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F. Amman: LONGITUDINAL INSTABILITY DUE TO BEAM-BEAM INTERACTION IN ELECTRON STORAGE RINGS. -

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LNF-71/82 - addendum

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Addendum

The relative change in the closed orbit length due to an angular perturbation at the azimuthal position s_o is very simply related to the off-energy function ψ at the same position s_o : eq. (14) can be written :

$$\eta = \frac{1}{2\pi R}\int_{s_o}^{s_o+2\pi R} \frac{x_\Delta}{\rho_m} ds = \frac{\Delta}{2\pi R}\psi(s_o) \tag{14'}$$

and the coefficient g, introduced in eq. (14), is therefore :

$$g = \frac{\psi}{\alpha R} \quad . \tag{14''}$$

The limit for the longitudinal stability, eq. (25), becomes :

$$\frac{\xi_{x,1}}{1+2\pi\xi_{x,1}\,\mathrm{ctg}(Q_x\frac{\pi}{m})} + \frac{\xi_{x,2}}{1+2\pi\xi_{x,2}\,\mathrm{ctg}(Q_x\frac{\pi}{m})} \leq \frac{\beta_x\alpha R}{2m\psi^2} \tag{25'}$$

or, when the terms $2\pi\xi\,\mathrm{ctg}(Q_x\frac{\pi}{m})$ can be neglected, as it is usually the case :

$$(\xi_{x,1} + \xi_{x,2}) \leq \frac{\beta_x \alpha R}{2m\psi^2} \quad . \tag{25''}$$

When the β function does not have wild variations ($\beta_{max} - \beta_{min} \gtrsim \langle\beta\rangle$), the off-energy function is approximately given by[1] :

$$\psi(s) \approx \left(\frac{\alpha R\beta(s)}{Q_x}\right)^{1/2} \tag{26}$$

in this case eq. (25") becomes :

$$(\xi_{x,1} + \xi_{x,2}) \lessapprox \frac{Q_x}{2m} \qquad \text{(approximate)} . \tag{25'''}$$

The approximation does not clearly apply to structures with low β insertions.

Eq. (14') can be derived as follows : in presence of a forcing term F(s), periodic with $2\pi R$, the closed orbit x_c, is the solution of the equation :

$$x_c'' + K(s)x_c = F(s) \qquad \text{where } K(s) = \frac{1}{\rho_m^2} + \frac{\frac{\partial B_z}{\partial x}}{B_o\rho_m} \tag{27}$$

which can be written as[1]:

$$(28) \qquad x_c(s) = \frac{\sqrt{\beta(s)}}{2\,\mathrm{sen}\,\pi Q_x} \int_s^{s+2\pi R} F(\bar{s}) \sqrt{\beta(\bar{s})}\, \cos\left\{\phi(\bar{s}) - \phi(s) - \pi Q_x\right\} d\bar{s}$$

where

$$\phi(s) = \int_0^s \frac{ds}{\beta} \quad .$$

If the forcing term is an angular perturbation Δ, $F(s) = \Delta \cdot \delta\,(s-s_o)$ and eq. (28) becomes:

$$(28') \qquad x_\Delta(s/s_o) = \frac{\sqrt{\beta(s)}}{2\,\mathrm{sen}\,\pi Q_x}\, \Delta \sqrt{\beta(s_o)}\, \cos\left\{\phi(s_o) - \phi(s) - \pi Q_x\right\}$$

where

$$s \leq s_o \leq s + 2\pi R \,.$$

For the periodicity of the closed orbit, and as $\phi(s-2\pi R) = -2\pi Q_x + \phi(s)$, we have also:

$$(28'') \qquad x_\Delta(s-2\pi R|\, s_o) \equiv x_\Delta(s|s_o) = \frac{\sqrt{\beta(s)}}{2 \sin Q_x \pi}\, \Delta \sqrt{\beta(s_o)} \cos\left\{\phi(s) - \phi(s_o) - \pi Q_x\right\}$$

with

$$s_o \leq s \leq s_o + 2\pi R \,.$$

Comparing eqs. (28') and (28'') we obtain $x_\Delta(s|s_o) \equiv x_\Delta(s_o|s)$; in eqs. (28') and (28'') the difference $\pm\left[\phi(s) - \phi(s_o)\right]$ must be taken mod. $2\pi Q_x$, bacause of the angular discontinuity in $s = s_o$.

The change in the closed orbit length due to the angular perturbation in $s=s_o$ is:

$$(29) \qquad \eta(s_o) = \frac{1}{2\pi R} \int_{s_o}^{s_o+2\pi R} \frac{x_\Delta(s|s_o)}{\rho_m} ds = \frac{\Delta}{2\pi R}\, \frac{\sqrt{\beta(s_o)}}{2 \sin \pi Q_x} \int_{s_o}^{s_o+2\pi R} \frac{\sqrt{\beta(s)}}{\rho_m} \cos\left\{\phi(s) - \phi(s_o) - \pi Q_x\right\} ds$$

From eqs. (28) and (29) one obtains that $\eta(s_o)$ is proportional to the solution of eq. (27) with a forcing term equal to the inverse of the radius of curvature in the bending magnets, $F(s) = 1/\rho_m$; this is, by definition, the off-energy function $\psi(s)$, solution of the equation:

$$(30) \qquad \psi'' + K(s)\psi = \frac{1}{\rho_m} \quad .$$

Eq. (14') is therefore proved.

I am indebted to J. Le Duff and D. Potaux for suggesting the connection between the quantities η and ψ that simplifies the equation giving the limit of the longitudinal instability due to beam-beam interaction.

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Abstract. -

The change in the beam orbit lengtn due to an angular perturbation introduces a coupling between transverse and longitudinal motion in a circular accelerator. This effect sets a limit on the maximum transverse density of the beams (and therefore on the luminosity) in a storage ring, that, in particular cases, can be lower than the transverse limit due to the betatron frequency shift; to make full use of the low β insertions additional conditions on the magnetic structure of the ring have to be satisfied.

1. - Equations of the longitudinal motion in a circular accelerator in presence of an orbit length perturbation. -

The equations of the longitudinal motion for the unperturbed machine, in the small amplitude approximation, are first derived, along the lines of ref. (1).

The time of passage of the particle in the RF cavity at the n-th revolution is:

$$t = t_s + nT_o - \tau \qquad (1)$$

T_o being the revolution time; t_s the passage of the reference particle (that, in the unperturbed machine, is synchronous with the RF field); τ is assumed to be positive when the particle considered passes the

RF cavity ahead of the synchronous particle.

The energy gain in the RF field, whose frequency is h times the angular revolution frequency ω_o and whose maximum integrated value over the particle path is V, is given by:

(2) $$\Delta E_{RF} = eV \sin h\omega_o t = eV \sin h\omega_o (t_s - \tau) .$$

To keep the usual convention for the synchronous phase φ_s let us define:

(3) $$h\omega_o t_s = \pi - \varphi_s ; \qquad h\omega_o \tau = \varphi$$

Eq. (2) becomes

(4) $$\Delta E_{RF} = eV \sin \left[\pi - (\varphi + \varphi_s)\right] = eV \sin(\varphi + \varphi_s)$$

The energy radiated in one revolution by a particle of energy $E = E_s(1 + p)$ is:

(5) $$u_r = u_s + \left(\frac{\partial u}{\partial p}\right)_s p = u_s + D \cdot p = eV \sin \varphi_s + D \cdot p$$

The energy balance in one revolution can be written as follows:

(6) $$E_s \cdot \Delta p = \Delta E_{RF} - u_r = eV \sin(\varphi + \varphi_s) - u_r$$

and, in the small amplitude approximation ($\varphi \ll 1$):

(7) $$E_s \cdot \Delta p \cong \varphi \cdot eV \cos \varphi_s - D \cdot p$$

Dividing both members of eq. (7) by the revolution period $2\pi/\omega_o$ and substituting the time derivative to the finite difference we obtain:

(8) $$\dot{p} = \frac{eV \cos \varphi_s}{E_s} \frac{\omega_o}{2\pi} \varphi - \frac{\omega_o}{2\pi} \frac{D}{E_s} p .$$

A second differential equation for the functions p and φ can be obtained writing the variation of φ in one revolution due to the difference in orbit length, with respect to the synchronous orbit, when $p \neq 0$:

(9) $$\Delta\varphi = - \alpha p \cdot 2\pi h$$

where α is the momentum compaction.

Using the time derivative instead of the finite difference we obtain :

(10) $$\dot{\varphi} = - \alpha h \omega_o p .$$

Eqs. (8) and (10) can be combined to obtain two second order differential equations for p and φ :

(11)
$$\ddot{p} + 2\rho\dot{p} + \Omega^2 p = 0 \qquad \rho = \frac{\omega_o}{4\pi}\frac{D}{E_s} ;$$
$$\ddot{\varphi} + 2\rho\dot{\varphi} + \Omega^2\varphi = 0 \qquad \Omega^2 = \omega_o^2 \frac{\alpha h}{2\pi} \frac{eV\cos\varphi_s}{E_s}$$

Remember that φ is defined positive when the particle considered is ahead of the synchronous particle, and $\dot{\varphi}$ has the opposite sign as compared to p (see eq. (10)).

Let us now introduce a perturbation in the orbit length not due to an energy variation and call η the orbit length difference relative to the unperturbed length. In the following the suffix s will refer to the synchronous particle in the unperturbed machine, which will not be anymore synchronous with the RF for $\eta \neq 0$; to avoid confusion we will speak of reference particle.

Eqs. (8) and (10) become :

(8') $$\dot{p} = \frac{eV\cos\varphi_s}{E_s}\frac{\omega_o}{2\pi}\varphi - \frac{\omega_o}{2\pi}\frac{D}{E_s}p - \frac{\omega_o}{2\pi}\frac{F}{E_s}\eta$$

(10') $$\dot{\varphi} = - \alpha h \omega_o p - h\omega_o\eta = -h\omega_o(\alpha p + \eta)$$

where
$$F = \left(\frac{\partial u_r}{\partial\eta}\right)_s .$$

In a ring where all the bending magnets have the same radius of curvature and index n the terms D and F can be expressed as follows, u_s being the energy radiated per turn by the reference particle[2] :

(12)
$$D = \left(\frac{\partial u_r}{\partial p}\right)_s = u_s\left[2 + (1-2n)\frac{R}{\rho_m}\alpha\right] = 2u_s + \alpha F$$
$$F = \left(\frac{\partial u_r}{\partial\eta}\right)_s = u_s(1-2n)\frac{R}{\rho_m}$$

where ρ_m and R are the magnetic radius and the average radius of the ring.

The corresponding damping coefficients can be defined :

(13)
$$\rho = \bar{\rho} + \rho' = \frac{\omega_o}{4\pi}\frac{D}{E_s} \; ; \qquad \bar{\rho} = \frac{\omega_o}{4\pi}\frac{2\,u_s}{E_s} \; ;$$
$$\rho' = \frac{\omega_o}{4\pi}\frac{u_s}{E_s}(1 - 2\,n)\frac{R}{\rho_m}\,\alpha$$

ρ' is proportional to αF, and is zero when $n = 0.5$.

Eqs. (8') and (10') can be combined to obtain :

(11')
$$\ddot{p} + 2\rho\dot{p} + \Omega^2 p = -\frac{\Omega^2}{\alpha}\,\eta - \frac{2\rho'}{\alpha}\,\dot{\eta}$$
$$\ddot{\varphi} + 2\rho\dot{\varphi} + \Omega^2\varphi = -h\,\omega_o \cdot 2\bar{\rho}\,\eta - h\,\omega_o\dot{\eta}$$

The particular solutions of eqs. (11') for $\dot{\eta} = 0$ and $\eta \neq 0$ give the energy difference and the synchronous phase variation of the particle synchronous with the RF field, in presence of a constant angular perturbation, with respect to the energy and synchronous phase of the reference particle in the unperturbed machine.

2. - Variation of the orbit length due to transverse forces. -

An angular perturbation in the radial plane produces, in general, a change in the orbit length; if Δ is the angular perturbation and x_Δ the radial displacement of the perturbed orbit with respect to the unperturbed one, the relative change in orbit length η is given by :

(14)
$$\eta = \frac{1}{2\pi R}\int \frac{x_\Delta}{\rho_m}\,ds = \frac{g}{2\pi}\,\alpha\,\Delta$$

the coefficient g must be determined through the integral and depends on the magnetic structure of the ring : in smooth approximation $g = 1$; in ADONE, when Δ is in the center of a straight section, $g = 1.85$; in particular cases g can be made equal to zero.

If there are m equal angular perturbations in equivalent positions along the ring η is given by :

(15)
$$\eta = \frac{g}{2\pi}\,\alpha\,m\Delta$$

Eqs. (15) and (11') show that, when the particle orbit length depends on an angular perturbation, a transverse force (the cause of Δ) has effects on the longitudinal motion.

Very strong transverse forces are present in the beam-beam interaction; if we have a bunch of N_2 particles, energy γ, transverse r.m.s. dimensions σ_x and σ_z (a gaussian distribution is assumed) and a counter rotating particle 1 of opposite sign, crossing the bunch at a radial distance $x_{1,2}$ from its center, in a position where the radial betatron reduced wave length is β_x, the angular perturbation induced on particle 1 is:

(16) $$\Delta_1 = - \frac{4\pi \xi_{x,2}}{\beta_x} x_{1,2} = A_2 x_{1,2}$$

with

(17) $$\xi_{x,2} = \frac{r_e N_2 \beta_x}{2\pi \sigma_x (\sigma_x + \sigma_z)\gamma}$$

r_e is the classical electron radius.

Notice that Δ_1 depends only on the transverse density of beam 2 (proportional to ξ/β) and not on the local β; the quantity ξ is introduced here, as it is the usual characteristic quantity related to the betatron frequency shift, that, in the so called "optical model", is conveniently used to express the transverse beam-beam limit.

As we are not here interested in the betatron oscillations, the radial distance $x_{1,2}$ can be taken as the sum of the closed orbit due to the energy difference from that of the reference particle and of the closed orbit due to the angular perturbation:

(18) $$x_{1,2} = \psi p_1 + x_{1\Delta} = \psi p_1 + \frac{\beta_x \Delta_1}{2} \operatorname{ctg}\left(Q_x \frac{\pi}{m}\right)$$

where m is the number of crossings in the ring, Q_x is the radial betatron wave number and ψ is the closed orbit function.

From eqs. (16) and (18) we obtain:

(19) $$\Delta_1 = \frac{A_2 \psi}{1 - \frac{A_2 \beta_x}{2} \operatorname{ctg}\left(Q_x \frac{\pi}{m}\right)} p_1 = \frac{- \frac{4\pi \xi_{x,2}}{\beta_x} \psi}{1 + 2\pi \xi_{x,2} \operatorname{ctg}\left(Q_x \frac{\pi}{m}\right)} p_1$$

Eqs. (15) and (19) give η as a function of p_1:

(20) $$\eta = \frac{-2 g \alpha m \, \xi_{x,2} \frac{\psi}{\beta_x}}{1 + 2\pi \xi_{x,2} \operatorname{ctg}\left(Q_x \frac{\pi}{m}\right)} p_2 = -\alpha k_2 p_1$$

Similar equations can be obtained for the effect on the longitudinal motion of a feedback used to stabilize the transverse instabilities ; in eqs. (16) and (19) for A_2 one must introduce the characteristic coefficient of the feedback, taking into account that the force (or the angular perturbation) at the time t is proportional to the displacement at an earlier time (one or more revolutions) ; it can be seen easily, following the same line as for the beam-beam interaction, that the effects are completely negligible, at least in normal cases.

3. - Longitudinal instability due to beam-beam interaction. -

Eq. (20) represents the variation of the orbit length of a single particle interacting with a counter rotating beam ; the energy equation (eq. (11')) becomes :

(21) $$\ddot{p}_1 + 2(\rho - k_2\rho')\dot{p} + \Omega^2(1 - k_2)p = 0$$

As $\rho' < \rho$ the single particle longitudinal stability condition is:

(22) $$k_2 = \frac{2\,m\,g\,\frac{\psi}{\beta_x}\,\xi_{x,2}}{1 + 2\pi\,\xi_{x,2}\,\mathrm{ctg}(Q_x\frac{\pi}{m})} \leq 1$$

This condition sets a limit on the transverse density (proportional to ξ/β) which is independent from the β at the crossing.

If we now consider the collective longitudinal motion of two rigid beams, we obtain the coupled equations :

(23) $$\ddot{p}_1 + 2\left[\rho_1 - k_2\rho_1'\right]\dot{p}_1 + \Omega_1^2(1-k_2)p_1 = -k_2\,\Omega_1^2 p_2 - 2\rho_1' k_2 \dot{p}_2$$

$$\ddot{p}_2 + 2\left[\rho_2 - k_1\rho_2'\right]\dot{p}_2 + \Omega_2^2(1-k_1)p_2 = -k_1\,\Omega_2^2 p_1 - 2\rho_2' k_1 \dot{p}_1$$

The Routh's stability conditions gives :

(24) $$k_1 + k_2 \leq 1$$

or

(25) $$\left(\frac{\xi_{x,1}}{1+2\pi\,\xi_{x,1}\mathrm{ctg}(Q_x\frac{\pi}{m})} + \frac{\xi_{x,2}}{1+2\pi\,\xi_{x,2}\mathrm{ctg}(Q_x\frac{\pi}{m})}\right) \leq \frac{\beta_x}{2\,m\,g\,\psi}$$

When the stability limit is reached, the synchrotron frequency of the relative mode of oscillation of the two beams becomes zero; as the linearized analysis breaks down at large amplitudes, the limit given by eq. (25) has to be considered a conservative estimate.

In ADONE we have $m = 6$; $g = 1.85$; $\beta_x = 8.9$ m; $\psi = 2.1$ m, and the correction terms $2\pi \xi \,\mathrm{ctg}(Q_x \pi/m)$ are completely negligible: the longitudinal stability limit should therefore be:

$$(\xi_{x,1} + \xi_{x,2}) \leq \frac{\beta_x}{2mg\psi} = 0.19 .$$

The maximum value experimentally observed for $(\xi_{x,1} + \xi_{x,2})$ is about 0.15 and, presumably, is not limited by the longitudinal effect.

4. - Conclusions. -

The influence of the transverse forces, due to beam-beam interaction, on the longitudinal motion stability sets a limit on the maximum transverse density of the beams in a storage ring which is independent from the radial β value at the crossing. If additional conditions are not satisfied (either low ψ at the crossing or low g) this effect may become the most severe limitation on the luminosity in a low β structure, more stringent than the usual "optical limit" on the betatron frequency shift.

In the rings with a low β matched insertion the condition of $g = 0$ can be easily obtained by having an integral number of betatron oscillations over the normal sectors and a distribution of the bending magnets with the proper simmetry.

References. -

(1) - M. Sands, The physics of electron storage rings. An introduction, Report SLAC-121 (1970).
(2) - C. Pellegrini, Suppl. Nuovo Cimento 22, 603 (1961).