S. Ferrara, R. Gatto and A. Grillo : MANIFESTLY CONFORMAL COVARIANT EXPANSION ON THE LIGHT CONE. -

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ABSTRACT. -

The isomorphism of the conformal algebra on space-time to the orthogonal O(4, 2) algebra is exploited to derive in a manifestly covariant way an operator product expansion on the light-cone in terms of irreducible operator representations of the conformal algebra. The expansion provides for a solution of the causality problem for operator expansion on the light-cone. Additional properties of the conformally covariant expansion, as well as its relation to the conformally invariant three-point function are discussed.

(x) - Istituto di Fisica dell'Università di Roma, and Istituto Nazionale di Fisica Nucleare, Sezione di Roma.

1. - INTRODUCTION. -

K. Wilson has advocated the relevance of scale invariance appli ed to an operator product expansion[2]. The possible relevance of the stro nger conformal invariance[3] for equal time commutators[4] and operator product expansions[5, 6] has recently been proposed. In particular in ref. (6) the following "improved" light-cone expansion was derived.

$$
A(x) B(0) = \sum_{n=0}^{\infty} c_n^{AB} (\frac{1}{x^2}) ^{\frac{1_A + 1_B + n - 1}{2}} x^{\alpha_1} \dots x^{\alpha_n} \cdot
$$
$$
x^2 \to 0
$$

$$
\cdot {}_1F_1 (\frac{1_A - 1_B + 1_n + n}{2} ; 1_n + n ; x \partial) O_{\alpha_1 \dots \alpha_n} (0)
$$

In the above equation A(x) and B(x) are local scalar (for simplicity) ope rators of dimensions 1_A and 1_B (in energy units), both annihilated by K_λ (the generator of special conformal transformations), i.e. satisfying $[K_\lambda , A(0)] = 0, [K_\lambda , B(0)] = 0; O_{\alpha_1} \dots \alpha_n (0)$ (symmetric traceless tensors of dimension 1_n) are those operators of the expansion basis which are an nihilated by K_λ ; c_n^{AB} are unknown constants; the hypergeometric function ${}_1F_1$ (a;c;z) arises from the structure of the conformal algebra.

In this paper we shall:

i) Offer a manifestly conformal covariant derivation of the im- proved expansion using the isomorphism of the conformal algebra to the orthogonal algebra O(4,2). In a subsequent paper[7] the proof is extended (in view of later applications) to derive a conformally covariant operator product expansion valid over the whole space-time.

ii) Present additional discussion on the properties of the impro- ved expansion. Besides offering a solution of the important causality pro

blem in operator expansions, and of translation invariance on hermitean basis, as extensively discussed in ref. (6), the improved expansion is directly related to the conformally covariant expressions for the vacuum expectation value of a product of three local operators and for the vertex function.

2. - CONFORMALLY COVARIANT FORMALISM. -

It is well-known[3] that the conformal algebra on space-time is isomorphic to the orthogonal algebra $O(4,2)$, whose generators constitute an antisymmetric tensor J_{AB} $(A,B = 0,1,2,3,5,6)$ with

(1) $$J_{\mu\nu} \equiv M_{\mu\nu} \, , \quad J_{65} = D, \quad J_{5\mu} = \frac{1}{2}(P_\mu - K_\mu), \quad J_{6\mu} = \frac{1}{2}(P_\mu + K_\mu)$$

where $M_{\mu\nu}$, D, P_μ and K_μ are the usual conformal generators.

Let us consider the pseudoeucledean space in six dimensions with metric tensor $g_{AA} = (+---,-+)$, $g_{AB} = 0$ for $A \neq B$. On the hypercone $\eta^A \eta_A = 0$, $\eta_A \equiv (\eta_\mu, \eta_5, \eta_6)$, one defines spinor functions $\psi_{\{\alpha\}}(\eta)$, which transform as

(2)
$$\delta \psi_{\{\alpha\}}(\eta) = -i\varepsilon^{AB} J_{AB}{}_{\{\alpha\}}{}^{\{\beta\}} \cdot \psi_{\{\beta\}}(\eta) =$$
$$\equiv -i\varepsilon^{AB}(L_{AB} \delta_{\{\alpha\}}{}^{\{\beta\}} + S_{\{\alpha\}}{}^{\{\beta\}}) \, \psi_{\{\beta\}}(\eta)$$

where

$$L_{AB} = i(\eta_A \partial_B - \eta_B \partial_A)$$

and $S_{\{\alpha\}\{\beta\}}$ is an irreducible representation of the spinor group $SU(2,2)$ (which is locally isomorphic to $O(4,2)$). Assuming such functions to be ho

4.

mogeneous, i.e.

(3)
$$\eta^A \partial_A \psi_{\{a\}}(\eta) = \lambda \psi_{\{a\}}(\eta)$$

one can show that the function $(k = \eta_5 + \eta_6, \; \pi_\mu = S_{6\mu} + S_{5\mu}, \; x_\mu = \frac{1}{k}\eta_\mu)$

(4)
$$O_{\{a\}}(x) = k^{-\lambda}(e^{-ix\pi}\psi)_{\{a\}}(\eta)$$

transforms according to a representation of the conformal algebra on spa ce-time, induced from a representation of the stability algebra at $x = 0$ by matrices

(5)
$$\Sigma_{\mu\nu} = S_{\mu\nu}, \quad \Delta = S_{65} - i\lambda, \quad k_\mu = S_{6\mu} - S_{5\mu}$$

on, in terms of spinor functions,

$$\left[O_{\{a\}}(x), P_\mu\right] = i\partial_\mu O_{\{a\}}(x)$$

$$\left[O_{\{a\}}(x), M_{\mu\nu}\right] = \left\{i(x_\mu \partial_\nu - x_\mu \partial_\nu)\partial_{\{a\}}{}^{\{\beta\}} + \Sigma_{\mu\nu\{a\}}{}^{\{\beta\}}\right\} O_{\{\beta\}}(x)$$

(6)
$$\left[O_{\{a\}}(x), D\right] = (ix_\nu \partial^\nu \delta_{\{a\}}{}^{\{\beta\}} + \Delta_{\{a\}}{}^{\{\beta\}}) O_{\{\beta\}}(x)$$

$$\left[O_{\{a\}}(x), k_\mu\right] = \left\{i(2x_\mu x_\nu \partial^\nu - x^2 \partial_\mu)\delta_{\{a\}}{}^{\{\beta\}} - 2ix^\nu\left[g_{\mu\nu}\Delta + \right.\right.$$
$$\left.\left. + \Sigma_{\mu\nu}\right]_{\{a\}}{}^{\{\beta\}} + K_{\mu\{a\}}{}^{\{\beta\}}\right\} O_{\{\beta\}}(x)$$

We shall be interested in irreducible representations of the conformal algebra which contain infinite towers of Lorentz tensors. It can be shown that such representations are uniquely specified from an irreducible re-

presentation with $K_\lambda = 0$ of the stability algebra; i.e. from an irreducible representation of SL(2,c)$\otimes$D, or, equivalently, from a Lorentz irreducible tensor $(\frac{n}{2}, \frac{n}{2})$ of scale dimensions 1_n[8]. In the covariant formalism in six dimensions such representations are specified in terms of an irreducible tensor representation of SU(2,2), $\psi_{A_1...A_n}(\eta)$ i.e. a representation for which n is the largest order of its Lorentz tensors , of degree of homogeneity $\lambda = -1_n$, and verifying the supplementary conditions[9].

$$(7) \qquad \eta^{A_1} \psi_{A_1...A_n}(\eta) = \partial^{A_1} \psi_{A_1...A_n}(\eta) = 0$$

3. - MANIFESTLY COVARIANT LIGHT-CONE EXPANSION. -

The light-cone limit, $(x-x')^2 \to 0$, of the product $A(x)B(x')$ corresponds to the covariant limit $A(\eta)B(\eta')$ for $\eta\eta' \to 0$, where $\eta\eta' = -\frac{1}{2}kk'(x-x')^2$. The most general expansion into irreducible tensor operators is of the form[10]

$$(8) \qquad A(\eta)B(\eta') = \sum_{\substack{n=0 \\ \eta\eta' \to 0}}^{\infty} E_n(\eta\eta')D^{(n)A_1...A_n}(\eta,\eta')\psi_{A_1...A_n}(\eta')$$

where

$$(9) \qquad E_n(\eta\eta') = c_n(\eta\eta')^{\frac{1}{2}(\lambda_A+\lambda_B-\lambda_n-n)}$$

with c_n constant and λ_A, λ_B, λ_n being the degrees of homogeneity of $A(\eta)$, $B(\eta)$, and $\psi_{A_1...A_n}(\eta)$. The general structure of $D^{(n)A_1...A_n}(\eta\eta')$ is

$$D^{(n)A_1 \ldots A_n}(\eta, \eta') =$$

(10)

$$= \sum_{m=0}^{n} \eta^{A_1} \ldots \eta^{A_{n-m}} \eta'^{A_{n-m+1}} \ldots \eta'^{A_n} D^{(n, m)}(\eta, \eta') c_{nm}$$

where c_{nm} are constants and $D^{(n, m)}(\eta, \eta')$ is a differential operator de_fined on the cones $\eta^2 = \eta'^2 = 0$, finite for $\eta\eta' = 0$, and homogeneous of de_gree $h = \frac{1}{2}(\lambda_A - \lambda_B + \lambda_n - n) + m$ in k/k'. The last condition follows from the covariance of the expansion under the group of dilatations $\eta_A \rightarrow \varrho\eta_A$ on the cone $\eta^2 = 0$. One can show that $D^{(n, m)}(\eta, \eta')$ is uniquely determined, and for $\eta\eta' = 0$ it reduces to

(11)
$$D^{(n, m)}(\eta, \eta') = (\eta \partial')^h$$

where by $(\eta \partial')^h$ we mean the application of $\eta \partial'$ h times, where h is in-teger, and its analytical continuation for non-integer h (as specified later). By simple algebraic steps one can recognize that the n+1 tensor covari-ants in Eq. (10) are all proportional and therefore one can consistently ta-ke

(12)
$$D^{(n)A_1 \ldots A_n}(\eta, \eta') = \eta'^{A_1} \ldots \eta'^{A_n}(\eta \partial')^{\frac{1}{2}(\lambda_A - \lambda_B + \lambda_n - n)}, \quad \eta\eta' = 0$$

To define the application of $(\eta \partial')^{\frac{1}{2}(\lambda_A - \lambda_B + \lambda_n - n)}$ on $\psi_{A_1 \ldots A_n}(\eta')$, we introduce the coordinates (x_μ, k) for a vector on the hypercone, and write

(13)
$$\eta \partial' = \frac{k}{k'}\left(k' \frac{\partial}{\partial k'} + (x-x')\partial' \right) \quad \text{at} \quad (x-x')^2 = 0$$

One has the following lemma$^{(10)}$: For β integer,

$$(14) \quad (\eta\, \partial')^{\beta} = (-\frac{k}{k'})^{\beta}\, L^{\beta} \sum_{J=0}^{\beta} \binom{\beta}{J} L^{-J} (x-x')^{\alpha_1} \ldots (x-x')^{\alpha_J}\, \partial'_{\alpha_1} \ldots \partial'_{\alpha_J}\, (-1)^J$$

where

$$(15) \quad L^J = 1_n (1_n+1) \ldots (1_n+J-1) = \Gamma(1_n+J)/\Gamma(1_n) \quad \text{and} \quad \partial'_{\mu} = \partial/\partial x'^{\mu}$$

One can write

$$(16) \quad (\eta\, \partial')^{\beta} = (-\frac{k}{k'})^{\beta}\, \frac{\Gamma(1_n+\beta)}{\Gamma(1_n)}\, {}_1F_1(-\beta; 1_n; (x-x')\,\partial')$$

where $_1F_1(a;c;z)$ is the hypergeometric confluent function and

$$\left[(x-x')\,\partial' \right]^J \quad \text{stands for} \quad (x-x')^{\alpha_1} \ldots (x-x')^{\alpha_J}\, \partial'_{\alpha_1} \ldots \partial'_{\alpha_J}\, .$$

Eq. (16) defines $(\eta\,\partial')^{\beta}$ also for β non-integer.

Making use of Eqs. (12) and (16), Eq. (8) can be rewritten as

$$(17) \quad A(x)\, B(x') = \sum_{\substack{n=0 \\ (x-x')^2 \to 0}}^{\infty} \left[\frac{1}{(x-x')^2} \right]^{1/2(1_A+1_B+n-1_n)}\, c_n^{AB}\, x^{A_1} \ldots x^{A_n} \cdot$$

$$\cdot\ {}_1F_1(\frac{1}{2}(1_A-1_B+1_n+n); 1_n; (x-x')\,\partial')\, \widetilde{\psi}_{A_1 \ldots A_n}(x')$$

where

$$(18) \quad \widetilde{\psi}_{A_1 \ldots A_n}(x') = (k')^{1_n}\, \psi_{A_1 \ldots A_n}(\eta'), \quad x^A \equiv (x_{\mu}, \frac{1+x^2}{2}, \frac{1-x^2}{2})$$

We note that in the limit $(x-x')^2 \to 0$ one can consistently put

(19)
$$\tilde{\psi}_{a_1 \ldots a_J, xxx \ldots}(0) = 0$$

where the crosses xxx.. stand for indices 5 or 6. In fact such compo-
nents of $\tilde{\psi}$ correspond to less dominant. contributions, of order
$\left[(x-x')^2 \right]^{n-J}$ with respect to the leading singularity of the particular tensor
representation.

Making use of the fundamental integral representation[16]

(20)
$$_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_o^1 du\, u^{a-1}(1-u)^{c-a-1} e^{uz}$$

and of the identity

(21)
$$e^{-ix\pi} \partial_\mu e^{ix\pi} = \partial_\mu + i\pi_\mu$$

one has

(22)
$$A(x)B(x') = \sum_{n=0}^{\infty} \left[\frac{1}{(x-x')^2} \right]^{\frac{1}{2}(1_A + 1_B + n - 1_n)} {}_c^{AB}{}_n x^{A_1} \ldots x^{A_n} e^{i x'\pi} \cdot$$
$$(x-x')^2 \to 0$$
$$\cdot\, _1F_1(\tfrac{1}{2}(1_A - 1_B + 1_n + n; \, 1_n; (x-x') \, \partial' + i\pi) O_{A_1 \ldots A_n}(x')$$

after having inserted Eq. (9) which reads as

(23)
$$\tilde{\psi}_{A_1 \ldots A_n}(x') = (e^{i x'\pi} O)_{A_1 \ldots A_n}(x')$$

Taking x' = 0 and inserting Eq. (20) one finds

(24)
$$x^{A_1} \ldots x^{A_n} {}_1F_1(a;c;x(\partial + i\pi)) O_{A_1 \ldots A_n}(0) =$$

$$= x^{A_1} \ldots x^{A_n} \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 du\, u^{a-1}(1-u)^{c-a-1} e^{iux\pi}\, O_{A_1 \ldots A_n}(ux) =$$

(24)

$$= x^{A_1} \ldots x^{A_n} \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 du\, u^{a-1}(1-u)^{c-a-1} \tilde{\psi}_{A_1 \ldots A_n}(ux)$$

One can show that[11]

$$x^{A_1} \ldots x^{A_n} \tilde{\psi}_{A_1 \ldots A_n}(x') = (x-x')^{\alpha_1} \ldots (x-x')^{\alpha_n} O_{\alpha_1 \ldots \alpha_n}(x')$$

at $(x-x')^2 = 0$. Therefore

$$x^{A_1} \ldots x^{A_n} \tilde{\psi}_{A_1 \ldots A_n}(ux) = (x-ux)^{\alpha_1} \ldots (x-ux)^{\alpha_n} O_{\alpha_1 \ldots \alpha_n}(ux) =$$

(25)

$$= (1-u)^n x^{\alpha_1} \ldots x^{\alpha_n} O_{\alpha_1 \ldots \alpha_n}(ux)\,.$$

and Eq. (22) can finally be written as

(26)

$$\begin{array}{l} A(x)\,B(0) = \sum_{n=0}^{\infty} \tilde{c}_n^{AB} \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A + l_B + n - l_n)} x^{\alpha_1} \ldots x^{\alpha_n} \\[4pt] x^2 \to 0 \end{array}$$

$$\cdot\ {}_1F_1\left(\tfrac{1}{2}(l_A - l_B + l_n + n;\ l_n + n;\ x\,\partial\,)\, O_{\alpha_1 \ldots \alpha_n}(0)\right.$$

where $\tilde{c}_n^{AB}$ are new constants, related to those in Eq. (22). Eq. (26) is the improved light cone expansion exhibited in the introduction and first derived in ref. (6) (and its Errata, where an algebraic mistake is corrected).

4. - PROPERTIES OF THE IMPROVED LIGHT-CONE EXPANSION. -

Our earlier derivation, in ref. (5) and (6), of the improved light-

-cone expansion, was directly obtained by commuting K_λ with both sides of the expansion

$$(27) \quad A(x)\,B(0) \underset{x^2 \to 0}{=} \sum_{n=0}^{\infty} \left(-\frac{1}{x^2}\right)^{\frac{1}{2}(1_A + 1_B + n - 1_n)} \sum_{m=n}^{\infty} c_{nm}^{AB}\, x^{\alpha_1} \cdots x^{\alpha_m}\, O_{\alpha_1 \cdots \alpha_m}^{nm}(0)$$

where

$$(28) \quad O_{\alpha_1 \cdots \alpha_m}^{n,m}(0) = \left[\cdots \left[O_{\alpha_1 \cdots \alpha_n}^{(n)}(0),\, P_{\alpha_{n+1}} \right], \cdots P_{\alpha_m} \right] (i)^{m-n}$$

are the operators of the expansion basis and K_λ commutes with $O_{\alpha_1 \cdots \alpha_n}^{n,n}(0)$.
In this way one obtains the recurrence relation

$$(29) \quad c_{n,n+k-1}^{AB} \left[\frac{1}{2}(1_A - 1_B + 1_n + n) + k - 1 \right] = c_{n,n+k}^{AB}\, k(1_n + n + k)$$

giving the solution

$$(30) \quad c_{n,n+k}^{AB} = \frac{\Gamma\left(-\frac{1}{2}(1_A - 1_B + 1_n + n) + k\right) \Gamma(1_n + n)}{k\, \Gamma\left(-\frac{1}{2}(1_A - 1_B + 1_n + n)\right) \Gamma(1_n + n + k)}\, c_{nn}^{AB}$$

Eq. (30) gives rise to the confluent hypergeometric function

$$(31) \quad {}_1F_1\left(\frac{1}{2}(1_A - 1_B + 1_n + n),\, 1_n + n;\, x\,\partial\right) = \sum_{k=0}^{\infty} \frac{c_{n,n+k}^{AB}}{c_{n,n}^{AB}}\, (x\,\partial)^k$$

Translation invariance on a hermitean basis (see ref. (6)) follows from the Kummer identity: ${}_1F_1(a;c;z) = {}_1F_1(c-a;c;-z)\,e^z$; together with the property $c_n^{AB} = (-1)^n c_n^{BA}$. Kummer's identity is reflected in the properties of the integrand in Eq. (20) under the exchange $u \leftrightarrows (1-u)$. We note that there is a strict formal analogy with the anologous behaviour that ensures of the validity of crossing in the Veneziano representation.

For $1_A = 1_B$ the function $_1F_1{}^{(18)}$ reduces essentially to a Bessel function I_ν and one thus obtains[6]

$$
(32) \quad \underset{x^2 \to 0}{A(x)\,B(0)} = \sum_{n=even} c_n (\frac{1}{x^2})^{\frac{1}{2}(2 1_A + n - 1_n)} x^{\alpha_1} \ldots x^{\alpha_n} (x\,\partial)^{\frac{1}{2}(1 - 1_n - n)} \cdot
$$

$$
\cdot \; I_{\frac{1}{2}(1_n + n - 1)}(\frac{1}{2} x\,\partial)\, O^{(n)}_{\alpha_1 \ldots \alpha_n}(\frac{1}{2} x)
$$

We also note that for $\frac{1}{2}(1_A - 1_B + n + 1_n) = -h$, with h non-negative integer, $_1F_1$ reduces essentially to a Laguerre polynomial and for given n only a finite number of tensors O^{nm} contribute to the expansion[18].

An important property, discussed in ref. (6), is the manifest cau_sality of the contribution to the expansion from each irreducible represen_tation

$$
(33) \quad (\frac{1}{x^2})^{\frac{1}{2}(1_A + 1_B + n - 1_n)} x^{\alpha_1} \ldots x^{\alpha_n} \int_0^1 du\, u^{\frac{1}{2}(1_A - 1_B + 1_n + n) - 1} \cdot
$$

$$
\cdot (1 - u)^{\frac{1}{2}(1_B - 1_A + 1_n + n) - 1} O_{\alpha_1 \ldots \alpha_n}(ux)
$$

In fact the vanishing of $[A(x)\,B(0), C(y)]$ for $y^2 < 0$, $(x-y)^2 < 0$, $x^2 = 0$ is entirely equivalent to the vanishing of $[O_{\alpha_1 \ldots \alpha_n}(ux), C(y)]$ for $(ux - y)^2 < 0$, $y^2 < 0$, $x^2 = 0$, provided $0 \le u \le 1$.

In Eq. (33) the functions

$$
f_n^{AB}(u) = u^{\frac{1}{2}(1_A - 1_B + 1_n + n) - 1} (1-u)^{\frac{1}{2}(1_B - 1_A + 1_n + n) - 1}
$$

are nothing but the Clebsch-Gordon coefficients for the decomposition of

the product of two representation, into irreducible components on a continuous basis, whereas the coefficients $c_{n,m}^{AB}$ refer to a discrete basis. The relation between the two sets is

$$(34) \qquad \int_0^1 du\, f_n^{AB}(u)\, u^k = k!\ \frac{c_{n,n+k}^{AB}}{c_{n,n}^{AB}}\ \frac{\Gamma(\frac{1_A-1_B+1_n+n}{2})\,\Gamma(\frac{1_B-1_A+1_n+n}{2})}{\Gamma(1_n+n)}$$

The relation analogous to the recurrence relation (29) in the case of the discrete basis for the Clebsch-Gordon coefficients $f_n^{AB}(u)$ of the continuous basis, is now a differential equation

$$(35) \qquad u(1-u)\frac{d}{du} f_n^{AB}(u) = \left[\frac{1}{2}(1_A-1_B+1_n+n) - 1 + u\,(2-1_n-n)\right] f_n^{AB}(u)$$

which can be deduced exactly as for Eq. (29). The causality condition is essentially implicit in the support properties of the Clebsch-Gordon coefficients $f_n^{AB}(u)$.

By similar reasoning one can deduce the most singular contribution on the light-cone to the product of two arbitrary tensors, irreducible under the conformal algebra. One obtains, for the expansion of $J^A_{\alpha_1 \dots \alpha_{n_A}}(x)\, J^B_{\beta_1 \dots \beta_{n_B}}(0)$ the solution

$$f_n^{AB}(u) \sim u^{\frac{1}{2}(1_A-1_B+1_n+n+n_B-n_A)-1}\,(1-u)^{\frac{1}{2}(1_B-1_A+1_n+n+n_A-n_B)-1}$$

or equivalently

$$(36) \qquad J^A_{\alpha_1 \dots \alpha_{n_A}}(x)\, J^B_{\beta_1 \dots \beta_{n_B}}(0) \underset{x^2 \to 0}{=} \sum_n \frac{1}{(x^2)^{\frac{1}{2}(1_A+1_B+n_A+n_B+n-1_n)}}\, c_n^{AB} \quad *$$

$$\bullet \; x_{\alpha_1} \ldots x_{\alpha_{n_A}} \; x_{\beta_1} \ldots x_{\beta_{n_B}} \; x^{\mu_1} \ldots x^{\mu_n} \; {}_1F_1(\tfrac{1}{2}(1_A - 1_B + 1_n + n + n_B - n_A); 1_n + n; x \, \partial) \, x$$

$$(36)$$

$$x \, O^{(n)}_{\mu_1 \ldots \mu_n}(0)$$

5. - RELATION TO VACUUM EXPECTATION VALUES AND TO VERTEX FUNCTIONS. -

It is well-known that in an exactly conformally invariant theory the vacuum expectation value $\langle 0 | C(y) A(x) B(z) | 0 \rangle$ of three conformal scalars of dimensions $1_C, 1_A$, and 1_B is uniquely fixed apart from a multiplicative constant[12]

$$\langle 0 | C(y) A(x) B(z) | 0 \rangle =$$

$$(37)$$

$$= c_{ABC} \left[\frac{1}{(x-y)^2} \right]^{\frac{1_A + 1_C - 1_B}{2}} \left[\frac{1}{(x-z)^2} \right]^{\frac{1_A + 1_B - 1_C}{2}} \left[\frac{1}{(y-z)^2} \right]^{\frac{1_A + 1_C - 1_A}{2}}$$

We put $z = 0$ in Eq. (37) and take $(x-z)^2 \to 0$, obtaining

$$\langle 0 | C(y) A(x) B(0) | 0 \rangle =$$
$$x^2 \to 0$$

$$(38)$$

$$= c_{ABC} \left(\frac{1}{x^2}\right)^{\frac{1}{2}(1_A + 1_B - 1_C)} \left(\frac{1}{y^2}\right)^{\frac{1}{2}(1_B + 1_C - 1_A)} \left(\frac{1}{y^2 - 2xy}\right)^{\frac{1}{2}(1_A + 1_C - 1_B)}$$

or by using a Feynman parametrization[20], following Migdal[13], one finds

$$(39) \qquad \langle 0 | C(y) A(x) B(0) | 0 \rangle = c_{ABC} \left(\frac{1}{x^2}\right)^{\frac{1}{2}(1_A + 1_B - 1_C)} \; .$$
$$x^2 \to 0$$

(39)
$$\cdot \int_0^1 du\, u^{\frac{1}{2}(1_A-1_B+1_C)-1} (1-u)^{\frac{1}{2}(1_B-1_A+1_C)-1} \langle 0|C(y)\,C(\lambda x)|0\rangle$$

Alternatively one can take the contribution to $A(x)\,B(0)$ from the term pro_

portional to

(40)
$$\left(\frac{1}{x^2}\right)^{\frac{1}{2}(1_A+1_B-1_C)} {}_1F_1\left(\frac{1_A-1_B+1_C}{2},1_C\,;x\,\partial\right)C(0)$$

on the light-cone, $x^2 = 0$; insert it into $\langle 0|C(y)\,A(x)\,B(0)|0\rangle$ and make use
of the selection rule[14]

(41)
$$\langle 0|C(y)\,O(0)|0\rangle \sim \left(\frac{1}{y^2}\right)^{1_C} \qquad \text{for} \quad 1_C = 1_0$$
$$= 0 \qquad \text{for} \quad 1_C \neq 1_0$$

One then gets exactly Eq. (39).

Finally we note that one can calculate the contribution of a gi-
ven tensor representation of the conformal algebra, contained in $A(x)\,B(0)$,
to the off-mass-shell vetex function. One has

(42)
$$V_n(x^2,xp) = \langle 0\,|A(x)\,B(0)|\,p\rangle_n = (\text{constant})\left(\frac{1}{x^2}\right)^{\frac{1}{2}(1_A+1_B+n-1_n)} \cdot$$
$$\cdot (xp)^n\, {}_1F_1\left(\frac{1}{2}(1_A-1_B+1_n+n);\ 1_n+n;\ -ixp\right)$$

In Eq.(42) $|p\rangle$ is a scalar state (or spin averaged) and the subscript n ap-
plies to the given representation.

6. - CONCLUSIONS. -

We have considered an expansion of a product of two local opera rators on the light-cone into irreducible representations of the conformal algebra. Such an expansion has been derived by different independent me- thods, and, perhaps most elegantly, by exploiting the isomorphism of the conformal algebra to the orthogonal $O(4,2)$ algebra. A most remarkable property of the obtained expansion is its explicit causality, which appears here as a property of support of the Clebsch-Gordon coefficients of the conformal algebra on a continuous basis. Each irreducible representation contributes a term which explicitly satisfies the causality requirement. The conformally covariant expansion also directly satisfies the require ments of translation invariance on a hermitean basis, a requirement which in an ordinary Wilson's type expansion is equivalent to an infinite set of algebraic conditions that the covariant expansion automatically takes into account. The relation of the covariant operator product expansion to the covariant three-point vacuum expectation value is also discussed in this paper and the two problems are seen to be directly related to one another.

REFERENCES AND FOOTNOTES. -

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(3) - See e. g. G. Mack and A. Salam, Annals of Phys., 53, 174 (1969); C. G. Callan Jr., S. Coleman and R. Jackiw, Annals of Phys. 59, 42 (1970); B. Zumino, in Brandèis Lectures on Particle Phys., MIT press (1970).

(4) - S. Ciccariello, R. Gatto, G. Sartori and M. Tonin, Annals of Phys., 65, 265 (1971).

(5) - S. Ferrara, R. Gatto and A. F. Grillo, Cern preprint TH-1311, to appear in Nuclear Phys. B.

(6) - S. Ferrara, R. Gatto and A. F. Grillo, Phys. Letters 36 B, 124 (1971); and Errata, ibidem.

(7) - S. Ferrara, G. Gatto and A. F. Grillo,. Lettere al Nuovo Cimento (to be published).

(8) - The three independent Casimirs C_I, C_{II}, C_{III} operators of $O(4,2)$ are fixed for such representations from the order n of the irreducible tensor and from l_n:

$$C_I \equiv J_{AB} \, J^{AB} = 2n(n+2) + 2\,l_n(l_n-1)$$

$$C_{II} \equiv \varepsilon_{ABCDEF} \, J^{AB} \, J^{CD} \, J^{EF} = 0$$

$$C_{III} \equiv J^B_A \, J^C_B \, J^D_C \, J^A_D = n(n+2)\left[l_n(l_n-4)+3\right]$$

(9) - The condition $\eta^{A_1} \psi_{A_1 \ldots A_n}(\eta) = 0$ insures of the property that the Lorentz tensor of highest order contained in the representation is annihilated by K_λ at $x = 0$. The condition $\partial^{A_1} \psi_{A_1 \ldots A_n}(\eta) = 0$ fixes the components $O_{\alpha_1 \ldots \alpha_{n-J} xx \ldots}$ (where $x = 5$ or 6) of the tensor in Eq. (4) in terms of the divergences $\partial_{\beta_1} \ldots \partial_{\beta_J} O_{\alpha_1 \ldots \alpha_{n-J}}^{\beta_1 \cdots \beta_J}$.

(10) - Proofs of the mathematical statements of this section will be given elsewhere.

(11) - This result follows from the identity: $x^{A_1} \ldots x^{A_n}(e^{ix'\pi})_{A_1 \ldots A_n}^{\alpha_1 \cdots \alpha_n} = (x-x')^{\alpha_1} \ldots (x-x')^{\alpha_n}$, expressing the action of π_μ as generator of coordinate translations[3].

(12) - Eq. (37) can be obtained in a straitforward and simple way by the six-dimensional covariant formalism.

(13) - A.A. Migdal, (to be published).

(14) - This rule is a special case of the selection rule of the conformal algebra on the light-cone $\langle 0|O_n(x) O_m(0)|0 \rangle \neq 0$ if and only if $l_n - l_m = n-m$. This result can be proven by requiring the conformal covariance of the identity representation (c-number) of the conformal algebra, contained in the operator product $O_{\alpha_1 \ldots \alpha_n}(x) O_{\beta_1 \ldots \beta_m}(0)$ for $x^2 = 0$.

(15) - Eq. (42) can be used for instance to derive asymptotic limits for form factors, see e.g. ref. (13).

(16) - Bateman Project: Vol. 1, Chapter VI, 19-255.

(17) - Bateman Project: Vol. 1, Chapter VI, 19-253.

(18) - $_1F_1(a, 2a;z) = \Gamma(a+\frac{1}{2}) (\frac{z}{4})^{1/2-a} e^{z/2} I_{a-1/2}(\frac{z}{2})$ (Bateman Project: Vol. 1, Chapter VI, 19-265).

(19) - $_1F_1(-h, c;z) = \Gamma(1+h) L_h^{(c-1)}(z)$ (Bateman Project: Vol. 1, Chapter VI, 19-268).

(20) - The integral representation in(39) is nothing but a Rieman-Liouille fractional integral (Bateman Integral transforms Vol. II, 19-186).