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Theoretical Predictions for Critical Exponents at the λ-Point of Bose Liquids.

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In a previous work (1) we pointed out an opportunity for theoretical predictions of critical exponents at the λ-point of Bose liquids. Some nonlinear integral equations which determine the critical behaviour can be given a solution covariant under conformal group transformations. This fact allows to transform the integral equations into equations in which the only unknowns are the indices, requiring only the evaluation of some integrals.

We have performed the calculation and we have found the following values for the exponents:

$$\nu = 0.6666\,, \qquad \eta = 0.2\cdot 10^{-6}\,. \tag{1}$$

Both these indices are not presently directly measurable. Scaling predictions (2) for ν, based upon the logarithmic singularity of the specific heat at the λ-point, yield $\nu = \frac{2}{3}$. This value is in very good agreement with our result. No measurement related to η is so far available in superfluid helium. We remark, however, that our enormously small value of η is related to the very small value of the difference $\nu - \frac{2}{3}$ through a relationship of the kind

$$\eta \simeq \text{const}\,(\nu - \tfrac{2}{3})^2\,, \tag{2}$$

where the constant is of the order of $0.5\cdot 10^2$. Taking the example of the three-dimen sional Ising model, where ν is known from computer calculations to be of the order of $0.62\div 0.63$, this relationship gives for η an estimate of the order of $5\cdot 10^{-2}$, *i.e.* noticeably different from 0, in agreement with computer estimates which yield $\eta \simeq 0.041$ (3).

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(1) G. PARISI and L. PELITI: *Lett. Nuovo Cimento*, **2**, 627 (1871).
(2) L. P. KADANOFF, W. GÖTZE, D. HAMBLEN, R. HECHT, E. A. S. LEWIS, V. V. PALCIAUSKAS, M. RAYL, J. SWIFT, D. ASPNES and J. KANE: *Rev. Mod. Phys.*, **39**, 395 (1967).
(3) M. A. MOORE, D. JASNOW and M. WORTIS: *Phys. Rev. Lett.*, **22**, 940 (1969).

In our calculation we neglected some more complicated diagrams. An evaluation of the error which may have been introduced in this way, as well as an application of the same theory to problems in high-energy physics, is in progress.

The integral equation involved is the following:

$$(3)\qquad V(X, Y, Z) = \int d^3x_1 d^3x_2 d^3y_1 d^3y_2 d^3z_1 d^3z_2 \cdot V(X, y_1, z_1)\, V(x_1, Y, z_2)\, V(x_2, y_2, Z)\, D(x_1 - x_2)\, G(y_1 - y_2)\, G(z_1 - z_2) ,$$

where V is the vertex part and G and D the particle and « phonon » propagator, respectively [1]. Denoting by $\bar{V}, \bar{G}, \bar{D}$ the analytic continuations the of Fourier transforms of V, G, D, respectively, to negative « momentum » squares, we add to (3) the following unitarity equations [4]:

$$(4)\qquad \operatorname{Im} \bar{G}^{-1}(p^2) = (2\pi)^{\frac{3}{2}} \int d^3K\, d^3q\, \delta^3(p - K - q) |\bar{V}(K^2, p^2, q^2)|^2 \cdot \theta(q_0) \operatorname{Im} \bar{G}(q^2) \cdot \theta(K_0) \operatorname{Im} \bar{D}(K_0),$$

$$(5)\qquad \operatorname{Im} \bar{D}^{-1}(p^2) = (2\pi)^{\frac{3}{2}} \int d^3K\, d^3q\, \delta^3(p - K - q) \cdot |\bar{V}(p^2, K^2, q^2)|^2 \cdot \theta(q_0) \operatorname{Im} \bar{G}(q^2) \cdot \theta(K_0) \operatorname{Im} \bar{G}(K^2) .$$

In [1] we emphasized that these equations allow for solutions of the form

$$(6)\qquad \begin{cases} G(\boldsymbol{x}) = g|\boldsymbol{x}|^{-1-\eta} , \qquad D(\boldsymbol{x}) = d|\boldsymbol{x}|^{-6+2/\nu} , \\ V(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = c|\boldsymbol{x} - \boldsymbol{y}|^{-1/\nu} \cdot |\boldsymbol{x} - \boldsymbol{z}|^{-1/\nu} |y - z|^{-5+\eta+1/\nu} . \end{cases}$$

Introducing eq. (6) into (3), (4) and (5), respectively, we obtain the following equations:

$$(7)\qquad 1 = f(\eta, \nu)(c^2 g^2 d) , \qquad 1 = g(\eta, \nu)(c^2 g^2 d) , \qquad 1 = h(\eta, \nu)(c^2 g^2 d) ,$$

where f, g, h are obtained by direct evaluation of the integrals. We note that the functions (6) may be not directly Fourier-transformable, in which case their Fourier transforms are to be understod as distributions.

The function f has been evaluated in exact way. It turned out to be

$$f(\eta, \nu) = \pi^9 \left[C\left(-\frac{1}{\nu}\right)\right]^3 \cdot [C(-1-\eta)]^4 \cdot \left[C\left(-5 + \eta + \frac{1}{\nu}\right)\right]^4 \cdot \left[C\left(-2 + \eta - \frac{1}{\nu}\right)\right]^3 C\left(-3 + \frac{1}{\nu}\right) \cdot C\left(-6 + \frac{2}{\nu}\right) \cdot C\left(2 - \eta - \frac{1}{\nu}\right) \cdot C(1 - 2\eta) ,$$

where

$$C(x) \equiv \Gamma\left(\frac{x+3}{2}\right) [\Gamma(-x/2)]^{-1} .$$

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[4] A. A. MIGDAL: *Sov. Phys. JETP*, **28**, 1036 (1969); A. M. POLYAKOV: *Sov. Phys. JETP*, **28**, 533 (1969).

The main tool involved was the identity, if $a + b + c = -6$,

$$\int d^3 t |t - x|^a |t - y|^b |t - z|^c = \pi^{\frac{3}{2}} \cdot C(a) \cdot C(b) \cdot C(c) \cdot |x - y|^{-3-c} \cdot |y - z|^{-3-a} \cdot |z - x|^{-3-b} . \tag{8}$$

The functions g and h were obtained by computer calculation with sufficient accuracy in a reasonable time. We note that the constant takes the value $\lambda \simeq 7 \cdot 10^{-12}$ on the solution.

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