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G. Parisi: THE DYNAMICS OF CONFORMAL FIELD THEORIES (I) EXACT INVARIANCE. -

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SUMMARY. -

In this work we write the costraints that exact conformal invariance puts on the various Green functions. We use conformal invariance to determine the solutions of the Dyson equation for the vertex and of the unitarity equation for the propagator. They are reduced to numerical equations and their solution fixes the value of the dimension. The relations with the Callan-Symanzik equation are briefly discussed.

INTRODUCTION. -

In the last years there has been great theoretical interest on the scaling invariance and on its breaking. The possibility that scale invariance is a weakly broken simmetry(1) and that can be treated on the same footing of broken SU(3) x SU(3) was introduced to explain SLAC results on deep inelastic electron proton scattering.

However a non trivial scaling invariant field theory has patologies like anomalous dimensions(2) and there is no simple connection between weakly broken scaling invariance an scaling in the deep inelastic scattering.

It was soon realized that at least formally a scaling invariant field theory is also invariant under the conformal group(3). This larger invariance group puts very hard restriction on the behaviour of the n-point functions, e. g. the vertex function is completely determined in a kinematic way(4, 5, 6). There are two ways to use this new costraint : the first is to study the restriction that conformal covariance puts on the Wilson expansion(7), the other is to try to find dynamically conformal covariant solutions of quantum field theory. This is possible because a great part of the momentum dependence of the Green fucntion is transferred from dynamics to kinematics. We shall follow this last dynamical approach.

In this work we show how in a conformally invariant theory the Dyson equation for the vertex and the unitarity equations for the propagator are reduced from non linear coupled integral equations to numerical equations(8). The solution of these equations fixes the value of the anomalous dimension of the field and of the effective coupling costant. In a forthcoming paper(9) we shall analize the deep interrelation between the standard perturbation expansion for a scaling invariant field theory, the renormalization group and conformal invariance. We shall also see how compute in a simple way many Feynmann diagrams.

In the first section of this paper we shall collect some standard result on the conformal group. Although we limit ourselves to the study of the conformal group constructed from a 4 dimensional space the extension to N-dimensional space is trivial. This remark is valid also for the other sections.

At this early stage of the theory we limit ourselves to the more simple case, the $\lambda\phi^3$ theory. In the second section we write the Dyson equation for the vertex without the inomogeneous term. This is equivalent to the hypothesis that this theory has a scaling invariant solution (this is true in perturbation theory only on a 6 dimensional space). We solve this simplified equation and this is possible because conformal invariance fixes the form of the three point function. We obtain in this way a relation between the dimension and the effective coupling costant(6).

In the third section we write down the unitarity condition for the propagator. The solution of this new equation produces another relation between the coupling costant and the dimension which turn out to be completelly fixed. At the end of this section we compare our work with the prediction of the Callan-Symanzik equation. We find no contradiction between them.

In the last section we briefly discuss our results and report the result of a similar analysis on the problem of II order phase transitions.

1. - THE CONFORMAL GROUP. -

The conformal group is the largest space time transformation group that leaves the light cone invariant.

It contains 15 generators :

P_μ (4) translations,

$M_{\mu\nu}$ (6) spatial rotations and special Lorentz transformations,

D (1) dilatation: $\delta x_\mu = \alpha x_\mu$

K_μ (4) special conformal transformations: $\delta x_\mu = \alpha(x_\mu x_\nu - x^2 g_{\mu\nu})$

The commutation relations between the generators are the following:

$$
(1) \qquad
\begin{aligned}
&[D, P_\mu] = -iP_\mu && [D, K_\mu] = iK_\mu \\
&[K_\mu, P_\nu] = -2i(g_{\mu\nu}D + M_\mu \;) && [K_\mu, K_\nu] = 0 \\
&[K_\rho, M_{\mu\nu}] = i[g_{\rho\mu}K_\nu - g_{\rho\nu}K_\mu] && [D, M_{\mu\nu}] = 0
\end{aligned}
$$

plus those of Poincarè algebra.

The action of this generators on a scalar field may be assumed to be:

$$
(2) \qquad
\begin{aligned}
&D\phi(x) = (ix_\nu \partial^\nu + i\Delta)\phi(x) \\
&K_\mu \phi(x) = i(2x_\mu x_\nu \partial^\nu - x^2 \partial_\mu + 2x^\nu g_{\mu\nu}\Delta)\phi(x)
\end{aligned}
$$

where Δ is the so called dimension of the field.

If ϕ is a free field or satisfies C.C.R. $\delta(x^0)[\phi(x), \dot{\phi}(0)] = \delta^4(x)$: $\Delta = 1$. If ϕ is an interaction field that does not satisfy C.C.R. $\Delta \gtrsim 1$. The impossibility of having $\Delta < 1$ follows from the Kallen-Lehemann reppresentation.

In a lagrangian field theory it is possible to write the generators of the conformal group in terms of the energy momentum tensor(10). The time dependency of the generators may be explicetely computed. It comes out that if D is time independent and there are no derivative coupling also K_μ is time independent(7).

This fact has as a consequence that in làgrangian field theory without derivative coupling exact scale invariance implies exact invariance under the whole conformal group. Of course this statement is obtained in a formal way and may be completely false in a non trivial scale invariant field theory.

We suppose that the generators of the conformal group are time independent and that they annihilate the vacuum. These assumptions have very sharp consequences(4, 6, 7):

$$\langle 0 | T(A(x)\,B(y)|0\rangle = 0 \qquad \text{if} \qquad \Delta_A \neq \Delta_B$$

$$\langle 0 | T(A(x)\,B(y)|0\rangle = (x^2 - i\varepsilon)^{-\Delta_A} \qquad \text{if} \qquad \Delta_A = \Delta_B$$

(3)

$$\langle 0 | T(A(x)\,A(y)\,A(z))|\,0\rangle =$$

$$= \left[(x-y)^2 - i\varepsilon\right]^{-\Delta/2} \left[(y-z)^2 - i\varepsilon\right]^{-\Delta/2} \left[(z-x)^2 - i\varepsilon\right]^{-\Delta/2}$$

Generally speaking the conformal group puts N scalar costraints on the N point function, which is function of $N(N-1)/2$ indipendent scalar variables. For $N = 1, 2, 3$ the function is completely determined and for $N \geq 4$ we find an arbitrary function of only $N(N-3)/2$ variables. We stress that till now we have imposed covariance of the field only for infinitesimal conformal transformations, not for finite one. The first does not imply the second because the action of finite conformal transformations is not a continuous one (a time like vector may become space like). All no-go theorems on a theory invariant under the whole conformal group are not relevant in our case.

The main result of this analysis is that in conformally covariant field theory the exact form of the 3 point function for the interacting field is determined not from dynamics but only from kinematics. This result is the essential tool we shall use in the next section to find a solution to the Dyson equation for the vertex.

2. - THE DYSON EQUATION. -

It is well known it is possible to derive from the perturbation expansion some integral equations for the propagator and the vertex which are known as Dyson equations. This equations seem to be more reliable than perturbation theory itself.

In a $\lambda\phi^3$ theory we can write the following Dyson equation for the renormalized vertex V and for the propagator D :

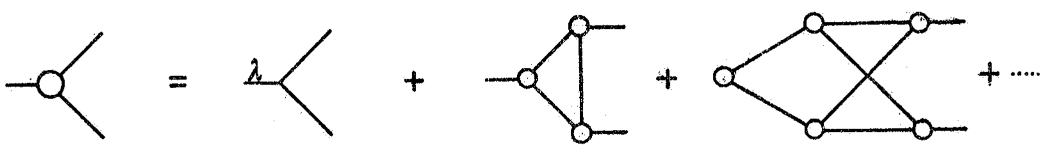

FIG. 1

where we must sum over all the skeletron diagrams and λ is the coupling costant.

If this equation has a scale invariant solution the term with λ scales in a different way from all the other terms and so λ must be zero : an exact scaling invariant solution of the Dyson equation is possible only in the limit $\lambda \doteq 0$. This condition is known also as the bootstrap condition(6).

It is very interesting to see that scale invariance is in some sense connected with bootstrap. This explains why in the next section we shall find determined and not arbitrary values of the ϕ-field dimension and of the effective coupling constant.

The equivalence in lagrangian field theory between scale and conformal invariance suggest that a Dyson equation having a scale invariant solution ($\lambda = 0$), may have also a conformal invariant solution.

The formal derivation of this result is due to Poliakov(4) in the case of $\lambda\phi^4$ interaction and can be readly extended to any other theory.

He has also shown that if we construct the N point vertex from the three point vertex in the standard way:

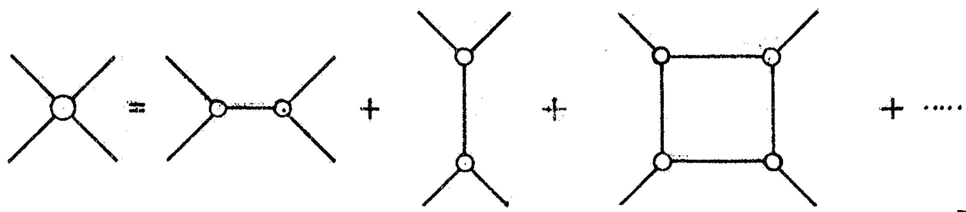

FIG. 2

the N point vertex will be conformal invariant.

We try now to see if the following functions may be solution of the Dyson equation:

(4)
$$D(x,y) = d(x-y)^{-2\Delta}$$
$$V(x,y,z) = g(x-y)^{-2+\Delta}(y-z)^{-2+\Delta}(z-x)^{-2+\Delta}$$

If we introduce them in the equation of Fig. 1 we find that each term of the r. h. s. is a conformal invariant function: this is due to the Poliakov theorem; but there exist only one conformal invariant 3 point fucntion: V itself. Any term of the r. h. s. well be clearly proportional to V ; The constant of proportionality will be clearly Δ dependent.

We can thus rewrite the Dyson equation in the following way:

(5)
$$g(x-y)^{-2+\Delta}(y-z)^{-2+\Delta}(z-x)^{-2+\Delta} = V(x,y,z) =$$
$$= g^3 d^3 f_1(\Delta)(x-y)^{-2+\Delta}(y-z)^{-2+\Delta}(z-x)^{-2+\Delta} +$$
$$+ g^5 d^6 f_2(\Delta)\left[(x-y)(y-z)(z-x)\right]^{-2+\Delta} + \cdots$$

If we divide by $V(x,y,z)$ we find:

$$1 = g^2 d^3 f_1(\Delta) + g^4 d^6 f_2(\Delta) + g^6 d^9 f_3(\Delta) \quad \cdots\cdots =$$
$$(6) \qquad = \alpha f_1(\Delta) + \alpha^2 f_2(\Delta) + \alpha^3 f_3(\Delta) \quad \cdots = F(\Delta, \alpha)$$

where $\alpha = g^2 d^3$ is the effective coupling costant.

If α and Δ are such that $F(\Delta, \alpha) = 1$ the functions (4) is a solution of the Dyson equation. In this way we succeded to transform the Dyson equation, a non linear integral equation to a numerical equation.

The function F is computed as a series in α. This series may be slowly convergent or not convergent at all. In this last case a possible may to give a meaning to the series is use of the Padè approximants. The coefficients of the single power of α clearly depend from Δ and may be expressed as integrals. For example:

$$(7) \qquad \begin{aligned} f_1(\Delta) = \int & dx_1\, dx_2\, dy_1\, dy_2\, dz_1\, dz_2\, (x-x_1)^{-2+\Delta} (x-x_2)^{-2+\Delta} \cdot \\ & \cdot (x_1-x_2)^{-2+\Delta} (x_2-y_2)^{-2+\Delta} (y_2-y)^{-2+\Delta} (y_1-y)^{-2+\Delta} - \\ & - (y_2-y_1)^{-2+\Delta} (y_1-z_2)^{-2\Delta} (z_2-z)^{-2+\Delta} (z-z_1)^{-2+\Delta} \cdot \\ & \cdot (z_1-z_2)^{-2+\Delta} (z_1-x_1)^{-2\Delta} \Big/ (x-y)^{-2+\Delta} (y-z)^{-2+\Delta} (z-x)^{-2+\Delta} \end{aligned}$$

The Poliakov theorem assures us that $f_1(\Delta)$ is independent from the choice of the points x, y, z.

If the integrals defining the r. h. s. of (6) are divergent, they can be computed in the region of Δ where they converge. They turn out to be analitic in Δ and consequently can be analitically continued to any value of Δ. The resulting functions have some poles at certain values of Δ and have no cuts: they are free both of ultraviolet and infrared divergences. This method of analitic continuation is commonly used in the framework of analitic renormalization(11).

The demostratiön of the existence of a region in Δ of convergence of the integrals and of the analitic properties of the functions will be the object of a forthcoming paper.

3. - THE UNITARITY CONDITION. -

As we have seen in the last section the Dyson equation for the vertex yelds a relation between the effective coupling and the dimension Δ of the interpolating field. In this section we shall use the unitarity condition for the propagator to derive a new relation between α and Δ .

The unitarity condition for the propagator may be written in the following diagrammatic way(12) :

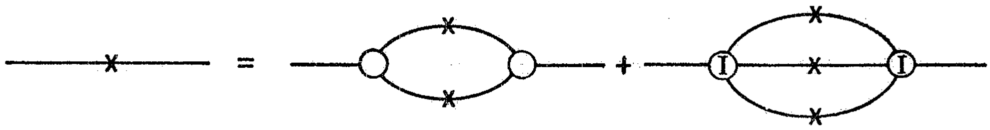

FIG. 3

where ——×—— is the immaginary part of the exact propagator and the bubbles are the exact one particle irrudicible N point vertex func tions. The absence of terms of the form :

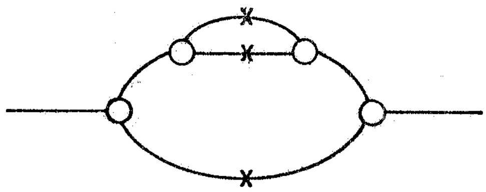

FIG. 4

is due to the use of the immaginary part of the exact propagator : $\theta(K_o)K^{2(-2+\Delta)}$, instead of the commonly used $\theta(K_o)\,\delta(K^2)$. The r. h. s. of the unitarity equation may be expanded in power of α :

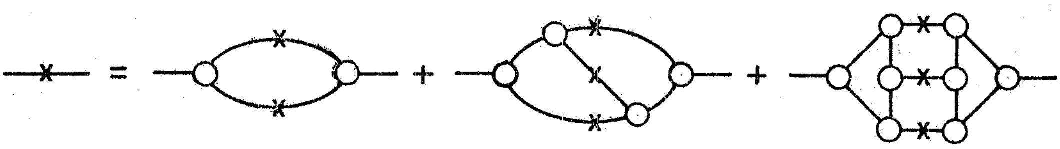

FIG. 5

If we introduce the functions (4) in equation the r. h. s. must be proportional to the left hand side because of scaling invariance; as in the last section the equation can be simplified to :

(8) $$d^{-1} = G_1(\Delta)\,d^2 g^2 + G_2(\Delta)\,d^5 g^4 + \cdots$$

Introducing $\alpha = d^3 g^2$:

(9) $$1 = \sum_{1}^{\infty}{}_n\, G_n(\Delta)\,\alpha^n = G(\Delta, \alpha)$$

Also in this case the functions can be computed by direct integration.

The whole discussion of last section on the convergence of the integrals defining $F_n(\Delta)$ applies also to $G_n(\Delta)$.

Using toghether the equation:

$$(10) \qquad F(\Delta, \alpha) = 1 \quad ; \qquad G(\Delta, \alpha) = 1$$

we get a system of two equations in two variables whose solution fixes the dimension of the field and the effective coupling costant .

This means that the values of the anomalous dimensions and of the effective coupling costant are not arbitrary, but fixed in a scaling and conformal invariant theory. This may seem strange but a similar result may be derived from Callan-Symanzik equation[13, 14]: there exist two functions $\beta(\lambda)$ and $\gamma(\lambda)$ such that the theory is exactly scaling invariant only at $\lambda = \lambda_c$, where λ_c is the first zero of $\beta(\lambda)$, the dimension of field ϕ term out to be: $\Delta = \gamma(\lambda_c)$.

Symanzik's renormalized coupling costant λ is proportional to our effective coupling costant α, so our results are very similar to those obtained in perturbation theory treated to all orders. We think that our equation may be a useful tool to determine the zero of the function $\beta(\lambda)$ and the dimension Δ.

4. - CONCLUSION. -

The main result of this work is that in a $\lambda\phi^3$ theory using conformal covariant functions, we are able to solve the Dyson and unitarity equations and to reduce the problem of computing the anomalous dimension to the solution of a numerical equation. The evaluation of the relevant integrals is cumbersome, but is in progress; the use of renormalization group techniques may be a useful tool for computing all the important contributions.

In a previous work[14] the equations have been used in a slight different context: the determination of the critical indices in the second order phase transition at the λ point of Helium: the two problems of determination of anomalous dimensions and of critical index have indeed many point of similarity[16]. The functions which are the equivalent of our F and G, were computed for the many body problem keepingonly the first order term in α, the effective coupling costant. The critical index ν, which is connected to the behaviour of the superfluid density near the transition turned out to be $\nu = 0.6666$ in very good agreement with the experiment[17] $\nu = 0.666 \pm 0.006$.

We hope that a similar results may be obtained in the next future for a realistic quantum field theory and that it will be possible to compare it with experimental data.

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