COMITATO NAZIONALE PER L'ENERGIA NUCLEARE
Laboratori Nazionali di Frascati

LNF-71/101
28 Dicembre 1971

S. Ferrara, R. Gatto, A. F. Grillo and G. Parisi: CANONICAL SCALING AND CONFORMAL INVARIANCE. -

Servizio Documentazione
dei Laboratori Nazionali di Frascati del CNEN
Casella Postale 70 - Frascati (Roma)

Laboratori Nazionali di Frascati del CNEN
Servizio Documentazione

LNF-71/101
28 Dicembre 1971

S. Ferrara, R. Gatto[(x)], A. F. Grillo and G. Parisi : CANONICAL SCALING AND CONFORMAL INVARIANCE. -

(To appear on Physics Letters B)

ABSTRACT

Arguments are given to show that a strictly conformal invariant skeleton theory is compatible with the observed scaling only if it possesses an infinite number of local tensors of spin $n = 2, 4, \ldots$ and scale dimension $l_n = 2 + n$, which are all conserved. The conclusion suggests that conformal invariance is spontaneously broken.

In this note we want to point out a remarkable conseguence of scaling, as observed at SLAC[(1)], and of a hypothetical conformal covariance[(2)] of the so-called skeleton theory[(3)]. The argument runs as follows. It is well-known that the observed scaling implies canonical dimensions for a family of symmetric traceless tensor operators contributing to the expansion of the product of two electromagnetic currents near the light-cone. Within the currently accepted theoretical frame[(3)]

(x) - Istituto di Fisica dell'Università di Roma, and Istituto Nazionale di Fisica Nucleare, Sezione di Roma.

one is dealing in such a limit with a scale invariant operator scheme, the skeleton theory. Such a scheme might exibit conformal invariance, beyond scale invariance[4, 5, 6].

A sequence of general mathematical properties (to be reported below) of an exact (not spontaneously broken) conformly covariant theory then shows that each of the tensors of the above family is divergenceless. That is, the skeleton theory possesses an infinite number of local conservation equations, to which one can think of associating an infinite number of conserved charges. Or, alternatively, strict conformal invariance does not apply to the skeleton theory.

In trying to make this note self-contained, as much as possible, we shall first review some of the basic properties of a field theory covariant under the conformal algebra of space-time. We briefly recall that the conformal algebra is a 15-dimensional Lie algebra, isomorphic to the orthogonal algebra 0(4, 2). Its generators are the Poincarè generators, P_μ and $M_{\mu\nu}$, the dilatation generator, D, and the generators of special conformal-transformations, K_μ, defined as $K_\mu = R P_\mu R^{-1}$ where R operates an inversion, $x_\mu \rightarrow -x_\mu/x^2$.

The commutation relations are : those of the Poincarè algebra ; those specifying the scalar and vector nature of D and K_μ respectively ; $[K_\mu, K_\nu] = 0$, obviously following from the definition ; and

(1) $$[D, K_\mu] = i K_\mu , \qquad [D, P_\mu] = -i P_\mu$$

(2) $$[P_\mu, K_\nu] = -i(g_{\mu\nu} D - M_{\mu\nu}) .$$

We consider irreducible conformal tensor fields $0_{\alpha_1 \dots \alpha_n}(x)$ which behave under the stability subalgebra at x = 0 as :

(3) $$[O_{\{\alpha\}}(0), M_{\mu\nu}] = \Sigma^{\{\beta\}}_{\{\alpha\}} O_{\{\beta\}}(0) ; \qquad [O_{\{\alpha\}}(0), D] = i l O_{\{\alpha\}}(0) ;$$
$$[O_{\{\alpha\}}(0), K_\mu] = 0$$

where l is the scale dimension and Σ is an irreducible tensor representation for $M_{\mu\nu}$. The irreducible representation to which $O_{\alpha_1 \ldots \alpha_n}$ belongs is thus characterized by n and l. We now state the following theorems[5].

a) <u>Degeneracy theorem</u> : The divergence, $\partial^{\alpha_1} O_{\alpha_1 \ldots \alpha_n}$, of an irreducible conformal tensor field $O_{\alpha_1 \ldots \alpha_n}$ of order n and scale dimension l_n equal to $2+n$ is an irreducible conformal tensor field[*].

Proof : Since $O_{\alpha_1 \ldots \alpha_n}$ is symmetric and traceless one obtains that $\left[\partial^{\alpha_1} O_{\alpha_1 \ldots \alpha_n}(0), K_\mu \right]$ is proportional to $(l_n - 2 - n) O_{\mu\alpha_2 \ldots \alpha_n}(0)$ and thus vanishes for $l_n = 2 + n$.

Corollary : A conserved irreducible conformal tensor has $l_n = 2+n$.

b) <u>Orthogonality theorem</u> : The vacuum expectation value of the product of two irreducible conformal tensor fields

$$W_{\alpha_1 \ldots \beta_m}(x) = \left\langle 0 \middle| O^{(n)}_{\alpha_1 \ldots \alpha_n}(x) \, O^{(m)}_{\beta_1 \ldots \beta_m}(0) \middle| 0 \right\rangle \tag{4}$$

vanishes unless $n = m$ and $l_n = l_m$, in an exactly (not spontaneously broken) conformal covariant theory.

Proof : From $\left[K_\mu, O^{(n)}_{\alpha_1 \ldots \alpha_n}(x) \right]$ as calculated from (1), (2) and (3) (induced representation) and from $\left[K_\mu, O^{(m)}_{\beta_1 \ldots \beta_m}(0) \right] = 0$, taking the vacuum expectation value of $\left[K_\mu, O^{n}_{\alpha_1 \ldots \alpha_n}(x) \, O^{(m)}_{\beta_1 \ldots \beta_m}(0) \right]$ one obtains :

$$(2 x_\mu x\partial - x^2 \partial_\mu + 2 l_n x_\mu - 2 i x^\nu \Sigma^{(n)}_{\mu\nu}) \, W(x) = 0$$

which can be seen to require $n = m$ and $l_n = l_m$. A more direct proof follows from the isomorphism of the Minkowski space with the homogeneous space $O(4,2)/\left[IO(3,1) \otimes D \right]$, but requires additional notions[5].

From a) and b) one derives the following corollary :

Corollary : An irreducible conformal tensor field $O_{\alpha_1 \ldots \alpha_n}$ with $l_n = 2 + n$ is necessarily conserved.

Proof: From a), $\partial^{\alpha_1} O_{\alpha_1 \ldots \alpha_n}$ satisfies

$$\left[\partial^{\alpha_1} O_{\alpha_1 \ldots \alpha_n}(0), K_\lambda \right] = 0$$

and has dimension 3 + n. Then, from b),

$$\langle 0 | \partial^{\alpha_1} O_{\alpha_1 \ldots \alpha_n}(x) O_{\beta_1 \ldots \beta_n}(y) | 0 \rangle = 0.$$

Therefore, $\langle 0 | \partial^{\alpha_1} O_{\alpha_1 \ldots \alpha_n}(x) \partial^{\beta_1} O_{\beta_1 \ldots \beta_n}(y) | 0 \rangle = 0$ and, more generally, $\langle 0 | \partial^{\alpha_1} O_{\alpha_1 \ldots \alpha_n}(x) A(y) \cdots Z(t) | 0 \rangle$ vanishes for arbitrary local operators $A(y) \cdots Z(t)$ which behave as irreducible conformal tensors. On the assumption that from such fields one can build up the entire Hilbert space, one then obtains, by causality, the result $\partial^{\alpha_1} O_{\alpha_1 \ldots \alpha_n}(x) = 0$. A direct proof based on the vanishing of the Green function for $\partial^{\alpha_1} O_{\alpha_1 \ldots \alpha_n}$ would require assumptions of positivity, which we have avoided.

It is now straightforward to apply the above results to the physical situation. The set $O_{\alpha_1 \ldots \alpha_n}$ of dimensions $l_n = 2 + n$, and $K_\lambda = 0$, is required from the observed scaling[(5)]. Namely, the most singular part of the product $j_\mu(x) j_\nu(0)$ when $x^2 \rightarrow 0$ can be expanded in terms of such operators. From the assumption that the skeleton theory is conformal invariant it then follows that $\partial^{\alpha_1} O_{\alpha_1 \ldots \alpha_n}(x) = 0$ for each n.

The conclusion of an infinite set of local conservation equations for the skeleton theory might turn out to be physically unacceptable, or in any case too strong a limitation. Among the various alternatives that one can take in such a case, one which seems to us rather suggestive is that of a spontaneously broken conformal invariance of the skeleton theory. We have stressed that the failure of the orthogonality theorem in this case unvalidates the conclusion on the infinite conservation laws. On a still more conjectural ground one may imagine that SU(3) x SU(3) remains spontaneously broken in the skeleton limit, implying spontaneous breaking of scale in that limit.

We would like to thank for discussions Professors G. De Franceschi, G. F. Dell'Antonio, and S. Doplicher.

REFERENCES AND FOONOTES. -

(1) - See for instance : M. Breidenback et al., Phys. Rev. Letters 23, 935 (1969).

(2) - H. A. Kastrup, Ann. Physik 7, 388 (1962); G. Mack and A. Salam, Ann. Phys. 53, 174 (1969).

(3) - K. Wilson, Phys. Rev. 179, 1499 (1969).

(4) - D. G. Boulware, L. S. Brown and R. D. Peccei, Phys. Rev. D2, 293 (1970).

(5) - S. Ferrara, R. Gatto and A. F. Grillo, Nuclear Phys. B34, 349 (1971); Frascati report LNF-71/79 (1971) (to appear in Springer Tracts).

(6) - S. Ciccariello, R. Gatto, G. Sartori and M. Tonin, Ann. Phys. 65, 265 (1971).

(*) - We call this theorem "degeneracy theorem" because it originates from the degeneracy, for $l_n = 2 + n$, of the representation of the stability algebra from which the irreducible tensor representation is induced. Namely, $O_{\alpha_1 \ldots \alpha_n}$ and $\partial^{\alpha_1} O_{\alpha_1 \ldots \alpha_n}$ behave irreducibly under the stability subalgebra (but not under the full algebra) and are both annihilated by K_λ .