regularized introducing an ultraviolet cutoff, a procedure which is not scale invariant. A possible way to bypass this difficulty is to use the C.S. eq. to reconstruct the pertubative serie computing only convergent integrals[3].

The C.S. eq. can be rewritten in the P space in an integrated form; in the case of the four points function we have:

$$G_4(p_1, p_2, p_3) = g - \int_0^1 \frac{d\alpha}{d} \left\{ \left[4\gamma(g) + \beta(g) \frac{d}{dg} \right] \cdot \right. \tag{5}$$
$$\left. \cdot\, G_4(\alpha p_1, \alpha p_2, \alpha p_3) + m^2 \eta(g) F_4(\alpha p_1, \alpha p_2, \alpha p_3) \right\}$$

where:

$$G_4(p_1, p_2, p_3) = \int dx_1\, dx_2\, dx_3\, e^{i(p_1 x_1 + p_2 x_2 + p_3 x_3)} \cdot$$
$$\cdot \left\langle 0 \right| T\left[\phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(0) \right] \left| 0 \right\rangle$$
(6)
$$F_4(p_1, p_2, p_3) = \int dx_1\, dx_2\, dx_3\, dy\, e^{i(p_1 x_1 + p_2 x_2 + p_3 x_3)} \cdot$$
$$\cdot \left\langle 0 \right| T\left[\phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(0) : \phi^2(y) : \right] \left| 0 \right\rangle$$

If we expand the r.h.s. and the l.h.s. of eq. (5) in power of g, we obtain G_4 at order N as an integral over F_4 computed at the same order plus G_4 computed at lower order (The first term in $\beta(g)$ is of order g^2 !). F_4 can be written as a sum of skeleton diagrams formed with the low order functions and in pertubation theory, according to the Weimberg theorem[12], all the integrals involved are convergent: eq. (5) yelds a finite expression for G_4 at order N in term of convergent integrals over the lower order functions.

A similar analysis may be performed on the two point function. This allows us to use the C.S. eq. to reobtain by an iterative procedure the pertubative expansion by only computing convergent integrals for the skeleton diagrams.

It is now clear that to obtain the proof of eq. (4) in perturbation theory, we need only to prove :

I) If an N-points function has a skeleton expantion in terms of functions that satisfy (4), the N-points function itself satisfies (4) ;

II) If the functions in the integrals on the r. h. s. of eq. (5) satisfy (4), also the l. h. s. shares the same property.

Assumptions I) and II) are true : they can be verified by explicit computation. A simple general argument for their validity is that at least in super renormalizable field theories, equation (4) must be true and also compatible with the C. S. eq. and with the skeleton expansion for the Green functions. This would be clearly impossible if assumptions I) and II) were false. The whole pertubation expantion may be obtained only using the C. S. eq. and computing skeleton diagrams : in this way eq. (4) is proved by indution at all order in pertubation theory.

It is know a simple task to prove conformal invariance in the deep euclidean region : from eq. (3) and (4) it follows that if $g = g_c$ (the zero of $\beta(g)$), the breaking of conformal invariance is a soft operator of low dimension and the theory is surely asymptotically conformal invariant. This seems not to happen for $g \neq g_c$, however the C. S. eq. theaches us that the form of the leading term of the Green function in the deep euclidean region is independent from the coupling costant, and asymptotic conformal invariance follows for arbitrary coupling costant.

The Green function of the zero mass theory normalized a là Gell-Mann Low may be obtained from the Green functions of the massive theory in the deep euclidean region and they are exactly conformal invariant.

We can therefore conclude that in the framework of renormalizable quantum field theory asymptotic conformal invariance is as reliable as asymptotic scale invariance. This strongly suggests that all the results recently obtained on asymptotic conformal symmetry may be applied to the physical word.

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