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G. Parisi and F. Zirilli: A SIMPLE METHOD FOR COMPUTING ELECTRODYNAMIC PROCESSES OF HIGH ORDER. -

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G. Parisi and F. Zirilli[(x)]: A SIMPLE METHOD FOR COMPUTING ELECTRODYNAMIC PROCESSES OF HIGH ORDER. -

SUMMARY. -

In this work we propose a general method for evaluate Feynman graphs; this method avoids the calculation of very long traces. Using this technique we write down the exact differential cross-section for the contribution of two photons annihilation to the process $e^+e^- \to e^+e^-e^+e^-$, where we take care of the Fermi statistics for the final particles. We compute also the polarization effects on this cross-sections.

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Everybody knows that it is very hard to extimate high order Feynman diagrams by means of the standard trace technique. The resulting expressions are very long and it is very easy to make a mistake given by a mark. For instance the differential cross section for $e^+e^- \to e^+e^-\gamma$ contains about 900 terms[(1)].

In this work we want to propose and to codified a completely different method, which is simpler than the older one. The final formulae are amenable to be used on a computer.

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The cross section for polarized particles can be found very easily.

In our approach we calculate directly the helicity amplitudes and sum their squared absolute values[2].

In Section I we give the details of this method and write a list of useful formulae. In Section II we apply this formalism to the processes $e^+e^- \rightarrow e^+e^-\gamma$. Finally we find the expression of the contribution of two photon annihilation to the differential cross section for the process $e^+e^- \rightarrow e^+e^- e^+e^-$.

This is the first exact calculation of this process. The indistinguibility of the final particles produces a lot of interference terms which make almost impossible the direct calculation of this process by standard techniques.

In Section III we discuss briefly the effect of initial polarization on colliding beam cross-sections.

I. - THE BASIC RELATIONS. -

As we said in the Section I,we find directly the helicity amplitudes. The main simplification involved is due to the fact that, if the number of the Feynman diagrams which contribute is N, there are N terms in the helicity amplitudes, but N^2 in the usual expression for $|A|^2$.

If the electrons are relativistic, we may obtain further simplifications; we neglecte amplitudes which violate the conservation of helicity along a line. The ratio between helicity violating and conserving amplitudes is of order m/E, due to the approximate γ_5 invariance.

When the differential cross section are found, it is very easy to get also the polarization of the final particles. We need only to use the well known formula:

$$P = \frac{\sigma_R - \sigma_L}{\sigma_R + \sigma_L} \qquad (1)$$

where σ_R and σ_L are the cross section for the production of positive or negative helicity particles. This cross sections can be readly obtained from the helicity amplitudes.

The general term contributing to the helicity amplitude is a product of a string of γ sandwiched by two spinors. It is possible to divide the calculation of such an expression in two steps:

In the first step the products of three or more γ are reduced to a linear combination of I, γ_5, γ_μ, $\gamma_5 \gamma_\mu$, $\sigma_{\mu\nu}$. This may be easily obtained by use of standard formulas like:

(2)
$$\gamma_\mu \gamma_\gamma \gamma_\sigma = g_{\mu\nu}\gamma_\sigma + g_{\nu\sigma}\gamma_\mu - g_{\mu\sigma}\gamma_\gamma + i\varepsilon_{\mu\nu\sigma\tau}\gamma_5\gamma^\tau$$
$$\gamma_\mu \not{p} \not{q} \not{K} \gamma^\mu = -2 \not{K} \not{q} \not{p}$$

In the second step we have only to evaluate expressions like $\bar{u}\,\Gamma\,u$, $\bar{V}\,\Gamma\,V$, etc.

Their values can be readly obtained by using the following formulas.

In our notation u^+, u^- are electron states, of positive or negative helicity, V^+V^- positron states of negative or positive helicity.

$u^+(\theta, E)$ is the positive helicity electron state of energy E, direction of motion (θ, φ) where θ and φ are the latitude and longitude of a polar coordinate system.

We define the following vectors:

(3)
$$\sigma_1^{++}(\bar\theta, \bar\varphi; \theta, \varphi) = -\sin\frac{\bar\theta}{2}\cos\frac{\theta}{2}\, e^{i\bar\varphi} - \cos\frac{\bar\theta}{2}\sin\frac{\theta}{2}\, e^{-i\varphi}$$
$$\sigma_2^{++}(\bar\theta, \bar\varphi; \theta, \varphi) = i\sin\frac{\bar\theta}{2}\cos\frac{\theta}{2}\, e^{i\bar\varphi} - i\cos\frac{\bar\theta}{2}\sin\frac{\theta}{2}\, e^{-i\varphi}$$
$$\sigma_3^{++}(\bar\theta, \bar\varphi; \theta, \varphi) = \sin\frac{\bar\theta}{2}\sin\frac{\theta}{2}\, e^{i(\bar\varphi - \varphi)} - \cos\frac{\theta}{2}\cos\frac{\bar\theta}{2}$$
$$\sigma_4^{++}(\bar\theta, \bar\varphi; \theta, \varphi) = \sin\frac{\bar\theta}{2}\sin\frac{\theta}{2}\, e^{i(\bar\varphi - \varphi)} + \cos\frac{\theta}{2}\cos\frac{\bar\theta}{2}$$
$$\sigma_\mu^{-+}(\bar\theta, \bar\varphi; \theta, \varphi) = \sigma_\mu^{++}(\theta + \pi, \bar\varphi; \bar\theta, \varphi)$$
$$\sigma_\mu^{+-}(\bar\theta, \bar\varphi; \theta, \varphi) = \sigma_\mu^{++}(\bar\theta, \bar\varphi; \theta + \pi, \varphi)$$
$$\sigma_\mu^{--}(\bar\theta, \bar\varphi; \theta, \varphi) = \sigma_\mu^{++}(\bar\theta + \pi, \bar\varphi, \theta + \pi, \varphi)$$

We define:

(4)

$$\alpha_1^+(E) = \frac{E+m}{\sqrt{2E(E+m)}} \qquad \alpha_2^+(E) = \frac{|P|}{\sqrt{2E(E+m)}}$$

$$\alpha_1^-(E) = \frac{E+m}{\sqrt{2E(E+m)}} \qquad \alpha_2^-(E) = \frac{-|P|}{\sqrt{2E(E+m)}}$$

$$\beta_1^+(E) = \frac{|P|}{\sqrt{2E(E+m)}} \qquad \beta_2^+(E) = \frac{E+m}{\sqrt{2E(E+m)}}$$

$$\beta_1^-(E) = \frac{-|P|}{\sqrt{2E(E+m)}} \qquad \beta_2^-(E) = \frac{E+m}{\sqrt{2E(E+m)}}$$

Using this notation we find:

(5)

$$\bar{u}^+(\bar{\theta}, \bar{\varphi}, \bar{E})\, u^+(\theta, \varphi, E) = \sigma_o^{++}(\bar{\theta}, \bar{\varphi}; \theta, \varphi) \left[\alpha_1^+(\bar{E})\ \alpha_1^+(E) - \alpha_2^+(\bar{E})\ \alpha_2^+(E)\right]$$

$$\bar{u}^+ \gamma_o u^+ = \sigma_o^{++}(\bar{\alpha}_1^+ \alpha_1^+ + \bar{\alpha}_2^+ \alpha_2^+)$$

$$\bar{u}^+ \gamma_k u^+ = \sigma_k^{++}\left[\bar{\alpha}_1^+ \alpha_2^+ + \bar{\alpha}_2^+ \alpha_1^+\right]$$

$$\bar{u}^+ \gamma_5 u^+ = \sigma_o^{++}\left[\bar{\alpha}_1^+ \alpha_2^+ - \bar{\alpha}_2^+ \alpha_1^+\right]$$

$$\bar{u}^+ \gamma_5 \gamma_o u^+ = \sigma_o^{++}\left[-\bar{\alpha}_2^+ \alpha_1^+ - \bar{\alpha}_1^+ \alpha_2^+\right]$$

$$\bar{u}^+ \gamma_5 \gamma_k u^+ = i\, \sigma_k^{++}\left[-\bar{\alpha}_2^+ \alpha_1^+ - \bar{\alpha}_1^+ \alpha_1^+\right]$$

$$\bar{u}^+ \sigma_{ok} u^+ = i\, \sigma_k^{++}\left[\bar{\alpha}_1^+ \alpha_2^+ - \bar{\alpha}_2^+ \alpha_1^+\right]$$

$$\bar{u}^+ \sigma_{ik} u^+ = \varepsilon_{ike}\, \sigma_e^{++}(\bar{\alpha}_1^- \alpha_1^+ - \bar{\alpha}_2^+ \alpha_2^+)$$

To obtain the corresponding formulae for different helicity one has only to substitute + with - and for those reagarding positron, α with β.

For example

$$(6)\quad \overline{V}^-(\overline{\theta},\overline{\varphi},\overline{E})\,\gamma_o\, u^+(\theta,\varphi,E) = \sigma_o^{-+}(\overline{\theta},\overline{\varphi};\theta,\varphi)\left[\beta_1^-(\overline{E})\,\alpha_1^+(E) - \beta_2^-(\overline{E})\,\alpha_2^+(E)\right]$$

It is easy to see that amplitudes helicity violating are of order m/E.

II. - APPLICATION OF THE FORMALISM TO THE PROCESS: e^+e^- $e^+e^- \to$; $e^+e^- \to e^+e^-e^+e^-$. -

Let us see low one can use the formalism of section two in order to compute differential cross sections.

We take under examination as first case the process $e^+e^- \to$ $\to e^+e^-\gamma$.

We denote by E the energy of the incoming electron in the c.m. frame, by E_+, E_-, E_γ the energy of the outgoing electron, positron and photon, by p_1 q_2 p_3 q_4 the four-momentum of the incoming e^-e^+ and the outgoing e^-e^+ and finally by K the four-momentum of the photon.

The differential cross section is

$$(7)\qquad \frac{d\sigma}{d^3p_3\, d^3q_4\, d^3K} = \frac{{}^3M^2}{2\pi K^o}\,\delta^4(p_1 + q_2 - p_3 - q_4 - K)$$

where

$$|M|^2 = \sum_{spin} |A|^2$$

The diagrams which contribute to this process at the third electromagnetic order are shown in Fig. 1.

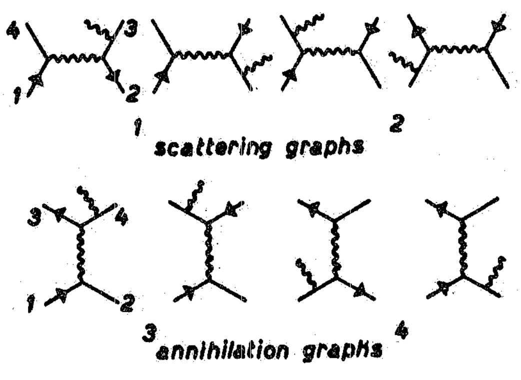

FIG. 1

We can write the amplitudes in the following way[3]

$$A = \frac{\bar{u}(p_3)\,\gamma^{\mu}\,u(p_1)\,\bar{V}(q_2)\,C_{\mu}\,(-q_2, -q_4)\,V(q_4)}{(p - p')^2} +$$

(8)

$$+ \frac{\bar{u}(p_3)\,C_{\mu}\,(p_3, p_1)\,u(p_1)\,\bar{V}(q_2)\,\gamma^{\mu}\,V(q_4)}{(q - q')^2} +$$

$$+ \frac{-\bar{V}(q_2)\,\gamma^{\mu}\,u(p_1)\,\bar{u}(p_3)\,C_{\mu}\,(p_3, -q_4)\,V(q_4)}{(p+q)^2} +$$

$$+ \frac{-\bar{V}(q_2)\,C_{\mu}\,(-q_2, p_1)\,u(p_1)\,\bar{u}(p_3)\,\gamma^{\mu}\,V(q_4)}{(p' + q')}$$

where:

(9) $$C_{\mu}\,(p, q) = \not{\varepsilon}\,\frac{\not{q} + \not{K} + m}{2(K\,q)}\,\gamma_{\mu} + \gamma_{\mu}\,\frac{\not{p} - \not{K} + m}{-2(K\,p)}\,\not{\varepsilon}$$

and ε is the photon polarization vector.

Using our rules we find for A the following expression:

$$A = \frac{\left|\varepsilon, -q_4+K, J^{31}, J^{24}\right| + i\left\|\varepsilon, -q_4+K, J^{31}, A^{24}\right\| + m\,T^{24}\cdot\varepsilon\cdot J^{31}}{-(p_1 - p_3)^2\;\; 2(q_4\,K)} +$$

(10)

$$+ \frac{\left|J^{31}, -q_2-K, \varepsilon, J^{24}\right| + i\left\|J^{31} -q_2-K, \varepsilon, A^{24}\right\| + m\,T^{24}\cdot J^{31}\cdot\varepsilon}{(p_1 - p_2)^2\;\; 2\,(q_2\,K)} +$$

$$+ \frac{\left|\varepsilon, p_3+K, J^{24}, J^{31}\right| + i\left\|\varepsilon\; p_3+K, J^{24}, A^{31}\right\| + m\,T^{31}\cdot\varepsilon\cdot J^{24}}{(q_2 - q_4)^2\;\; 2\,(p_3\,K)} +$$

$$+ \frac{\left|J^{24}, p_1-K, \varepsilon, J^{31}\right| + i\left\|J^{24}, p_1-K, \varepsilon, A^{31}\right\| + m\,T^{31}\cdot J^{24}\cdot\varepsilon}{-(q_2 - q_4)^2\;\; 2\,(p_1\,K)} +$$

$$
(10)\quad
\begin{aligned}
&-\frac{\left|\varepsilon, -q_4+K, J^{21}, J^{34}\right| + i\left\| \varepsilon, -q_4+K, J^{21}, A^{34}\right\| + m\,T^{34}\cdot\varepsilon\cdot J^{21}}{-(q_2+p_1)^2\, 2\,(q_4 K)} + \\
&-\frac{\left| J^{21}, p_3-K, \varepsilon, J^{34}\right| + i\left\| J^{21}, p_3-K, \varepsilon, A^{34}\right\| + m\,T^{34}\cdot J^{21}\cdot\varepsilon}{-(q_2+p_1)^2\, 2\,(p_3 K)} + \\
&-\frac{\left|\varepsilon, p_1+K, J^{34}, J^{21}\right| + i\left\| \varepsilon, p_1+K\; J^{34}, A^{21}\right\| + m\,T^{21}\cdot\varepsilon\cdot J^{34}}{(p_3+q_4)^2\, 2\,(p_1 - K)} + \\
&-\frac{\left| J^{34}, \varepsilon, -K-q_2, J^{21}\right| + i\left\| J^{34}, -K-q_2, \varepsilon, A^{21}\right\| + m\,T^{21}\cdot J^{34}\cdot\varepsilon}{(p_3+q_4)^2\, 2\,(q_2 K)}
\end{aligned}
$$

where:

$$
(11)\quad
\begin{aligned}
&J^{31}_{\mu} = \bar{u}(p_3)\gamma_\mu\; u(p_1);\; A^{31}_{\mu} = \bar{u}(p_3)\;\gamma_5\gamma_\mu u(p_1);\; T^{31}_{\mu\nu} = \bar{u}(p_3)\gamma_\mu\,\gamma_\nu\; u(p_1) \\
&J^{24}_{\mu} = \bar{V}(q_2)\gamma_\mu\; V(q_4);\; A^{24}_{\mu} = \bar{V}(q_2)\;\gamma_5\gamma_\mu\; V(q_4);\; T^{24}_{\mu\nu} = V(q_2)\gamma_\mu\gamma_\nu\; V(q_4) \\
&J^{21}_{\mu} = \bar{V}(q_2)\gamma_\mu\; u(p_1);\; A^{21}_{\mu} = \bar{V}(q_2)\;\gamma_5\gamma_\mu\; u(p_1);\; T^{21}_{\mu\nu} = \bar{V}(q_2)\gamma_\mu\,\gamma_\nu\; u(p_1) \\
&J^{34}_{\mu} = \bar{u}(p_3)\gamma_\mu\; V(q_4);\; A^{34}_{\mu} = \bar{u}(p_3)\;\gamma_5\gamma_\mu\; V(q_4);\; T^{34}_{\mu\nu} = \bar{u}(p_3)\gamma_\mu\,\gamma_\nu\; V(q_4)
\end{aligned}
$$

and:

$$
(12)\quad
\begin{aligned}
&\left| J, A, V, q\right| = (J\cdot A)\,(V\cdot q) + (J\cdot q)\,(A\cdot V) - (J\cdot K)\,(A\cdot q) \\
&\left\| J, A, V, q\right\| = \varepsilon_{\mu\nu\varrho\sigma}\; J^{\mu} A^{\nu}\; V^{\varrho}\, q^{\sigma} \\
&T\cdot A\cdot\varepsilon = T_{\mu\nu}\;\; A^{\mu}\,\varepsilon^{\nu}
\end{aligned}
$$

Of course a depends from spin and we have to sum the different helicity amplitudes. The expression of A looks at first not easy to handle, but is simpler to obtaine, than the usual one.

We now compute the contribution of two photon annihilation to the process $e^+e^- \rightarrow e^+e^-e^+e^-$ (4); the others diagrams, i.e. bremsstrahlung of Dalitz pair, give a smaller contribution. It is important to know the exact differential cross section of this process: its total cross section is very big at the energy of the colliding beams machines $\sim 10^{-26}$ cm^2; so this process may give a relevant contribution to the production of electrons at very small angles.

The relevant diagrams are shown in Fig. 2.

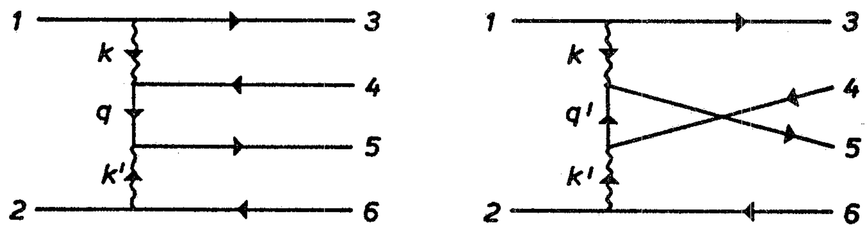

FIG. 2

The differential cross section is:

$$\frac{d\sigma}{d^3 p_3\, d^3 p_4\, d^3 p_5\, d^3 p_6} = \frac{\alpha^4}{8\pi^4} \frac{|M|^2\, \delta^4(p_1+p_2-p_3-p_4-p_5-p_6)}{E_1 E_2 E_3 E_4 E_5 E_6} \tag{13}$$

where $M^2 = \sum_{spin} |A|^2$.

The amplitude dependent from the spin is:

$$A = \left\{ \frac{1}{K^2} \frac{1}{K'^2} \left[\frac{1}{q^2-m^2} \left(|J^{26}, q, J^{31}, J^{54}| + i \, \| J^{26}, q, J^{31}, A^{54} \| + m T^{54} \cdot J^{26} \cdot J^{31} \right) - \frac{1}{q'^2-m^2} \left(|J^{31}, q', J^{26}, J^{54}| + i \, \| J^{31}, q', J^{26}, A^{54} \| + m T^{54} \cdot J^{31} \cdot J^{26} \right) \right] \right\} - \{3 \leftrightarrow 5\} - \{6 \leftrightarrow 4\} + \{3 \leftrightarrow 5, 6 \leftrightarrow 4\} \tag{14}$$

where

$$K = p_3 - p_1,\ K' = p_6 - p_2;\ q = p_1 - p_3 - p_4;\ q' = p_2 - p_6 - p_4$$

(15) $$J_\mu^{31} = \bar{u}(p_3)\gamma_\mu u(p_1);\ J_\mu^{26} = \bar{V}(p_2)\gamma_\mu V(p_6)$$

$$J_\mu^{54} = \bar{u}(p_5)\gamma_\mu V(p_4);\ A_\mu^{54} = \bar{u}(p_5)\gamma_5\gamma_\mu V(p_4);\ T_{\mu\nu}^{54} = \bar{u}(p_5)\gamma_\mu\gamma_\nu V(p_4)$$

With the same method we can compute other processes and also radiative corrections with closed loops.

III. - THE EFFECT OF POLARIZATION IN COLLIDING BEAMS. -

The cross section for polarized particles can be easily computed from the helicity amplitudes. As an example we study the effect of polarization of the initial states on the angular distributions in the process $e^+e^- \to$ final state.

We suppose that in the c.m. sistem the e^- is polarized along the direction ($\theta = \pi/2, 0$) orthogonal to the direction of motion ($\theta = 0$, $\varphi = 0$) and the e^+ is polarized in the opposite direction. It is possible that the magnetic field present in the induces such a polarization in the electrons and the positrons circulating in the storage ring.

With our standard method we compute the helicity amplitudes $A^{\pm\pm}$ where the first $\pm$ is related to the helicity of the e^- and the second one to the helicity of the e^+.

The matrix element for the differential cross section for unpolarized particles turns out to be:

(16) $$|M|^2 = \frac{1}{4}\left\{|A^{++}|^2 + |A^{--}|^2 + |A^{+-}|^2 + |A^{-+}|^2\right\}$$

It is easy to find that when the e^- and e^+ are polarized in as we just explained, this matrix element it changes into:

(17) $$|M|^2 = \frac{1}{4}\left|A^{++} + (A^{-+} + A^{+-})e^{i\varphi} + A^{--}e^{2i\varphi}\right|$$

From this formulae it is clear that no addition work is needed for computing the differential cross sections for polarized particles.

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