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CONFORMAL INVARIANCE ON THE LIGHT CONE AND CANONICAL DIMENSIONS

S. FERRARA *
CERN, Geneva

R. GATTO
Istituto di Fisica dell'Università, Padova,
Istituto Nazionale di Fisica Nucleare, Sezione di Padova

A. F. GRILLO
Laboratori Nazionali di Frascati del CNEN, Frascati

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Abstract: Covariance under infinitesimal transformations of the spinor group SU(2,2), the covering group of the conformal group, can place significant restrictions on the Wilson's type analysis of operator products on the light cone. For a discussion of the expansion it is relevant to analyze under such transformations the infinite set of local operators providing a basis for the expansion. An infinite ladder of irreducible representations provides such a basis. The existence of a scaling function places relations between the dimensions of tensors, belonging to inequivalent representations, which are all annihilated under K_λ, the generator of infinitesimal special conformal transformations. These relations are not deducible from conformal invariance alone. We establish a theorem which fixes the scale dimension of an irreducible symmetric local operator which is, together with its divergence, annihilated by K_λ. The theorem (or some possible extensions of it) may be useful to build up an algebraic scheme satisfying canonical dimensions. Finally, as an example of a mathematical mechanism providing for the required correlation of dimensions, we discuss an algebraic scheme based on enlarging the conformal algebra.

1. INTRODUCTION

It is well known that the convergence of the Bjorken-Callan-Gross sum sules [1, 2] provides stringent restrictions on the structure of the underlying theory [3-5]. These restrictions can be analyzed in terms of equal-time commutators [4, 5] or on the basis of operator product expansions [3, 6, 7].

Following Bjorken's analysis [2] one has general sum rules of the kind

* On leave of absence from 'Laboratori Nazionali di Frascati del CNEN', Frascati.

$$f_n = \int_0^2 \mathrm{d}\omega\, \omega^n F_t(\omega) = \lim_{P_z \to \infty} \tfrac{1}{2}(-i)^n \int \frac{\mathrm{d}^3x}{P_0{}^n} \langle P_z | \left[\frac{\partial^n J_x(x,0)}{\partial t^n}, J_x(0) \right] | P_z \rangle \,,$$

$$n = 1, 3, 5, \ldots ,$$

where $F_t(\omega)$ is the transverse scaling function and $J_\mu(x)$ the electromagnetic current. The positivity of $F_t(\omega)$ ensures that $0 < f_n < 2f_{n-1}$. Each f_n must be finite and non-vanishing. The matrix element of $[\partial^n J_x/\partial t^n, J_x]$ carries n+1 momenta. Therefore, the operator $O_{\mu_1 \ldots \mu_{n+1}}$, which contributes the leading term of the commutator, must have spin $s = n+1$ whereas its scale dimension [3] is $l = n+3$, providing the relation $l = s+2$. Canonical dimensions [7, 8] would provide a way of satisfying this relation. The subject of renormalization of dimensions has been discussed by many authors and it appears as a very difficult dynamical problem [9-11].

The relevance of approximate dilatation invariance [3, 4, 12-15] has led to the exploration of the stronger conformal invariance [16-18]. Implications of conformal invariance in the analysis using equal time commutators have already been discussed [4]. Covariance under the infinitesimal transformations of the SU(2, 2) group (the covering group of the conformal group [19]) can consistently be required on operator product expansions on the light cone of the type first proposed by Wilson [3]. For a discussion of the ensuing restrictions it is essential to analyze the transformation properties of the infinite basis of local operators which allow for the realization of the expansion. We shall be interested in the region in configuration space close to the light cone (leading singular terms) [7], as it is well known that it is that region which is relevant for the Bjorken asymptotic limit [3, 4, 6, 7]. The imposition of covariance under the action of the SU(2, 2) generators has been proposed in such a situation and justified under some assumptions (we refer to Wilson's paper for such a discussion, in the case of dilatation, ref. [3]).

In the present work we illustrate the role of an infinite ladder of irreducible representations of the algebra in classifying the basis of local operators on which the expansion is made. In each irreducible representation a set of operators which are annihilated by K_λ, which generates special conformal transformations, contribute to the matrix element defining the limit of the structure function. The existence of a scaling limit for such function is equivalent to requiring definite and related dimensions (canonical dimensions) [3-5, 7].

The conformal structure is uncapable alone of providing a justification for such a striking regularity; its basis seems to lie in the dynamics. On the other hand, one may base the discussion on introducing indecomposable (rather than irreducible) representations, as advocated by Mack [5]. Here again, however, the assumption required to produce the required spectrum of dimensions goes beyond the assumption of conformal invariance on the light cone.

In order to understand the possible dynamical content of the fundamental scaling relation $l = s+2$ we demonstrate a theorem which shows that, under our main assumption of conformal invariance on the light cone such a relation is equivalent to an infinite set of 'generalized partial conservation equations'. By this we mean that those tensors which contribute to the structure functions are such that their divergences are annihilated by K_λ. If such a generalized partial conservation equation is satisfied, the scaling relation between spin and dimension follows. But it is also true that if the scaling relation holds there must be such an infinite set of generalized partial conservations. In particular the right-hand side of a partial conservation equation may vanish giving rise to an exact conservation. The latter situation obtains if the theory approaches on the light cone a free field theory of massless particles [20] (for instance free quark commutators on the light cone, etc.). In such a case the scaling relation follows directly. But it also holds, as our theorem shows, in the presence of interactions, provided they maintain at least a structure of generalized partial conservations in place of the original exact conservations.

To gain a different intuition of the problem one can try to visualize the product $A(x)B(0)$ of two local fields $A(x)$ and $B(x)$ as a non-local operator, describing a composite system.

One can develop, for instance, an analogy with the hydrogen atom *. The conformal algebra can then be seen as providing the degeneracy algebra of the composite system. The Casimir operators have on the system an infinite discrete spectrum. Correspondingly, there are infinite towers of operators with increasing and related spin and dimensions. Their couplings are determined by product composition rules. However, different towers (inequivalent irreducible representations) are not a priori related unless a dynamical scheme is imposed. This may be achieved, for instance, by postulating a dynamical group, like for the hydrogen atom.

In the last section of the paper we shall also examine such a possibility in a simplest model. We have not, however, carried out a general discussion of possible spectrum generating algebras, in view of the apparent arbitrariness. Nevertheless, we think that further study of these points should be useful.

2. CONFORMAL ALGEBRA

The algebra associated to the conformal group is a 15-dimensional Lie algebra with the following commutation rules: the commutators of the Poincaré algebra among P_μ and $M_{\mu\nu}$, and

$$[K_\sigma, M_{\mu\nu}] = i(g_{\sigma\mu} K_\nu - g_{\sigma\nu} K_\mu) \,, \qquad [D, M_{\mu\nu}] = 0 \,,$$

$$[K_\mu, P_\nu] = -2i(g_{\mu\nu} D + M_{\mu\nu}) \,, \qquad [D, P_\mu] = -iP_\mu \,,$$

$$[K_\mu, K_\nu] = 0 \,, \qquad [K_\mu, D] = -iK_\mu \,. \tag{2.1}$$

* For a review of dynamical groups and composite systems see, for instance, ref. [21].

In terms of $J_{AB}\,(A, B = 0, \ldots, 6)$,

$$J_{\mu\nu} = M_{\mu\nu}, J_{65} = D, J_{5\mu} = \tfrac{1}{2}(P_\mu - K_\mu), J_{6\mu} = \tfrac{1}{2}(P_\mu + K_\mu) \;, \tag{2.2}$$

one has

$$[J_{KL}, J_{MN}] = i(g_{KN}J_{LM} + g_{LM}J_{KN} - g_{KM}J_{LN} - g_{LN}J_{KM}) \;,$$
$$g_{AA} = (+---,-+) \;,$$
$$g_{AB} = 0 \;, \quad A \neq B. \tag{2.3}$$

Eqs. (2.3) are those of the orthogonal algebra O(4, 2). The Casimir operators are

$$C_{\rm I} = J_{AB}J^{AB};\; C_{\rm II} = \epsilon_{ABCDEF}J^{AB}J^{CD}J^{EF};\; C_{\rm III} = J^{AB}J_{BC}J^{CD}J_{DA} \;. \tag{2.4}$$

In particular,

$$C_{\rm I} = M_{\mu\nu}M^{\mu\nu} + 2P\cdot K + 8iD - 2D^2 \;. \tag{2.5}$$

On the Minkowsky space, special conformal transformations

$$x'_\mu = \frac{x_\mu - c_\mu x^2}{1 - 2\,c\cdot x + c^2x^2} \;, \tag{2.6}$$

can map finite points to infinity; they can violate causality. The action of the conformal group in Minkowsky space is equivalent to that of the connected part of $O(4,2)/C_2$ (where $C_2 = (\mathbf{1}, -\mathbf{1})$ on the cone in six-dimensions (i.e., the homegeneous space $O(4,2)/IO(3,1)$). The transformation which maps the six-dimensional cone in Minkowsky space is a projective transformation and some points are mapped to infinity. The conformal group is locally causal: for each x in Minkowsky space, there is a neighbourhood of the identity for which causality is preserved. The conformal algebra is the algebra of the covering group of O(4, 2), the spinor group SU(2, 2). We are interested in covariance under infinitesimal SU(2, 2) transformations. Therefore, multivalued representations of the factor group $O(4,2)/C_2$ are a priori allowed and can be related to physical quantities.

3. LIGHT-CONE EXPANSION AND REDUCIBLE LADDER REPRESENTATION OF THE CONFORMAL GROUP

For simplicity we shall confine our discussions to the expansion of the product of two local operators $A(x)$, $B(x)$, which are Lorentz scalars, have well defined dimensions l_A and l_B and satisfy $[A(0), K_\mu] = [B(0), K_\mu] = 0$.

We assume the existence of an infinite set of Hermitian local tensor operators $O(x) = O^{nm}_{\alpha_1 \dots \alpha_m}(x)$ which transform according to

$$[O(x), P_\lambda] = i\partial_\lambda O(x) ,$$

$$[O(x), D] = (ix_\nu \partial^\nu + \Delta) O(x) ,$$

$$[O(x), M_{\mu\nu}] = [i(x_\nu \partial_\mu - x_\mu \partial_\nu) + \Sigma_{\mu\nu}] O(x) ,$$

$$[O(x), K_\lambda] = \{i(2x_\lambda x_\nu \partial^\nu - x^2 \partial_\lambda - 2ix^\nu [g_{\lambda\nu}\Delta + \Sigma_{\lambda\nu}]) + \kappa_\lambda\} O(x) , \qquad (3.1)$$

where $\Sigma_{\mu\nu}$, Δ, K_μ form a representation of the stability group at $x = 0$. The index n (to be called principal quantum number) labels an indecomposable infinite-dimensional representation of the stability algebra ($\Sigma_{\mu\nu}$, Δ, K_μ). Within the representation, $O^{nn}_{\alpha_1 \dots \alpha_n}(0)$ is the tensor of lowest dimension ($= l_n$) and satisfies

$$[O^{nn}_{\alpha_1 \dots \alpha_n}, K_\lambda] = 0 . \qquad (3.2)$$

Under K_λ, which acts, as well-known, as a dimension lowering operator, $O^{nm}_{\alpha_1 \dots \alpha_m}$ goes into a suitable superposition of $O^{n, m-1}_{\alpha_1 \dots \alpha_{m-1}}$, implying

$$[O^{nm}_{\alpha_1 \dots \alpha_m}(0), \Delta] = i(l_n + m - n) O^{nm}_{\alpha_1 \dots \alpha_m}(0) . \qquad (3.3)$$

Using conformal invariance it can be shown (see sect. 4) that, by possibly redefining the operators in terms of suitable combinations, P_μ can be taken to act as a rising dimension operator within each representation. As a consequence each string of operators $O^{nm}_{\alpha_1 \dots \alpha_m}$ (labelled by the principal quantum number n and the starting dimension l_n) realizes an irreducible (infinite-dimensional) representation of the full conformal algebra.

For definiteness we take

$$\underset{\{\alpha_1 \dots \alpha_{m+1}\}}{S} i [O^{nm}_{\alpha_1 \dots \alpha_m}, P_{\alpha_{m+1}}] = O^{nm+1}_{\alpha_1 \dots \alpha_{m+1}} . \qquad (3.4)$$

In sect. 4 we shall give explicit examples of such families of operators. We write a generalized Wilson's expansion

$$\lim_{x^2 \to 0} A(x) B(0) = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} F^{AB; \alpha_1 \dots \alpha_m}_{nm}(x) \, O^{nm}_{\alpha_1 \dots \alpha_m}(0) , \qquad (3.5)$$

where $O^{nm}_{\alpha_1 \dots \alpha_m}$ are completely symmetric. After commuting with D one obtains by a simple argument

$$F_{nm}^{AB;\,\alpha_1\ldots\alpha_m}(x) = \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B-l_n+n)}$$
$$\times \sum_{k=0,2\ldots}^{[\frac{1}{2}m]+1} C_{nmk}^{AB}\, g_{\alpha_1\alpha_2}\ldots g_{\alpha_{k-1}\alpha_k}\, g\cdot(x^2)^{\frac{1}{2}k}\, x^{\alpha_{k+1}}\ldots x^{\alpha_m}\,. \tag{3.6}$$

In the leading contribution on the light cone the terms with $k \neq 0$ can be neglected and we have

$$F_{nm}^{AB;\,\alpha_1\ldots\alpha_m}(x) = \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B-l_n+n)} C_{nm}^{AB}\, x^{\alpha_1}\ldots x^{\alpha_m}\,, \tag{3.7}$$

where $C_{nm}^{AB} = C_{nm_0}^{AB}$. Thus only the irreducible $(\frac{1}{2}m, \frac{1}{2}m)$ highest spin part of $O_{\alpha_1\ldots\alpha_m}^{nm}$ enters at the leading order in x^2. The $[\frac{1}{2}m]$ traces are irrelevant

$$A(x)B(0) = \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B)} \sum_{n=0}^{\infty} \left(\frac{1}{x^2}\right)^{\frac{1}{2}(n-l_n)} \sum_{m=n}^{\infty} C_{nm}^{AB}\, x^{\alpha_1}\ldots x^{\alpha_m}\, O_{\alpha_1\ldots\alpha_m}^{nm}\,. \tag{3.8}$$

After translating eq. (3.8) of $-x$ and by taking the Hermitian conjugate of both sides one has

$$B(x)A(0) = \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B)} \sum_{n=0}^{\infty} \left(\frac{1}{x^2}\right)^{\frac{1}{2}(n-l_n)} \sum_{m=n}^{\infty} C_{nm}^{BA}\, x^{\alpha_1}\ldots x^{\alpha_m}\, O_{\alpha_1\ldots\alpha_m}^{nm}$$
$$= \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B)} \sum_{n=0}^{\infty} \left(\frac{1}{x^2}\right)^{\frac{1}{2}(n-l_n)}$$
$$\times \sum_{m=n}^{\infty} \left(\sum_{h=0}^{m-n} (-1)^{m-h}\, \frac{1}{h!}\, C_{n,m-h}^{AB}\right) x^{\alpha_1}\ldots x^{\alpha_m}\, O_{\alpha_1\ldots\alpha_m}^{nm}\,. \tag{3.9}$$

We thus obtain

$$C_{nm}^{BA} = \sum_{h=0}^{m-n} (-1)^{m-h}\, \frac{1}{h!}\, C_{n,\,m-h}^{AB}\,, \tag{3.10}$$

and, in particular,

$$C_{nn}^{AB} = (-1)^n\, C_{nn}^{BA}\,. \tag{3.11}$$

From eq. (3.8) we also obtain ($L = -iD$)

$$(-i)[A(x)B(0),K_\lambda] = \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B)} \sum_{n=0}^{\infty} \left(\frac{1}{x^2}\right)^{\frac{1}{2}(n-l_n)}$$

$$\times \sum_{m=n}^{\infty} C_{nm}^{AB}\{2[l_A + m - \tfrac{1}{2}(l_A + l_B + n - l_n)]\, x^{\alpha_1} \dots x^{\alpha_m} x_\lambda$$

$$- x^2 \sum_{i=1}^{m} g_\lambda^{\alpha_i} x^{\alpha_1} \dots x^{\hat{\alpha}_i} \dots x^{\alpha_m}\} O_{\alpha_1 \dots \alpha_m}^{nm}(0) \,, \qquad (3.12)$$

(the notation $x^{\hat{\alpha}_i}$ means that x^{α_i} is to be left out in the product). The coefficients C_{nm}^{AB} are thus determined from the part of $[O_{\alpha_1 \dots \alpha_m}^{nm}, K_\lambda]$ proportional to $g_{\lambda\alpha_i}$ which we write as

$$[O_{\alpha_1 \dots \alpha_m}^{nm}, K_\lambda] = ib(n,m) \underset{\{\alpha_1 \dots \alpha_m\}}{S} g_{\lambda\alpha_i} O_{\alpha_1 \dots \hat{\alpha}_i \dots \alpha_m}^{nm-1} \,, \quad \text{for} \quad m \geqslant n+1 \,, \qquad (3.13)$$

$$= 0 \,, \qquad \text{for} \quad m \leqslant n \,. \qquad (3.14)$$

in terms of known coefficients $b(n,m)$.
Eqs. (3.12) and (3.13) give

$$C_{nm}^{AB}(\tfrac{1}{2}(l_A - l_B + l_n + n) + m - n) = \tfrac{1}{2} b(n, m+1) C_{n,m+1}^{AB} \,, \qquad (3.15)$$

or

$$\frac{C_{nm}^{AB}}{C_{nm}^{BA}} = \frac{C_{nm+1}^{AB}}{C_{nm+1}^{BA}} \, \frac{\tfrac{1}{2}(l_B - l_A + l_n + n) + m - n}{\tfrac{1}{2}(l_A - l_B + l_n + n) + m - n} \,, \qquad (3.16)$$

and by iteration

$$\frac{C_{nm}^{AB}}{C_{nm}^{BA}} = \frac{C_{nn}^{AB}}{C_{nn}^{BA}} \, \frac{B(\tfrac{1}{2}(l_B - l_A + l_n + n), m-n)}{B(\tfrac{1}{2}(l_A - l_B + l_n + n), m-n)} \,, \qquad (3.17)$$

where $B(x,y)$ is the β function. To derive eq. (3.17) only the tensor properties of $O_{\alpha_1 \dots \alpha_m}^{nm}$ under the stability algebra at $x = 0$ have been used. By combining eqs. (3.10) and (3.16) one also has

$$C^{AB}_{nm} = (-1)^n \frac{B(\frac{1}{2}(l_B - l_A + l_n + n), m-n)}{B(\frac{1}{2}(l_A - l_B + l_n + n), m-n)} \sum_{h=0}^{m-n} (-1)^{m-h} \frac{1}{h!} C^{AB}_{nm-h} . \quad (3.18)$$

Knowledge of $b(n, m)$, after computation of the commutator

$$\begin{aligned} i[O^{nm}_{\alpha_1 \dots \alpha_m}(0), K_\lambda] = &-(m-n)(2l_n+m-n-1) \underset{\{\alpha\}}{S}\, g_{\alpha_m \lambda} O_{\alpha_1 \dots \alpha_n, \alpha_{n+1} \dots \alpha_{m-1}} \\ &- (m-n)(m-n-1) \underset{\{\alpha\}}{S} (g_{\alpha_m \lambda} O_{\alpha_1 \dots \alpha_n, \alpha_{n+1} \dots \alpha_{m-1}} \\ &- g_{\alpha_{m-1} \alpha_m} O_{\alpha_1 \dots \alpha_n, \alpha_{n+1} \dots \alpha_{m-2}, \lambda}) \\ &+ 2(m-n)n \underset{\{\alpha\}}{S} (g_{\alpha, \alpha_m} O_{\alpha_1 \dots \alpha_n \lambda, \alpha_{m+1} \dots \alpha_{m-1}} \\ &- g_{\lambda \alpha_1} O_{\alpha_2 \dots \alpha_n \alpha_m, \alpha_{n+1} \dots \alpha_{m-1}}) \end{aligned} \quad (3.19)$$

allows us to rewrite (3.15) as

$$C^{AB}_{n,n+K-1}(\tfrac{1}{2}(l_A - l_B + l_n + n) + K - 1) = C^{AB}_{n,n+K} \cdot K(l_n + n + K) \quad (3.20)$$

and to obtain by iteration the solution of (3.18) as

$$C^{AB}_{n,n+K} = C^{AB}_{n,n} \Gamma(\tfrac{1}{2}(l_A - l_B + l_n + n) + K) \Gamma(l_n + n) / K! \Gamma(\tfrac{1}{2}(l_A - l_B + l_n + n)) \Gamma(l_n + n + K) . \quad (3.21)$$

Using (3.21), we sum over m in (3.8) and obtain

$$A(x)B(0) \underset{x^2 \to 0}{=} \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A + l_B)}$$

$$\times \sum_{n=0}^{\infty} \left(\frac{1}{x^2}\right)^{\frac{1}{2}(n - l_n)} C^{AB}_n x^{\alpha_1} \dots x^{\alpha_n} \Phi(\tfrac{1}{2}(l_A - l_B + l_n + n), l_n + n; x\partial) O_{\alpha_1 \dots \alpha_n}(0) \quad (3.22)$$

in terms of the confluent hypergeometric function $\Phi(a, c, z)$. The improved light cone expansion in eq. (3.22) has the advantage of explicitly satisfying causality and translation invariance.

4. SOME CONSEQUENCES OF CONFORMAL INVARIANCE ON THE LIGHT CONE

When $A = B$, eq. (3.11) implies $C_{nn} = 0$ for odd n. In such a case, only those irreducible representations which start with an irreducible tensor of even order n, which is annihilated by K_λ contribute. Tensors of odd n appearing in the expansion must not be such as to be annihilated by K_λ; thus they will belong to some irreducible representation which starts with a tensor of even order.

As a consequence of eq. (3.19) we observe that, when $m-n = l_B - l_A - l_n$, only the first m-n Clebsch-Gordan coefficients C^{AB}_{nm} of the corresponding representation are different from zero. Then, in the case of 'correlated dimensions', only the first $n = l_B - l_A - l$ representations have a finite number of terms.

With 'correlated dimensions', $l_n = l+n$, translation invariance could, in principle, give rise to relations between irreducible representations of different 'principal quantum number' n. This phenomenon can never happen in other cases; translation invariance tells us that P_μ is a rising operator in each inequivalent representation; in fact, this operator cannot connect with different n as they have a different overall power in x^2.

In the first case, translation invariance gives

$$\sum_{m=0}^{n} [C^{BA}_{n-m,n} + (-1)^{n+1} C^{AB}_{n-mn}]\, O^{n-mn}_{\alpha_1 \dots \alpha_n}$$

$$= \sum_{m=1}^{n} \frac{(-1)^{n-m}}{m!} \sum_{h=0}^{n-m} C^{AB}_{n-m-h,n-m}\, O^{n-m-h,n-m}_{\alpha_1 \dots \alpha_{n-m}, \dots \alpha_n} . \tag{4.1}$$

Eq. (4.1) can be solved by induction. One has

$$C^{AB}_{00} = C^{BA}_{00} ,$$

$$C^{AB}_{00}\, i[O^{00}, P_{\alpha_1}] = (C^{AB}_{11} + C^{BA}_{11}) O^{11}_{\alpha_1} + (C^{BA}_{10} + C^{AB}_{10}) O^{01}_{\alpha_1} , \tag{4.2}$$

etc.

Commuting eq. (4.2) with K_λ one sees that the commutators of $[O^{00}, P_{\alpha_1}]$ and $[O^{01}_{\alpha_1}, P_{\alpha_1}]$ are both proportional to the same operator O^{00}. In fact, O^{11} does not contribute since $[O^{11}_{\alpha_1}, K_\lambda] = 0$. One can, therefore, always define $\tilde{O}^{11}_\alpha$ and $\tilde{O}^{01}_\alpha$ such that

$$[\tilde{O}^{11}_\alpha, K_\lambda] = 0 , \tag{4.3}$$

where $[\tilde{O}^{01}_\alpha, K_\lambda]$ gives $g_{\lambda\alpha} O^{00}$ and $[O^{00}, P_\lambda]$ gives $\tilde{O}^{01}_\lambda$. Extending such a procedure to the higher tensors one concludes that it is always possible by means of translation invariance to redefine the basis such that P_λ acts within each inequivalent representation as a step-up operator.

Eq. (4.1) becomes an identity for $A = B$ and even n. To diagonalize the corresponding even operators one can use the equation for $n+1$ and commute both sides with K_λ.

The above discussion shows, in particular, that a single indecomposable representation (representations which cannot be separated into irreducible components but having invariant non-trivial subspaces) is not sufficient to expand a product of operators near the light cone. In such a case, conformal invariance would tell that it must also be irreducible. An infinite set of operators, each annihilated by K_λ, occurs in the light cone expansion. The coefficients C_{nn}^{AB} cannot be derived from conformal invariance. The condition $l_n = l+n$ assuring scaling of the measured scaling functions is not to be considered as a consequence of conformal invariance alone.

In connection with the problem of 'correlated dimensions' or, particularly, of 'canonical dimensions' ($l = 2$) we can add the following remarks. For a tensor $(\frac{1}{2}n, \frac{1}{2}n), T_{\alpha_1 \ldots \alpha_n}$, satisfying

$$[T_{\alpha_1 \ldots \alpha_n}, P_{\alpha_1}] = 0 \ , \tag{4.4}$$

and annihilated by K_λ, use of the Jacobi identity tells that

$$[[T_{\alpha_1 \ldots \alpha_n}, P_{\alpha_1}], K_\lambda]$$

is proportional to $(l_n - n - 2) T_{\lambda, \alpha_2 \ldots \alpha_n}$. From eq. (4.4) one has

$$l_n = n+2 \ . \tag{4.5}$$

Eq. (4.5) follows from eq. (4.4) but the reverse is not true: it is sufficient that

$$\mathcal{C}_{\alpha_2 \ldots \alpha_n} = [T_{\alpha_1 \ldots \alpha_n}, P_{\alpha_1}] \tag{4.6}$$

realizes an irreducible representation of the conformal algebra* ($K_\lambda = 0$ and well-defined dimension). Note that some care has to be exercised in the application of these results. For instance, for the e.m. field one would derive $l = 3$ from $\partial_\lambda A^\lambda = 0$ and $l = 1$ from $\Box A_\lambda = 0$. But both equations are not conformally invariant and they must be replaced by the Maxwell equation written in a general gauge so that conformal invariance holds.

In conclusion, we can mention that the above results can easily be extended to examine the implications of conformal invariance to expansions of more complicated operators, belonging to more complicated representations of the stability algebra (at $x = 0$).

* This is a particular case of the requirement that a local operator, in order to satisfy a conformal invariant equation, must have a well-defined relation between spin and dimension.

5. EXPLICIT REALIZATIONS IN PARTICULAR MODELS

We first discuss the explicit operator basis in terms of which products of a (supposed conformally invariant) scalar field theory. A set of operators of the order n is provided by the expressions

$$:\partial_{\alpha_1}\dots\partial_{\alpha_{n-m}}(\varphi\partial_{\alpha_{n-m+1}}\dots\partial_{\alpha_n}\varphi):\,. \tag{5.1}$$

Such a basis is neither Hermitian nor does it exhibit simple transformation properties. We, therefore, define

$$\begin{aligned}
O^{00} &= :\varphi\varphi: \\
O^{01}_{\alpha_1} &= :\partial_{\alpha_1}(\varphi\varphi): \\
O^{11}_{\alpha_1} &= :\varphi\overleftrightarrow{\partial}_{\alpha_1}\varphi: \\
O^{02}_{\alpha_1\alpha_2} &= :\partial_{\alpha_1}\partial_{\alpha_2}(\varphi\varphi): \\
O^{12}_{\alpha_1\alpha_2} &= \underset{\{\alpha_1\alpha_2\}}{S} :\partial_{\alpha_1}(\varphi\overleftrightarrow{\partial}_{\alpha_2}\varphi): \\
O^{22}_{\alpha_1\alpha_2} &= \underset{\{\alpha_1\alpha_2\}}{S} [:\partial_{\alpha_1}\partial_{\alpha_2}(\varphi\varphi):-:\varphi\overleftrightarrow{\partial}_{\alpha_1}\overleftrightarrow{\partial}_{\alpha_2}\varphi:]
\end{aligned} \tag{5.2}$$

etc.

The set $O^{nm}_{\alpha_1\dots\alpha_m}$ symmetrized and made traceless provides us with a ladder of irreducible representations of the conformal algebra with the properties we have requested. The spectrum of the Casimir operators C_{I} and C_{III} is given by

$$\tfrac{1}{4}C_{\mathrm{I}}(n) = n(n+1)-2\ , \quad C_{\mathrm{III}}(n) = n(n+1)-3\ , \tag{5.3}$$

where we have inserted the canonical value $l = 2$ (e.g., free field theory). At each n a new operator comes in, which is annihilated by K_λ; in fact, under K_λ every choice of n+1 independent operators of the order n goes into linear combinations of n operators of the order n-1, implying that one linear combination exists which is annihilated by K_λ. The operators $\mathcal{C}_{\alpha_1\dots\alpha_n}$, see eq. (4.6), are annihilated by K_λ, since eq. (4.5) holds.

It is useful to refer to the triangular scheme in fig. 1. The tensors O^{00}, O^{11}, O^{22}, ... on the lowest diagonal are annihilated by K_λ. On the vertical lines are the towers of irreducible Lorentz tensors which build up irreducible representations of the conformal algebra. In such representations other tensors are present, according to the discussion of sect. 3, but they contribute non-leading terms on the light cone. As noted by Mack [5] there

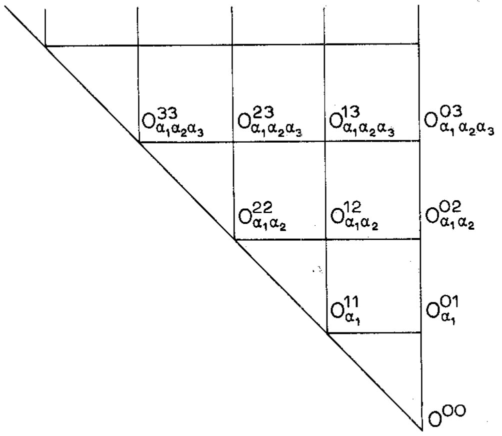

Fig. 1. Triangular scheme classifying the set of operators forming a basis for light-cone expansion.

are, besides irreducible representations, also indecomposable representations (i.e., representations with non-trivial invariant subspace, but which cannot be reduced) which can be used to provide a classification of the sets of operators providing a basis for light-cone expansions. One might think that the set of irreducible representations of the conformal algebra, relevant on the light cone, constitutes an indecomposable representation of the little gorup. One can show that this is not the case. The constraints implied by translation invariance, see eq. (4.1), cannot be satisfied in such a case. Nevertheless, the whole analysis could be reformulated in terms of an infinite set of indecompensable representations. However, the correlation of dimensions required from conformal invariance would in any case have to rest in the dynamics alone.

As an illustration of some of the stated points let us consider in massless free field theory the expansions

$$:\phi^3(x)::\phi(0): \cong \frac{c}{-x^2+i\epsilon x_0} :\phi^2(x):$$

$$:\phi^2(x)::\phi(0): \cong \frac{c}{-x^2+i\epsilon x_0} :\phi(x)\phi(0): + c\text{-number} + \text{h.c.} ,$$

where c is a constant. The left-hand sides behave undistinguishably under dilatations; however, commuting with K_λ brings in the difference $l_A - l_B$. On the right-hand sides, one has a single representation, in the first case, a ladder, as those we have discussed, in the second case.

One can develop a similar discussion of the basis starting from spin-$\frac{1}{2}$ fields. The lowest spin interesting representation starts with the operator $:\bar{\psi}\gamma_\mu\psi:$ (which is annihilated by K_λ]. For what concerns the second-order symmetric tensors, K_λ acting on

$$\underset{\{\alpha\mu\}}{S} :\partial_\alpha(\bar{\psi}\gamma_\mu\psi):$$

leads to $g_{\alpha\mu} :\bar{\psi}\gamma_\lambda\psi:$ while

$$\underset{\{\alpha\mu\}}{S} :\bar{\psi}\gamma_\mu\gamma_5\partial_\alpha\psi:$$

is annihilated by K_λ.

Otherwise we could interchange γ_μ with $\gamma_\mu\gamma_5$ and obtain similar results. Correspondingly, one has two different types of representations (distinguished by parity). For the rest, no essential differences arise.

Expansions of more general products are easily discussed. For example, one could expand products of operators with $\Sigma_{\mu\nu}$, $K_\lambda \neq 0$. For instance, for the product of a conserved current and a scalar operator, one has

$$J_\mu(x)\phi(0) = \sum_{h=1}^{\infty} \sum_{m=n}^{\infty} F^{nm}_{\mu\alpha_1\dots\alpha_m}(x) O^{nm}_{\alpha_1\dots\alpha_m} ,$$

$$F^{nm}_{\mu\alpha_1\dots\alpha_m}(x) = \underset{\{\alpha_1\dots\alpha_m\}}{S} (\partial_\mu\partial_{\alpha_1} - g_{\mu\alpha_1}\Box) F(x^2) x_{\alpha_2}\dots x_{\alpha_m} C_{nm} ,$$

$$F(x^2) = \left(\frac{1}{x^2}\right)^{1+\frac{1}{2}(l_\phi - l)} ,$$

where l_ϕ, l are the dimensions of ϕ and O^{00}, respectively. Conformal invariance implies in this case the absence of the $n = 0$ representation.

6. EXAMPLE OF ALGEBRAIC REALIZATIONS OF CANONICAL DIMENSIONS

In this section, we shall discuss the possibility of connecting the occurrence of canonical dimensions to a possible algebraic structure of the non-local operator $A(x)B(0) = \psi(x)$. As remarked in sect. 1, the convergence of the Bjorken-Callan-Gross sum rules gives a relation between the dimensions of the Lorentz irreducible tensors annihilated by K_λ, in the light-cone expansion, and which belong to inequivalent irreducible representations of the conformal algebra. Such relations are not implied by conformal invariance. Perhaps, one can describe the situation in more transparent terms by considering the non-local operator $\psi(x)$ as providing a description of a hypothetical composite system. On the system, the Casimir operator C_I of the conformal algebra has an infinite spectrum. The conformal algebra plays the role of the degeneracy algebra. Of course, it does not define the dynamics.

For each eigenvalue of the Casimir there is (apart from possible trivial degeneracies) an infinite tower of operators with known dimensions and increasing spins. The coefficients which fix their couplings on the light cone are supposedly obtainable as Clebsch-Gordon coefficients for the conformal algebra (as we have already remarked this implies general dynamical restrictions on the theory). But, in any case, operators which belong to different eigenvalues of the conformal Casimir cannot be related, unless through dynamics. An exceptional case would be one in which only an irreducible representation appears in the expansion. In this case, the structure functions will be a singular δ like distribution.

The analogy with a composite system can be extended further. For instance, for the H atom, we know that the dynamical group (spectrum algebra) is O(4, 1). The energy spectrum of the system can be accommodated in an infinite-dimensional irreducible representation O(4, 1). In the light-cone analysis, we are discussing here, we know, as a main dynamical information, that the operators with $K_\lambda = 0$ have so-called 'canonical dimensions', precisely,

$$\dim O^{nn}_{\alpha_1 \dots \alpha_n} = n+2 \ . \tag{6.1}$$

We can look for a spectrum generating algebra which reproduces the correct spectrum of the Casimir and the correct classification of the operators (such as in eq. (5.3) and in fig. 1, respectively). We recall (see sect. 2) that the conformal algebra acting on the Minkowsky space is isomorphic to the O(4, 2) algebra acting on the light cone in six dimensions. The operator C_{I} is essentially an angular momentum in six dimensions for the composite system described by the 'wave function' $\psi(x)$

$$C_{\mathrm{I}} = J_{AB} J^{AB} \ . \tag{6.2}$$

We can try to enlarge the O(4, 2) algebra in order to induce transitions between states of different six-dimensional angular momentum. The new algebra will have to contain a lowering operator L_ρ that transforms $O^{nn}_{\alpha_1 \dots \alpha_n}$ into $O^{n-1, n-1}_{\alpha_1 \dots \alpha_{n-1}}$. Moreover, it will be appealing to recover the whole spectrum of C_{I} within one single irreducible representation of the algebra.

As an example, we can choose $\mathcal{G} = \mathrm{O}(4,2) \otimes \mathrm{O}(4,2)$ as a possible dynamical algebra for the non-local operator $\psi(x)$. We call J'_{AB} and J''_{AB} the two sets of 15 generators for each of the two factor algebras O(4, 2). In addition, we define

$$J_{AB} = J'_{AB} + J''_{AB} \ , \qquad \tilde{J}_{AB} = J'_{AB} - J''_{AB} \ . \tag{6.3}$$

The generators span an O(4, 2) subalgebra of $\mathcal{G}$, which we identify with the conformal algebra, following eqs. (2.2). We also introduce

$$T_{\mu\nu} = J'_{\mu\nu} - J''_{\mu\nu} \ , \qquad S = J'_{65} - J''_{65} \ ,$$

$$H_\mu = J'_{5\mu} + J'_{6\mu} - (J''_{5\mu} + J''_{6\mu}) \ , \qquad L_\mu = J'_{6\mu} + J''_{5\mu} - (J'_{5\mu} + J''_{6\mu}) \ . \tag{6.4}$$

We then have: for D, K_μ, $M_{\mu\nu}$, P_μ the conformal algebra commutators, and:

$$[D, T_{\mu\nu}] = 0 \ , \qquad [D, S] = 0 \ , \qquad [D, L_\mu] = iL_\mu \ , \qquad [D, H_\mu] = -iH_\mu \ ,$$

$$[P_\mu, H_\sigma] = 0 \ , \qquad [P_\mu, S] = iH_\mu \ , \qquad [P_\mu, L_\sigma] = 2i(g_{\mu\sigma}S + T_{\mu\sigma}) \ ,$$

$$[P_\mu, T_{\rho\sigma}] = i(g_{\mu\rho}H_\sigma - g_{\mu\sigma}H_\rho) \ , \qquad [K_\mu, H_\sigma] = -2i(g_{\mu\sigma}S + T_{\mu\sigma}) \ ,$$

$$[K_\mu, S] = iL_\mu \ , \qquad [K_\mu, L_\rho] = 0 \ , \qquad [K_\mu, T_{\rho\sigma}] = i(g_{\mu\rho}L_\sigma - g_{\mu\sigma}L_\rho) \ ,$$

$$[M_{\rho\sigma}, T_{\mu\nu}] = 0 \ , \qquad [M_{\mu\nu}, S] = 0 \ , \qquad [M_{\mu\nu}, L_\rho] = -i(g_{\rho\mu}L_\nu - g_{\rho\nu}L_\mu) \ ,$$

$$[M_{\mu\nu}, H_\rho] = -i(g_{\rho\mu}H_\nu - g_{\rho\nu}H_\mu) \ ,$$

$$[S, L_\nu] = +iK_\nu \ , \qquad [S, H_\nu] = -iP_\nu \ , \qquad [H_\nu, L_\mu] = +2i(g_{\nu\mu}D + M_{\mu\nu}) \ ,$$

$$[S, T_{\rho\nu}] = 0 \ , \qquad [H_\nu, H_\mu] = 0 \ , \qquad [L_\nu, L_\mu] = 0 \ ,$$

$$[H_\nu, T_{\rho\sigma}] = i(g_{\nu\rho}P_\sigma - g_{\nu\sigma}P_\rho) \ , \qquad [L_\nu, T_{\rho\sigma}] = i(g_{\nu\rho}K_\sigma - g_{\nu\sigma}K_\rho) \ . \tag{6.5}$$

In particular, L_μ is a dimension-lowering operator which, besides, maps states which are annihilated by K_μ into similar states. We can, therefore, employ L_μ to induce transitions among inequivalent representations within the basis. We have

$$[C_{\mathrm{I}}, L_\rho] = -6L_\rho + 4iM_{\rho\nu}L^\nu + 2[P_\mu, L_\rho]K^\mu - 8L_\rho - 4iDL_\rho - 2L_\rho \ , \tag{6.6}$$

and when applied to a state which is annihilated by K_μ,

$$[C_{\mathrm{I}}, L_\rho] = (-6g_{\rho\nu} + 4iM_{\rho\nu})L^\nu + (4L - 10)L_\rho \ . \tag{6.7}$$

We note that, due to the presence of the term $4iM_{\rho\nu}L^\nu$ in eq. (6.7), L_ρ does not transform an eigenstate of C_{I} into another eigenstate. In fact, because of its vector character L_ρ induces two kinds of transitions

$$(\tfrac{1}{2}n, \tfrac{1}{2}n) \to (\tfrac{1}{2}(n-1), \tfrac{1}{2}(n-1)) \ , \qquad (\tfrac{1}{2}n, \tfrac{1}{2}n) \to (\tfrac{1}{2}(n+1), \tfrac{1}{2}(n+1)) \ . \tag{6.8}$$

We note that one must restrict oneself to representations which contain an operator (called O^{00} in our scheme, see fig. 1) which is annihilated by L_ρ. For such representations the second transition in eq. (6.8) is irrelevant, similarly to what happens for the operator K_λ. The operator C_{I} takes on the eigenvalue $C_{\mathrm{I}}(n-1)$ when acting on the $(\frac{1}{2}(n-1), \frac{1}{2}(n-1))$ irreducible component of the state $L_\rho|n\rangle$. By iteration one can thus conclude that there exist representations of O(4, 2) $\otimes$ O(4, 2) which contain the spectrum of representations of the conformal algebra which is relevant for the light-cone expansion. The spectrum may indeed be richer, but not all representations contribute leading terms on the light cone.

The following relation holds between one Casimir operator, to be called $\mathcal{C}$, of the algebra O(4, 2) ⊗ O(4, 2) and C_{I}

$$\mathcal{C} = C_{\mathrm{I}} - 2J'_{AB}J''^{AB} . \tag{6.9}$$

We can evaluate $\mathcal{C}$ on the irreducible representation of an operator basis whose lowest element is the scalar O^{00} satisfying

$$[M_{\mu\nu}, O^{00}] = 0 , \qquad [K_\rho, O^{00}] = 0 ,$$

$$[L_\rho, O^{00}] = 0 , \qquad [D, O^{00}] = ilO^{00} . \tag{6.10}$$

The eigenvalue of $\mathcal{C}$ is $l(l-8) + \frac{1}{2}(T^2 - 2S^2)^{00}$ where the superscript 00 denotes the eigenvalue on O^{00} and $T^2 = T_{\mu\nu}T^{\mu\nu}$. The acceptable representations of O(4, 2) ⊗ O(4, 2) thus differ in the eigenvalue of $\frac{1}{2}(T^2 - 2S^2)$ on the scalar object. However, they are also restricted by the fact that they have to contain representations of the conformal algebra with $C_{\mathrm{II}} = 0$ (see sect. 3).

The algebraic scheme obtained realizes the tower structure illustrated in fig. 1. The canonical value ($l = 2$) is then fixed if the octet currents J_μ and $\theta_{\mu\nu}$ are located within the chosen representations. Eq. (6.4) is then satisfied and complete scaling assured. Besides, we note that for each irreducible Lorentz operator of the basis annihilated by K_μ we have a kind of partial conservation equation: the operator

$$O^{nn}_{\alpha_2 \ldots \alpha_n} = [O^{nn}_{\alpha_1 \ldots \alpha_n}, P_{\alpha_1}]$$

is a local operator annihilated by K_μ.

7. CONCLUSIONS

We have made one essential assumption, that invariance under the infinitesimal generators of the conformal group applies on the light cone, and deduced its consequences. As to the legitimacy of the assumption we can only point out that: (a) the conformal group is in fact the invariance group of the light cone; and that, (b), by a well-known result (see for instance ref. [16]) a Lagrangian theory which is invariant under dilatation is also invariant under conformal transformations in rather general situations (it is sufficient for instance, although not necessary, that it contains no derivative couplings). The result (b) is of relevance if one likes for instance the viewpoint that the theory, which as we know, cannot be exactly scale-invariant, approaches on the light cone some specific scale-invariant theory, which then one may reasonably assume to be also conformally invariant.

Based on our assumption of conformal invariance on the light cone we have discussed the properties of the local operators which appear in the light-cone expansion, by showing that they belong to different irreducible representations of the conformal group, and that in each representation the operator of lowest dimension (in energy units) commutes with the generator of the special conformal transformations and contributes to the scaling function. We have also shown how the coefficients of the light-cone expansion are fixed within each irreducible tower, such that the entire dynamical content is contained in the expansion coefficients multiplying the operators of lowest dimensionality.

We have then attacked the problem that we consider of principal interest in this whole field, namely that of understanding the dynamical basis of the fundamental scaling relation $l = s+2$, among the dimensions l of the operators in the light-cone expansion and their spin. Such a relation follows from the existence of the scaling functions and our preceding analysis eliminates any doubt that it may simply follow from conformal invariance *alone* without having to add some additional dynamical assumption. To this regard we have essentially one result and one suggestion. The result is a theorem which shows that the scaling relation is equivalent to an infinite set of what we call 'generalized partial conservation equations'. A special case, trivial, but in fact coincident with the one which usually is advocated, is that of an infinite number of exact conservations (free massless field theory on the light cone, etc.). The suggestion we have is the following. The light-cone expansion of the product of two local operators may be subjected to the kind of group-theoretical analysis that has proven to be useful for compound systems (dynamical groups). One is in fact dealing with bilocal operators which may be thought of as describing compound systems. The example we offer is that of a rather straightforward extension of the conformal algebra, but we think that much more is to be done on this approach.

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