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B. D'Ettorre Piazzoli: DIFFERENTIAL CROSS SECTION FOR THE PROCESS $e^+e^- \to \mu^+\mu^-\gamma$ AND RADIATIVE EVENTS. -

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1 - INTRODUCTION -

In this paper we establish a general formula for the differential cross section of the bremsstrahlung process $e^+e^- \to \mu^+\mu^-\gamma$ taking into account the hard photon emission by both initial and final states. We assume that the muon charge is not measured. The calculations are made with the standard perturbation theory in the one photon exchange approximation (Sect. 2).

In Sect. 3 we check our results by integrating the differential cross section over the final muon states in order to compare the resulting total cross section with that found by V.N. Baier and V.A. Khoze(1), and U. Mosco(2) who use a different approach.

Then we apply these results to establish the cross section for "radiative events" in which only the directions of the final muons - noncollinear and in general non-coplanar with the e^+e^- beams - are detected (Sect. 4).

2. - DIFFERENTIAL CROSS SECTION FOR $e^+e^- \to \mu^+\mu^-\gamma$. -

2.1. - The matrix element. -

We use the metric $\delta_{11} = \delta_{22} = \delta_{33} = 1$, $\delta_{44} = -1$, the system of units $\hbar = c = 1$, $\alpha = e^2/4\pi$. The scalar product $(ab) = \vec{a}\cdot\vec{b} - a_4 b_4$.

The process under consideration - $e^+e^- \to \mu^+\mu^-\gamma$ - is represented by the four annihilation graphs of Fig. 1, where $p_-(\vec{p}_-, E_-)$, $p_+(\vec{p}_+, E_+)$, $\mu_-(\vec{\mu}_-, E_{\mu-})$, $\mu_+(\vec{\mu}_+, E_{\mu+})$, $K(\vec{K}, K)$ are the $e^-, e^+, \mu^-, \mu^+, \gamma$ four-momenta, respectively, and

$$p_-^2 = p_+^2 = -m_e^2, \qquad \mu_-^2 = \mu_+^2 = -m_\mu^2$$

where m_e and m_μ are the electron and muon masses.

(x) - Laboratorio di Cosmo-Geofisica del CNR - Torino.

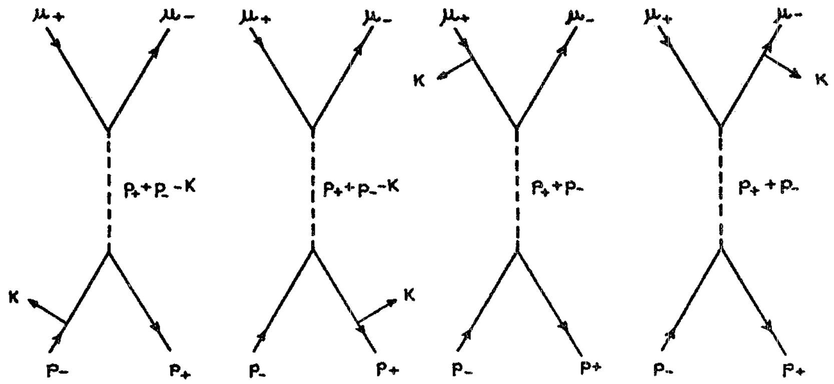

FIG. 1

The differential cross section is given in invariant form by :

$$d\sigma = \frac{1}{\left[(p_+p_-)^{1/2} - m_e^4\right]^{1/2}} \frac{e^6}{2(2\pi)^5} m_e^2 m_\mu^2 \int \overline{\sum}_{r^-r^+,\, s^-s^+,\, \text{pol.}} \left| T_{r^-r^+,\, s^-s^+} \right|^2 \times$$

$$\times \frac{d\vec{K}}{K} \frac{d\vec{\mu}_+}{E_{\mu+}} \frac{d\vec{\mu}_-}{E_{\mu-}} \delta^4 (p_+ + p_- - K - \mu_+ - \mu_-)$$

The transition matrix element is :

$$T_{r^-r^+s^-s^+} = \frac{(\overline{V}_{r^+}(p_+) C_i^\nu U_{r^-}(p_-))\,(\overline{U}_{s^-}(\mu_-)\, \gamma_\nu\, V_{s^+}(\mu_+))}{(p_+ + p_- - K)^2} +$$

$$+ \frac{(\overline{V}_{r^+}(p_+)\, \gamma_\nu\, U_{r^-}(p_-))\,(\overline{U}_{s^-}(\mu_-)\, C_f^\nu V_{s^+}(\mu_+))}{(p_+ + p_-)^2}$$

where (we set $\hat{a} = a_\nu \gamma^\nu$):

$$C_i^\nu = \gamma^\nu \frac{i(\hat{p}_- - \hat{K}) - m_e}{-2(p_-K)} \hat{e} + \hat{e} \frac{i(-\hat{p}_+ + \hat{K}) - m_e}{-2(p_+K)} \gamma^\nu = \gamma^\nu f(\hat{p}_-, \hat{K})\hat{e} + \hat{e}\, g(-\hat{p}_+, \hat{K})\gamma^\nu$$

$$C_f^\nu = \gamma^\nu \frac{i(-\hat{\mu}_+ - \hat{K}) - m_\mu}{2(\mu_+K)} \hat{e} + \hat{e} \frac{i(\hat{\mu}_- + \hat{K}) - m_\mu}{2(\mu_-K)} \gamma^\nu = \gamma^\nu f(-\hat{\mu}_+, \hat{K})\hat{e} + \hat{e}\, g(\hat{\mu}_-, \hat{K})\gamma^\nu$$

$U_{r^-}(p_-)$, $V_{r^+}(p_+)$, $U_{s^-}(\mu_-)$, $U_{s^+}(\mu_+)$ are the Dirac spinors for $e^-, e^+, \mu^-\mu^+$, and e is the polarization vector of the photon.

We notice that C_i^ν and C_f^ν are connected by the interchanges :

(1) $p_- \leftrightarrow -\mu_+$, $p_+ \leftrightarrow -\mu_-$, $m_e \leftrightarrow m_\mu$.

$\overline{\sum}_{r^-r^+,\, s^-s^+,\, \text{pol.}}$ represents the average over the spins of the initial electrons, the sum over the spins of the final muons and the sum over the polarizations of the photon.

We now define the following tensors:

$$(A_e)^{\mu\nu} = m_e^2 \sum_{r^-, r^+, \text{pol.}} (\overline{V}_{r^+}(p_+) C_i^{\mu} U_{r^-}(p_-)) \cdot (\overline{V}_{r^+}(p_+) C_i^{\nu} U_{r^-}(p_-))^+$$

$$(J_e)^{\mu\nu} = m_e^2 \sum_{r^- r^+} (\overline{V}_{r^+}(p_+) \gamma^{\mu} U_{r^-}(p_-))(\overline{V}_{r^+}(p_+) \gamma^{\nu} U_{r^-}(p_-))^+$$

$$(I_e)^{\mu\nu} = m_e^2 \sum_{r^- r^+} (\overline{V}_{r^+}(p_+) C^{\mu} U_{r^-}(p_-))(\overline{V}_{r^+}(p_+) \gamma^{\nu} U_{r^-}(p_-))^+$$

and

$$\left.\begin{array}{l}(A_\mu)^{\mu\nu} \\ (J_\mu)^{\mu\nu} \\ (I_\mu)^{\mu\nu}\end{array}\right\} \begin{array}{l}\text{obtained from the foregoing ones by the exchanges} \\ V_{r^+}(p_+) \to U_{s^-}(\mu_-); \quad U_{r^-}(p_-) \to V_{s^+}(\mu_+); \quad C_i^{\mu} \to C_f^{\mu}; \\ m_e \to m_\mu ; \quad \sum_{r^- r^+} \to \sum_{s^- s^+}\end{array}$$

so that it results

$$m_e^2 m_\mu^2 \overline{\sum}_{r^- r^-, s^- s^+, \text{pol.}} \left| T_{r^- r^+, s^- s^+} \right|^2 = \frac{1}{4} \left[\frac{(A_e)^{\mu\nu}(J_\mu)_{\mu\nu}}{\Delta^4} + \frac{(A_\mu)^{\mu\nu}(J_e)_{\mu\nu}}{\Lambda^4} + \frac{2\sum_{\text{pol.}}(I_e)^{\mu\nu}(I_\mu)_{\mu\nu}}{\Delta^2 \Lambda^2} \right] \tag{2}$$

with

$$\Delta = \left[\Delta_\mu \Delta^\mu\right]^{1/2} = \sqrt{(p_+ + p_- - K)^2} = \sqrt{(\mu_+ + \mu_-)^2} = 2E(1 - K/E)^{1/2}$$

$$\Lambda = \left[\Lambda_\mu \Lambda^\mu\right]^{1/2} = \sqrt{(p_+ + p_-)^2} = 2E$$

where E is the electron energy in the rest system of the electrons $\vec{p}_+ + \vec{p}_- = 0$. We have set $\Delta_\mu = (p_+ + p_- - K)_\mu$ and $\Lambda_\mu = (p_+ + p_-)_\mu$.

The terms in (2) represent the initial "bremsstrahlung, the final" bremsstrahlung and the interference term.

The calculation of the tensors is performed by the standard techniques. We get(x):

$$(A_e)^{\mu\nu} = -\frac{1}{4} \mathrm{Tr}\Big\{(\gamma^{\mu} f(\hat{p}_-, \hat{K})\gamma^{\rho} + \gamma^{\rho} g(-\hat{p}_+, \hat{K})\gamma^{\mu})(i\hat{p}_- - m_e) \times$$

$$\times (\gamma^{\rho} f(\hat{p}_-, \hat{K})\bar{\gamma}^{\nu} + \bar{\gamma}^{\nu} g(-\hat{p}_+, \hat{K})\gamma^{\rho})(i\hat{p} + m_e)\Big\} = \tag{3a}$$

(x) - We can put $\bar{\gamma}^{\nu} = -\gamma^{\nu}$, though $\bar{\gamma}^K = -\gamma^K$ (K = 1, 2, 3), $\bar{\gamma}^4 = \gamma^4$; cfr. ref.[3].

$$= - \frac{m_e^2}{(p-K)^2}\left[\frac{\Delta^2}{2}\delta^{\mu\nu} - 2\, p_+^\mu p_+^\nu\right] - \frac{m_e^2}{(p+K)^2}\left[\frac{\Delta^2}{2}\delta^{\mu\nu} - 2\, p_-^\mu p_-^\nu\right] +$$

$$+ \left(\frac{p_+K}{p_-K} + \frac{p_-K}{p_+K}\right)\delta^{\mu\nu} -$$

(3a)

$$- \frac{1}{(p_-K)(p_+K)}\left[\Delta^2(p_+^\mu p_+^\nu + p_-^\mu p_-^\nu + (p_+p_-)\delta^{\mu\nu}) + 2m_e^2(p_+^\mu p_-^\nu + p_-^\mu p_+^\nu)\right] +$$

$$+ \left[A_e\right]^{\mu\nu}$$

where

$$\left[A_e\right]^{\mu\nu} = - \frac{m_e^2}{(p_-K)^2}\left[\Delta^\mu p_+^\nu + p_+^\mu \Delta^\nu\right] - \frac{1}{(p-K)}\left[\Delta^\mu p_+^\nu + p_+^\mu \Delta^\nu\right] -$$

$$- \frac{m_e^2}{(p_+K)^2}\left[\Delta^\mu p_-^\nu + p_-^\mu \Delta^\nu\right] - \frac{1}{(p_+K)}\left[\Delta^\mu p_-^\nu + p_-^\mu \Delta^\nu\right] +$$

$$+ \frac{1}{(p_+K)(p_-K)}\left\{\left[2m_e^2 - (p_+p_-)\right]\left[\Delta^\mu(p_-^\nu + p_+^\nu) + (p_-^\mu + p_+^\mu)\Delta^\nu\right] - 2m_e^2\Delta^\mu\Delta^\nu\right\}$$

$[A_e]^{\mu\nu}$ doesn't contribute to the differential cross section as a consequence of the conservation of the electromagnetic current

$$\Delta^\mu (J_\mu)_{\mu\nu} = \Delta^\nu (J_\mu)_{\mu\nu} = 0$$

In addition

(4a)

$$(J_e)^{\mu\nu} = \frac{1}{4}\mathrm{Tr}\left\{\gamma^\mu (i\hat{p}_- - m_e)\bar{\gamma}^\nu (i\hat{p}_+ + m_e)\right\} =$$

$$= p_+^\mu p_-^\nu + p_-^\mu p_+^\nu + \frac{\Lambda^2}{2}\delta^{\mu\nu}$$

(3b) $$(A_\mu)^{\mu\nu} = I\left\{(A_e)^{\mu\nu}\right\}$$

(4b) $$(J_\mu)^{\mu\nu} = I\left\{(J_e)^{\mu\nu}\right\}$$

where by writing $(M) = I\{(N)\}$ we mean that (M) is obtained from (N) by the substitutions (1). Of course, these substitutions imply the interchange $\Delta \leftrightarrow \Lambda$. We don't report the interference tensors. In effect, the interchange $\mu_+ \longleftrightarrow \mu_-$ changes the sign of $(I_\mu)^{\mu\nu}$ because

$$C_f^\nu(\mu_+, \mu_-, m_\mu) = -C_f^\nu(\mu_-, \mu_+, -m_\mu) = -C_f^\nu(\mu_-, \mu_+, m_\mu)$$

The last equality is due to the fact that m_μ and m_μ^3 are associated in the trace to an odd number of γ matrices. So the interference term does not contribute when the muons' charge is not measured.

Saturating (3a)-(4b) and (3b)-(4a) we have

$$(Ae)^{\mu\nu}(J_\mu)_{\mu\nu} = -\frac{m_e^2}{(p_-K)^2}\left[\frac{\Delta^2}{2}(\Delta^2+2m_e^2+2m_\mu^2) - 4(p_+\mu_+)(p_+\mu_-)\right] -$$

$$-\frac{m_e^2}{(p_+K)^2}\left[\frac{\Delta^2}{2}(\Delta^2+2m_e^2+2m_\mu^2) - 4(p_-\mu_+)(p_-\mu_-)\right] +$$

$$+\left(\frac{p_+K}{p_-K}+\frac{p_-K}{p_+K}\right)(\Delta^2+2m_\mu^2) -$$

(5a)

$$-\frac{1}{(p_-K)(p_+K)}\Big[\Delta^2\Big\{2(p_+\mu_+)(p_+\mu_-)+2(p_-\mu_+)(p_-\mu_-)-\Delta^2 m_e^2 +$$

$$+(p_+p_-)(\Delta^2+2m_\mu^2)\Big\} +$$

$$+2m_e^2\Big\{2(p_+\mu_-)(p_-\mu_+)+2(p_+\mu_+)(p_-\mu_-)+\Delta^2(p_+p_-)\Big\}\Big] \equiv$$

$$\equiv \sum_{j=1..4}(F_i)_j$$

where we have called $(F_i)_j$ the j-th expression in the right-hand side of the equality,

(5b) $$(A_\mu)^{\mu\nu}(J_e)_{\mu\nu} = I\left\{(A_e)^{\mu\nu}(J_\mu)_{\mu\nu}\right\} \equiv \sum_{j=1..4}(F_f)_j$$

2.2. - Kinematics. -

We notice first that, if the muons charge is not measured, the matrix element is symmetric for the interchange $\mu_+ \leftrightarrow \mu_-$ (cfr. (5)). So in the following we refer to μ_1 and μ_2, where μ_1 is the muon detected at the angle $\Omega_{\mu_1} = (\theta, \phi_\theta)$. (see Fig. 2).

In the rest system of the electrons, $\vec{p}_+ + \vec{p}_- = 0$, we orient the z-axis along the e^+ direction, and define the angles as follows. (see Fig. 2):

$$\theta = (\vec{\mu}_1, \vec{p}_+); \qquad \gamma = (\vec{K}, \vec{p}_+); \qquad \alpha = (\vec{\mu}_1, \vec{K}).$$

The angular relation holds

$$\cos\alpha = \cos\theta\cos\gamma + \sin\theta\sin\gamma\cos(\phi_\theta - \phi_\gamma)$$

where ϕ_θ and ϕ_γ are the azimuthal angles of $\vec{\mu}_1$ and $\vec{K}_1$ respectively.

In terms of K and α we have:

(6a) $$\left|\vec{\mu}_1(\alpha, K)\right| = \frac{1}{2}\,\frac{(2E-K)\sqrt{\Delta^4-4m_\mu^2(\Delta^2+K^2\sin^2\alpha)} - \Delta^2 K\cos\alpha}{\Delta^2+K^2\sin^2\alpha}$$

$$E_{\mu_1}(\alpha, K) = \left[\left|\vec{\mu}_1(\alpha, K)\right|^2 + m_\mu^2\right]^{1/2}$$

$$E_{\mu_2}(\alpha, K) = 2E-K-E_{\mu_1}(\alpha, K)$$

(6b) $$\left|\vec{\mu}_2(\alpha, K)\right| = \left[E^2_{\mu_2}(\alpha, K) - m_\mu^2\right]^{1/2}.$$

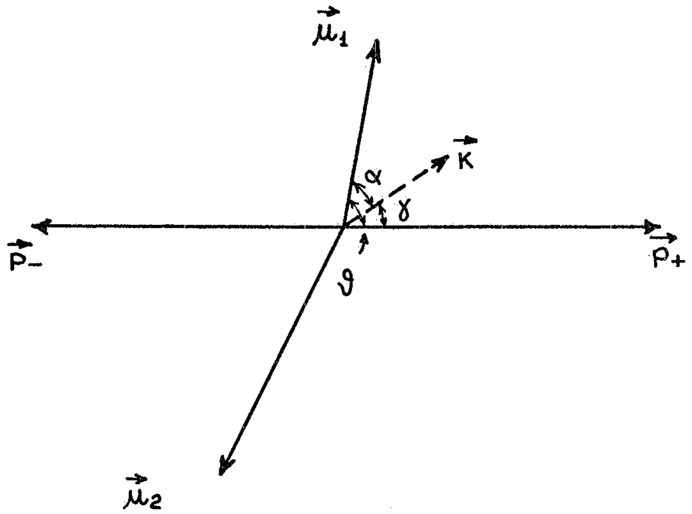

FIG. 2

All the invariants involved are defined in the electrons' rest system as follows:

$$(p_+p_-) = m_e^2 - \frac{\Delta^2}{2}$$

$$(\mu_+\mu_-) = m_\mu^2 - \frac{\Delta^2}{2}$$

$$(p_+K) = EK(\beta_e\cos\gamma - 1)$$

$$(p_-K) = -EK(\beta_e\cos\gamma + 1)$$

$$(\mu_+K) = K(|\vec{\mu}_1|\cos\alpha - E_{\mu_1})$$

$$(\mu_-K) = -K(|\vec{\mu}_1|\cos\alpha - E_{\mu_1} + 2E)$$

$$(p_+\mu_+) = E(|\vec{\mu}_1|\,\beta_e\cos\theta - E_{\mu_1})$$

$$(p_+\mu_-) = -E(|\vec{\mu}_1|\beta_e\cos\theta - E_{\mu_1}) - \frac{\Lambda^2}{2} - (p_+K)$$

$$(p_-\mu_+) = -E(|\vec{\mu}_1|\beta_e\cos\theta + E_{\mu_1})$$

$$(p_- \mu_-) = E(|\vec{\mu}_1| \beta_e \cos\theta + E_{\mu 1}) - \frac{\Lambda^2}{2} - (p_- K)$$

where β_e is the electrons velocity.

The phase space of one photon emission is

$$\int \frac{d\vec{K}}{K} \frac{d\vec{\mu}_1}{E_{\mu 1}} \frac{d\vec{\mu}_2}{E_{\mu_2}} \delta^4 (p_+ + p_- - K - \mu_1 - \mu_2) =$$

$$= \frac{d\vec{K}}{K} \frac{(\Delta^2 - 4m_\mu^2)^{1/2}}{2\Delta} d\Omega^c_{\mu 1} \text{ in the rest system of the muons}$$

$$= \frac{d\vec{K}}{K} \frac{|\vec{\mu}_1|}{(2E-K) + \frac{E_{\mu 1}}{|\vec{\mu}_1|} \cdot K\cos\alpha} d\Omega_{\mu 1} \text{ in the rest system of the electrons.}$$

2.3. - Differential cross-section for $e^+e^- \longrightarrow \mu^+\mu^-\gamma$. -

From the foregoing results we obtain the differential cross section in the electrons' rest system for a muon in $d\Omega_{\mu 1}$ and a photon in $d\vec{K} = K^2 dK\, d\Omega_K$:

$$\frac{d\sigma}{d\Omega_{\mu 1} d\vec{K}} = 2 \left[\frac{d\sigma_i}{d\Omega_{\mu 1} d\vec{K}} + \frac{d\sigma_f}{d\Omega_{\mu 1} d\vec{K}} \right] =$$

(7)
$$= \frac{\alpha^3}{E^2 \beta_e} \frac{1}{(2\pi)^2} \frac{1}{K} \frac{|\vec{\mu}_1|}{(2E-K) + \frac{E_{\mu 1}}{|\vec{\mu}_1|} K\cos\alpha} \left\{ \frac{1}{\Delta^4} \sum_{j=1,4} (F_i)_j + \frac{1}{\Lambda^4} \sum_{j=1,4} (F_f)_j \right\}$$

$$0 \leq \theta \leq \pi \qquad 0 \leq \gamma \leq \pi \qquad K \geq \varepsilon$$

$$0 \leq \phi_\theta \leq \pi \qquad 0 \leq \phi_\gamma \leq 2\pi$$

where the factor 2 in the right-hand side of the first equality comes from the non--identification of the charge of the muon detected and ε is the soft-photon cut-off.

We have called $(d\sigma_i)/(d\Omega_\mu d\vec{K})$ and $(d\sigma_f)/(d\Omega_{\mu 1} d\vec{K})$ the differential cross sections for electron and muon bremsstrahlung, respectively (cfr. (2)).

To take into account any possible breakdown of Q.E.D. by means of a form factor $F(4E^2)$ modifying the amplitude of the process, we must merely make the following substitutions

$$\frac{1}{\Delta^4} \sum_j (F_i)_j \rightarrow \frac{|F(\Delta^2)|^2}{\Delta^4} \sum_j (F_i)_j$$

$$\frac{1}{\Lambda^4}\sum_j (F_f)_j \rightarrow \frac{|F(\Lambda^2)|^2}{\Lambda^4}\sum_j (F_f)_j .$$

3. - TOTAL CROSS SECTION FOR $e^+e^- \longrightarrow \mu^+\mu^-\gamma$. -

We check the differential cross section by integrating it over the final muon states. To simplify the calculation, we integrate in the muons' rest system.

We have

$$\frac{d\sigma_i}{d\vec{K}} = \frac{\alpha^3}{E^2\beta_e}\frac{1}{(2\pi)^2}\frac{(\Delta^2-4m_\mu^2)^{1/2}}{4\Delta}\frac{1}{\Delta^4}\sum_j\int (F_i)_j\, d\Omega^c_{\mu_1} \tag{8}$$

The integrals involved are given in Appendix. The result is

$$\int (F_i)_1 d\Omega^c_{\mu_1} = \frac{4\pi}{3}(\Delta^2+2m_\mu^2)\left\{-\frac{m_e^2(\Delta^2+2m_e^2)}{(p_-K)^2} - \frac{2m_e^2}{(p_-K)} + \frac{2m_e^2}{\Delta^2}\right\}$$

$$\int (F_i)_2 d\Omega^c_{\mu_1} = \int (F_i)_1 d\Omega^c_{\mu_1} \quad \text{with} \quad (p_-K) \rightarrow (p_+K)$$

$$\int (F_i)_3 d\Omega^c_{\mu_1} = 4\pi(\Delta^2+2m_\mu^2)\left(\frac{p_+K}{p_-K}+\frac{p_-K}{p_+K}\right)$$

$$\int (F_i)_4 d\Omega^c_{\mu_1} = \frac{4\pi}{3}\frac{(\Delta^2+2m_\mu^2)}{(p_-K)(p_+K)}\left\{-2(p_+p_-)(\Delta^2+2m_e^2) - \left[(p_+K)^2+(p_-K)^2\right] - 4m_e^2\frac{(p_-K)(p_+K)}{\Delta^2} + 2m_e^2\left[(p_+K)+(p_-K)\right]\right\}$$

Summing and putting in (8) we find

$$\frac{d\sigma_i}{d\vec{K}} = \frac{\alpha^3}{(2\pi)\cdot 6E^2\beta_e}\frac{\Delta^2+2m_\mu^2}{\Delta^4}\frac{(\Delta^2-4m_\mu^2)^{1/2}}{\Delta}\left\{\left(\frac{p_+}{p_+K}-\frac{p_-}{p_-K}\right)^2(\Delta^2+2m_e^2) + 2\left(\frac{p_+K}{p_-K}+\frac{p_-K}{p_+K}\right)\right\}$$

which is the total cross-section obtained by V.N. Baier and V.A. Khoze[1] without making the calculation of the differential cross-section. This coincides also with the cross section calculated by U. Mosco[2].

Similarly, we have

$$\frac{d\sigma_f}{d\vec{K}} = \frac{\alpha^3}{E^2\beta_e}\,\frac{1}{(2\pi)^2}\,\frac{(\Delta^2-4m_\mu^2)^{1/2}}{4\Delta}\quad\frac{1}{\Lambda^4}\sum_j\int(F_f)_j\,d\Omega^c_{\mu 1}$$

Now the integration is more complicated (cfr. Appendix). We get

$$\int(F_f)_1\,d\Omega^c_{\mu 1} = -\frac{4\pi}{K^2}\left\{\frac{(p_-K)(p_+K)}{K^2}\left[2m_\mu^2\,L\left(1+2\frac{\Delta^2}{\Lambda^2}\right)-\Delta^2-8m_\mu^2\frac{\Delta^2}{\Lambda^2}\right]-\right.$$

$$\left.-\frac{4m_\mu^2}{\Lambda^2}(p_-K)(p_+K)+(\Lambda^2+2m_e^2)\left[\frac{2\Delta^2}{\Lambda^2}m_\mu^2+\frac{\Delta^2}{2}-\frac{\Delta^2}{\Lambda^2}m_\mu^2\,L\right]\right\}$$

$$\int(F_f)_2\,d\Omega^c_{\mu 1} = \int(F_f)_1\,d\Omega^c_{\mu 1}$$

$$\int(F_f)_3\,d\Omega^c_{\mu 1} = 8\pi\;(\Lambda^2+2m_e^2)(L-1)$$

$$\int(F_f)_4\,d\Omega^c_{\mu 1} = \frac{8\pi}{K^2}\left\{\frac{(p_-K)(p_+K)}{K^2}\left[L\left(2m_\mu^2\frac{\Delta^2}{\Lambda^2}-2m_\mu^2+\frac{4m_\mu^4}{\Lambda^2}-\Delta^2\right)-\right.\right.$$

$$\left.-6\,m_\mu^2\frac{\Delta^2}{\Lambda^2}+3\Delta^2\right]+$$

$$+2(p_-K)(p_+K)\left(1-L-2\frac{m_\mu^2}{\Lambda^2}\right)+$$

$$\left.+(\Lambda^2+2m_e^2)\left[L\left(\frac{\Delta^2}{2}-\frac{2m_\mu^4}{\Lambda^2}\right)+\frac{\Delta^2}{\Lambda^2}m_\mu^2-\frac{\Delta^2}{2}\right]\right\}$$

with $L = (1/\beta_\mu^c)\log\left[(1+\beta_\mu^c)/(1-\beta_\mu^c)\right]$, where β_μ^c is the muons velocity in their rest system

$$\beta_\mu^c = \frac{(\Delta^2-4m_\mu^2)^{1/2}}{\Delta}$$

We find

$$\frac{d\sigma_f}{d\vec{K}} = \frac{\alpha^3}{(2\pi)\cdot 32E^6\beta_e}\;\frac{1}{K^3}\;\frac{(\Delta^2-4m_\mu^2)^{1/2}}{\Delta}\left[\frac{2(p_-K)(p_+K)}{K^2}(A_1+A_2)-(\Lambda^2+2m_e^2)\,A_1\right]$$

where we have set

$$A_1 = 2\Delta^2+2m_\mu^2\frac{\Delta^2}{\Lambda^2}+2K^2-L\left[\Delta^2+2m_\mu^2\frac{\Delta^2}{\Lambda^2}+2K^2-\frac{4m_\mu^4}{\Lambda^2}\right]$$

$$A_2 = 2(\Delta^2-2m_\mu^2\,L)\,.$$

Also in this case we obtain the total cross section found in ref. (1).

4. - RADIATIVE EVENTS -

4.1. - Experimental photon phase space. -

In the present experiments with colliding beams the energies of the final muons are not detected. Therefore the emission of hard photon is in general admitted. Because of this emission the momenta of the final muons are not collinear. Moreover, the plane $\vec{\mu}_1$ - $\vec{\mu}_2$ contains the e^+e^- beams only in the case in wich the photon is emitted in the directions along either e^+e^- - coplanar events, otherwise we have non-coplanarity.

Using the results of the sections 2 and 3 we obtain the exact differential cross-section for these "radiative events".

We consider the following experimental features, see Fig. 3:

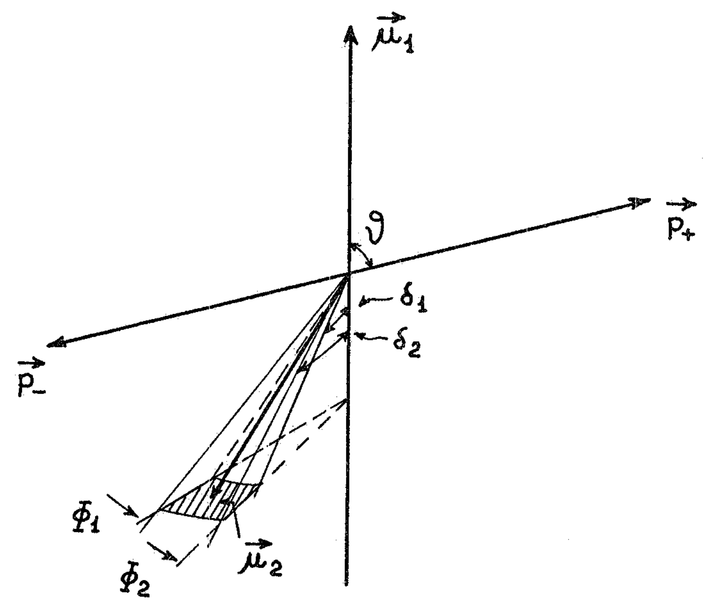

FIG. 3

- μ_1 is detected at the angle $\Omega_{\mu_1} = (\theta, \phi_\theta)$
- the angular restrictions on μ_2 are:

$$\delta_1 \leq (\vec{\mu}_2, -\vec{\mu}_1) \leq \delta_2; \qquad \phi_1 \leq \phi_{\mu 2} \leq \phi_2$$

where ϕ_{μ_2} is the azimuthal angle of $\vec{\mu}_2$ around $\vec{\mu}_1$.
- the energy restrictions on μ_1 and μ_2 are

$$|\vec{\mu}_1| > |\vec{\mu}_1|_{MIN}; \qquad |\vec{\mu}_2| > |\vec{\mu}_2|_{MIN}$$

These experimental constraints on the phase space of μ_1 and μ_2 define the allowed region for the momentum of the undetected photon. The experimental photon phase space - for a given ϕ_{μ_2} - is sketched in Fig. 4, shaded area.

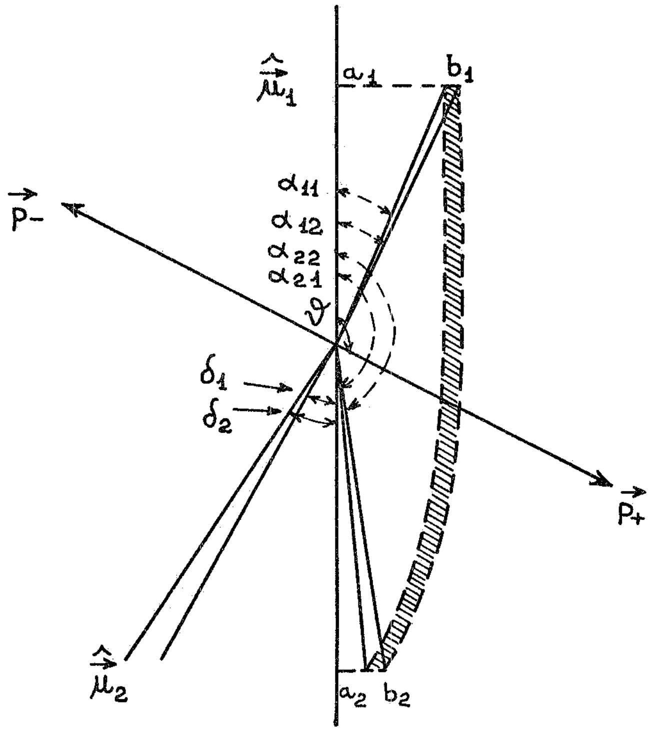

FIG. 4

The lines a_1b_1 and a_2b_2 are defined by the relations

(9a) $$\left|\vec{\mu}_1(\alpha, K)\right| = \left|\vec{\mu}_1\right|_{MIN}$$

(9b) $$\left|\vec{\mu}_2(\alpha, K)\right| = \left|\vec{\mu}_2\right|_{MIN}$$

where, for any α, the relation which gives the minimum K holds. The functions $|\vec{\mu}_1(\alpha, K)|$ and $|\vec{\mu}_2(\alpha, K)|$ are given by (6a, b). Practically, since $|\vec{\mu}_1|_{MIN} \approx |\vec{\mu}_2|_{MIN}$ for symmetric apparatus, the equation (9a) defines the line a_1b_1, the equation (9b) the line a_2b_2. These boundaries establish, for any photon angle α, the maximum photon momentum K allowed by the energy limitations.

The equation satisfied by line b_1b_2 is

(10) $$\left[K^2(\alpha,\delta)\times\frac{\sin^2(\alpha-\delta)}{\sin^2\delta}+m_\mu^2\right]^{1/2}+\left[K^2(\alpha,\delta)\times\frac{\sin^2\alpha}{\sin^2\delta}+m_\mu^2\right]^{1/2}=2E-K(\alpha,\delta)$$

which defines the relation between the non-collinearity angle δ and the momentum K, for any photon angle α. In the $m_\mu = 0$ limit we have

$$K(\alpha,\delta) = 2E\,\frac{\sin\delta}{\sin(\alpha-\delta)+\sin\alpha+\sin\delta}$$

The limiting angles α_{ij}, Fig. 4, are defined by the relation

(11) $$\left|\vec{\mu}_i(\alpha_{ij}, K(\alpha_{ij},\delta_j))\right| = \left|\vec{\mu}_i\right|_{MIN} \qquad i,j = 1,2$$

with $K(\alpha_{ij}, \delta_j)$ given by (10).

The limiting values for the photon energy are defined as follows

$$K_{MIN} = K(\alpha, \delta_1) \quad \text{from (10)} \quad \text{for any } \alpha$$

$$K_{MAX} \begin{cases} = K_1 & \text{from (9a)} & \text{for } \alpha_{11} \leq \alpha \leq \alpha_{12} \\ = K(\alpha, \delta_2) & \text{from (10)} & \text{for } \alpha_{12} \leq \alpha \leq \alpha_{22} \\ = K_2 & \text{from (9b)} & \text{for } \alpha_{22} \leq \alpha \leq \alpha_{21} \end{cases}$$

Then we orient along $\vec{\mu}_1$ the z - axis for the integration over $\vec{K}$, so

$$d\Omega_K = \sin\alpha \, d\alpha \, d\phi_\alpha$$

(12)
$$\cos\gamma = \cos\theta \cos\alpha + \sin\theta \sin\alpha \cos(\phi_\alpha - \phi_e)$$

where ϕ_α and ϕ_e are the azimuthal angles of $\vec{K}$ and $\vec{p}_+$ around $\vec{\mu}_1$, respectively. Of course, $\phi_{\mu_2} - \phi_\alpha = \pm\pi$, the sign being unimportant. The reference plane $\phi = 0$ is quite arbitrary; for instance, we can choice for it the plane $\vec{\mu}_1$-$\vec{p}_+$, so $\phi_e = 0$.

Thus we get

(13)
$$\int_{\text{Exp. photon phase space}} d\vec{K} = \int_{\phi_1-\pi}^{\phi_2-\pi} d\phi_\alpha \left\{ \int_{\alpha_{11}}^{\alpha_{12}} \sin\alpha \, d\alpha \int_{K(\alpha,\delta_1)}^{K_1} K^2 dK + \int_{\alpha_{12}}^{\alpha_{22}} \sin\alpha \, d\alpha \int_{K(\alpha,\delta_1)}^{K(\alpha,\delta_2)} K^2 dK + \int_{\alpha_{22}}^{\alpha_{21}} \sin\alpha \, d\alpha \int_{K(\alpha,\delta_1)}^{K_2} K^2 dK \right\}$$

4.2. - Cross section for radiative events. -

From the foregoing results we obtain the cross section for the case in which a muon is detected at the angle $\Omega_{\mu 1} = (\theta, \phi_0)$, the other muon is detected in a finite solid angle defined by $\Delta\delta = \delta_2 - \delta_1$, $\Delta\phi = \phi_2 - \phi_1$, the charge is not measured. We refer to it as the cross section for "radiative events", $d\sigma/d\Omega_{\mu_1}(\Delta\delta, \Delta\phi)$.

The result is simply:

(14)
$$\frac{d\sigma}{d\Omega_{\mu 1}}(\Delta\delta, \Delta\phi) = \int_{\text{Exp. photon phase space}} \left(\frac{d\sigma}{d\Omega_{\mu_1} d\vec{K}}\right) d\vec{K}$$

where $(d\sigma)/(d\Omega_{\mu_1} d\vec{K})$ is given by (7) with $\cos\gamma$ related to the angles around $\vec{\mu}_1$ by the relation (12), and the integration over the photon variables has to be carried out following the prescription (13).

We remark that the formula (14) is rigorous, because the differential cross section $(d\sigma)/(d\Omega_{\mu_1} d\vec{K})$ (in one photon exchange approximation) and the photon phase space have been derived exactly.

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APPENDIX -

In the rest system of the muons we have

$$E_+^c = \frac{\Lambda^2+2(p_+K)}{2\Delta} = \frac{\Delta^2-2(p_-K)}{2\Delta}$$

$$E_-^c = \frac{\Lambda^2+2(p_-K)}{2\Delta} = \frac{\Delta^2-2(p_+K)}{2\Delta}$$

$$K^c = \frac{\Lambda^2-\Delta^2}{2\Delta} = K\,\frac{\Lambda}{\Delta} = \frac{K}{(1-K/E)^{1/2}}$$

$$E^c_{\mu_1} = E^c_{\mu_2} = \frac{\Delta}{2} = E(1-K/E)^{1/2}$$

and

$$\left|\vec{\mu}_1^{\,c}\right| = \left|\vec{\mu}_2^{\,c}\right| = \frac{1}{2}\,(\Delta^2-4m_\mu^2)^{1/2}$$

For the evaluation of $\sum_j \int (F_i)_j d\Omega^c_{\mu_1}$ the z-axis is oriented along p_+^c. The non-trivial integral is

$$\int \cos\theta^c \cos\theta'^c d\Omega^c_{\mu_1} = \frac{4}{3}\pi\cos(\vec{p}_+^{\,c}, \vec{p}_-^{\,c}) = \frac{4}{3}\pi\,\frac{1}{|\vec{p}_+^{\,c}||\vec{p}_-^{\,c}|}\left[(p_+p_-)+E_+^cE_-^c\right]$$

where

$$\theta^c = (\vec{\mu}_1^{\,c}, \vec{p}_+^{\,c})$$

$\theta'^c = (\vec{\mu}_1^{\,c}, \vec{p}_-^{\,c})$ related to θ^c by

$$\cos\theta'^c = \cos\theta^c\cos(\vec{p}_+^{\,c}, \vec{p}_-^{\,c}) + \sin\theta^c\sin(\vec{p}_+^{\,c}, \vec{p}_-^{\,c})\cos(\phi_{\mu_1}-\phi_{p_-})$$

with ϕ_{μ_1} and ϕ_{p_-} azimuthal angles of $\vec{\mu}_1^{\,c}$ and $\vec{p}_-^{\,c}$. But the last term depends only linearly on $\cos(\phi_{\mu_1}-\phi_{p_-})$, and by integrating on ϕ_{μ_1} it disappears.

The evaluation of $\sum_j \int (F_f)_j d\Omega^c_{\mu_1}$ is harder because the denominators - which contain (μ_+K) and (μ_-K) - also depend on the $\vec{\mu}_1$ - variables. Now the polar axis is directed along the photon momentum $\vec{K}^c$ and the integration is performed on $\alpha^c = (\vec{\mu}_1^{\,c}, \vec{K}^c)$ and ϕ^c_α, azimuthal angle: $d\Omega^c_{\mu_1} = \sin\alpha^c d\alpha^c d\phi^c_\alpha$.

The following angular relations hold

(A1) $$\cos\theta^c = \cos\alpha^c\cos\gamma^c + \sin\alpha^c\sin\gamma^c\cos\phi^c_\alpha$$

(A2) $$\cos\theta'^c = \cos\alpha^c\cos\gamma'^c - \sin\alpha^c\sin\gamma'^c\cos\phi^c_\alpha$$

where we have set

$$\gamma^c \equiv (\vec{K}^c, \vec{p}_+^{\,c}); \qquad \gamma'^c \equiv (\vec{K}^c, \vec{p}_-^{\,c})$$

and the reference plane $\phi = 0$ coincides with the plane $\vec{K}^c - \vec{p}_+^c$.

The non trivial integrals are

(A3a) $$\int \frac{d\Omega^c_{\mu_1}}{(\mu_\pm, K)^2} = 2\pi \frac{\Delta^2}{K^2\Lambda^2} \frac{2}{m_\mu^2}$$

(A3b) $$\int \frac{\cos\alpha^c d\Omega^c_{\mu_1}}{(\mu_\pm, K)^2} = \mp \frac{2\pi}{ab} \frac{\Delta^2}{K^2\Lambda^2} \left\{ L - \frac{\Delta^2}{2 m_\mu^2} \right\}$$

(A3c) $$\int \frac{\cos^2\alpha^c d\Omega^c_{\mu_1}}{(\mu_\pm, K)^2} = \frac{2\pi}{b^2} \frac{\Delta^2}{K^2\Lambda^2} \left\{ 2 - 2L + \frac{\Delta^2}{2 m_\mu^2} \right\}$$

(A3d) $$\int \frac{\sin^2\alpha^c \cos^2\phi^c_\alpha d\Omega^c_{\mu_1}}{(\mu_\pm, K)^2} = \frac{2\pi}{b^2} \frac{\Delta^2}{K^2\Lambda^2} \left\{ L - 2 \right\}$$

$$\int \frac{d\Omega^c_{\mu_1}}{(\mu_+ K)(\mu_- K)} = \frac{2\pi}{a^2} \frac{\Delta^2}{K^2\Lambda^2} L$$

$$\int \frac{\cos^2\alpha^c d\Omega^c_{\mu_1}}{(\mu_+ K)(\mu_- K)} = \frac{2\pi}{b^2} \frac{\Delta^2}{K^2\Lambda^2} \left\{ L - 2 \right\}$$

$$\int \frac{\sin^2\alpha^c \cos^2\phi^c_\alpha d\Omega^c_{\mu_1}}{(\mu_+ K)(\mu_- K)} = \frac{2\pi}{b^2} \frac{\Delta^2}{K^2\Lambda^2} \left\{ 1 - L \frac{2m_\mu^2}{\Delta^2} \right\}$$

where we have called a and b the quantities $|\vec{\mu}_1^c|$ and $E^c_{\mu_1}$.

As an example we describe in some detail the angular integration of $(F_f)_2$,

$$(F_f)_2 = - \frac{m_\mu^2}{(\mu_- K)^2} \left[\frac{\Lambda^2}{2} (\Lambda^2 + 2m_e^2 + 2 m_\mu^2) - 4(\mu_+ p_-)(\mu_+ p_+) \right] =$$

$$= - \frac{m_\mu^2}{(\mu_- K)^2} \left[\frac{\Lambda^2}{2} (\Lambda^2 + 2m_e^2 + 2m_\mu^2) - \Delta^2 E^c_+ E^c_- \right] -$$

$$- \frac{m_\mu^2}{(\mu_- K)^2} \left[4ab \left\{ |\vec{p}^c_-| \cos\theta'^c E^c_+ + |\vec{p}^c_+| \cos\theta^c E^c_- \right\} - 4b^2 |\vec{p}^c_+| |\vec{p}^c_-| \cos\theta^c \cos\theta'^c \right]$$

Using (A1), (A2) and (A3a, b, c, d) we get

$$(F_f)_2 d\Omega^c_{\mu 1} = -\frac{4\pi}{K^2}\left[\frac{\Delta^2}{2}(\Lambda^2 + 2m_e^2 + 2m_\mu^2) - \frac{\Delta^4}{\Lambda^2}E^c_+E^c_-\right] -$$

$$-\frac{4\pi}{K^2}\,2m_\mu^2\,\frac{\Delta^2}{\Lambda^2}\left\{\left(\left|\vec{p}^{\,c}_-\right| \cos\gamma'^c E^c_+ + \left|\vec{p}^{\,c}_+\right|\cos\gamma^c E^c_-\right)\left(L - \frac{\Delta^2}{2m_\mu^2}\right) -\right.$$

$$\left.-\left[\left|\vec{p}^{\,c}_+\right|\left|\vec{p}^{\,c}_-\right|\cos\gamma^c\cos\gamma'^c\left(2-2L+\frac{\Delta^2}{2m_\mu^2}\right) - \left|\vec{p}^{\,c}_+\right|\left|\vec{p}^{\,c}_-\right| \sin\gamma^c\sin\gamma'^c(L-2)\right]\right\}$$

The relations which allow us to put this result into the form given in Sect. 3 are as follows

$$\Delta^2 E^c_+ E^c_- = \frac{\Lambda^2\Delta^2}{4} + (p_+K)(p_-K)$$

$$\left|\vec{p}^{\,c}_+\right| \cos\gamma^c E^c_- + \left|\vec{p}^{\,c}_-\right|\cos\gamma'^c E^c_+ = (p_+K)(p_-K)\left[\frac{1}{K^2}\left(\frac{\Lambda^2}{\Delta^2} - 1\right) - \frac{2}{\Delta^2}\right]$$

$$\left|\vec{p}^{\,c}_+\right|\left|\vec{p}^{\,c}_-\right|\cos\gamma^c\cos\gamma'^c = (p_+K)(p_-K)\left[\frac{1}{K^2} + \frac{1}{\Delta^2}\right] - \frac{\Lambda^2}{4}$$

$$-\left|\vec{p}^{\,c}_+\right|\left|\vec{p}^{\,c}_-\right| \sin\gamma^c\sin\gamma'^c = m_e^2 - \frac{(p_+K)(p_-K)}{K^2}.$$