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IMPROVED LIGHT-CONE EXPANSION

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An improved light-cone expansion is obtained by explicitly summing up towers of derivative operators which transform irreducibly under the conformal algebra. The improved expansion explicitly exhibits the causality restrictions and conformal invariance on the light cone.

It is the purpose of this note to derive an improved light-cone expansion, eq. (3) below, (for arbitrary dimensions of the operators involved) essentially by summing up the derivative operators which appear in the formula. Indeed, operator "towers" (a basic operator and all its derivatives) occur in the expansion and the coefficients within each tower are determined by the assumption of conformal invariance on the light cone*. The improved expansion has the advantage (besides being more compact) of making the causality requirement formally satisfied by each term, and of explicitly exhibiting the restrictions from translation invariance and hermiticity. To illustrate the first point, let us consider the expansion $A(x)B(0) = \sum_n c_n(x) O_n(0)$ of the product of two local operators A, B in terms of the set $\{O_n\}$ [2]. Commuting with a local operator C formally gives: $[A(x), C(y)]\,B(0)+A(x)[B(0), C(y)] = \sum_n c_n(x)[O_n(0), C(y)]$. For y spacelike, i.e. $y^2 < 0$, each single term $[O_n(0), C(y)]$ vanishes; but the left-hand-side of the above equation does not vanish necessarily if only $y^2 < 0$ (it vanishes if in addition $(x-y)^2 < 0$, which amounts on the light cone to $y^2 < 2xy$). Clearly, there is no real paradox: nevertheless an improved form of the expansion in which each single term exhibits the correct causality properties would be more useful. As to the second point, translation invariance for hermitean basis, let us take, as a simplest example, a light-cone expansion

$$j(x)j(0) \approx (x^2 - i\epsilon x_0)^{-1} \sum x^{\alpha_1} \dots x^{\alpha} \;\; T_{\alpha_1 \dots \alpha_n}(0) \tag{1}$$

where $j(x)$ is a scalar hermitean current (of dimensions -2, in length) and $T_{\alpha_1 \dots \alpha_n}(x)$ are symmetric traceless tensors (of dimensions $-(n+2)$). The tensors $T_{\alpha_1 \dots \alpha_n}$ are hermitean, thus guaranteeing the correct causality support for the commutator $[j(x), j(0)]$. One next translates eq. (1) by $-x$, changes x into $-x$, expands $T_{\alpha_1 \dots \alpha_n}(x)$ in power series at $x = 0$, and takes the hermitean conjugate of the resulting expression. Comparison with the original expansion, eq. (1), gives the infinite set of relations

$$T_{\alpha_1 \dots \alpha_n}(x) = \sum_{m=0}^{m=n} \frac{(-1)^m}{(n-m)!} \partial_{\alpha_{m+1}} \dots \partial_{\alpha_n} \; T_{\alpha_1 \dots \alpha_m}(x) \; . \tag{2}$$

From eq. (2) one sees that $T_{\alpha_1 \dots \alpha_n}$ for n odd is a sum of derivatives of $T_{\alpha_1 \dots \alpha_n}$ with m even and $< n$. The set of relations (2) will be automatically satisfied by our improved light-cone expansion.

We shall first exhibit our improved light-cone expansion, examine some of its properties, and then briefly report on its derivation. It will be convenient to deal with a slightly more general situation than

* The assumption of dilation invariance on the light-cone is by now widely used. However, in Lagrangian formalism dilation invariance implies conformal invariance in conventional models without derivative couplings (see e.g.[1]). This result may suggest assuming, *on* the light cone, invariance under (infinitesimal) conformal transformations.

usually assumed. For two scalar local operators A, B of dimensions $-l_A$, $-l_B$ we write our improved light-cone expansion as

$$A(x)\,B(0) \approx (x^2-\mathrm{i}\epsilon x^0)^{-1/2(l_A+l_B)} \sum_{n=0}^{\infty} (x^2-\mathrm{i}\epsilon x^0)^{1/2(l_n-n)} \times c_n^{AB}\, x^{\alpha_1}\ldots x^{\alpha_n}\, \Phi(l_A-l_B+l_n,\ 2l_n;\ x\partial)\ O_{\alpha_1\ldots\alpha_n}(0) \tag{3}$$

where: $O_{\alpha_1\ldots\alpha_n}(x)$ are local, Lorentz irreducible, $(\frac{1}{2}n, \frac{1}{2}n)$, tensors, of dimensions $-l_n$, satisfying $[K_\lambda, O_{\alpha_1\ldots\alpha_n}(0)] = 0$ (K_λ generates special conformal transformations); $\Phi(a,c;x)$ is the confluent hypergeometric function ([3] p. 248); and the expansion coefficients c_n^{AB} satisfy *

$$c_n^{AB} = (-1)^n\, c_n^{BA}\ . \tag{4}$$

For $l_n-n = l$ (where l is fixed), as presumably happens in the expansion for $j_\mu(x)\,j_\nu(0)$ [6], the x^2-dependence in eq. (3) factorizes out, and one regains an expansion of the kind in eq. (1) except that all derivative operators have been formally summed up by means of the hypergeometric function Φ **. Having written down our expansion we come back to the causality restriction eq. (3) can formally be written as ***

$$A(x)\,B(0) \approx (x^2-\mathrm{i}\epsilon x^0)^{-1/2(l_A+l_B)} \sum_{n=0}^{\infty} (x^2-\mathrm{i}\epsilon x^0)^{1/2(l_n-n)}\, \tilde{c}_n^{AB}\, x^{\alpha_1}\ldots x^{\alpha_n} \times \int_0^1 \mathrm{d}u\, u^{l_A-l_B+l_n-1} (1-u)^{l_B-l_A+l_n-1}\, O_{\alpha_1\ldots\alpha_n}(ux) \tag{5}$$

$$\tilde{c}_n^{AB} = c_n^{AB} \frac{\Gamma(2l_n)}{\Gamma(l_A-l_B+l_n)\,\Gamma(l_B-l_A+l_n)}\ . \tag{6}$$

Commuting with $C(y)$ one obtains, on the right, commutators $[O_{\alpha_1\ldots\alpha_n}(ux),\ C(y)]$, $0 \leqslant u \leqslant 1$, which vanish for $(ux-y)^2 < 0$ or $y^2 < 2uxy$, when x is on the light cone. The term on the left, $[A(x),\ C(y)]B(0) + A(x)[B(0),\ C(y)]$, certainly vanishes if $y^2<0$ and $(x-y)^2<0$, or $y^2<2xy$. Clearly if $y^2<2uxy$ holds, $0\leqslant u\leqslant 1$, than both $y^2<0$ and $y^2<2xy$ hold, and viceversa q.e.d.

We next verify that translation invariance holds automatically. We repeat the steps leading to eq. (2). We translate eq. (3) of $-x$, change x into $-x$, and take the hermitean conjugate, obtaining

$$B(x)\,A(0) \approx (x^2-\mathrm{i}\epsilon x^0)^{-1/2(l_A+l_B)} \sum_{n=0}^{\infty} (x^2-\mathrm{i}\epsilon x^0)^{1/2(l_n-n)}\, (-1)^n\, c_n^{AB} x^{\alpha_1}\ldots x^{\alpha_n}\, \Phi(l_A-l_B+l_n, 2l_n; -x\partial)\ O_{\alpha_1\ldots\alpha_n}(x). \tag{7}$$

However, using Kummer transformation $\Phi(a,c;x) = \Phi(c-a, c; -x)\mathrm{e}^x$, we have that

$$\Phi(l_A-l_B+l_n)\, 2l_n; -x\partial)\ O(x) = \Phi(l_B-l_A+l_n; 2l_n, x\partial)\ \mathrm{e}^{-x\partial}\ O(x) = \Phi(l_B-l_A+l_n;\ 2l_n; x\partial)\ O(0);$$

recalling also eq. (4) one sees that eq. (7) is identical to eq. (3) except for the interchange $A \leftrightarrow B$. q.e.d.

We note that for $A = B$ eq. (4) tells that $c_n^{AA} = 0$ for n odd, as we know in particular from our discussion of eq. (2). Also, for $A = B$ we notice that eq. (3) becomes

$$A(x)\,A(0) = (x^2-\mathrm{i}\epsilon x_0)^{-l_A} \sum_{n=0}^{\infty} (x^2-\mathrm{i}\epsilon x_0)^{1/2(l_n-n)}\, c_n'\, x^{\alpha_1}\ldots x^{\alpha_n} (x\partial)^{1/2-l_n}\ I_{P_n-1/2}(x\partial)\ O_{\alpha_1\ldots\alpha_n}(\tfrac{1}{2}x)\ , \tag{8}$$

* Eq. (4) follows only from translation invariance and hermicity of the basis [4].

** The relations among derivative and non-derivative operators implied by eq. (3) is automatically satisfied in models based on free quark commutators [6]. These models correspond to special choices of $O_{\alpha_1\ldots\alpha_n}$ and c_n^{AB} in eq. (3), for two currents: the summed up expansion corresponds to a bilocal operator. We also note that the invariance (by construction) under infinitesimal conformal transformations of the improved expansion, for arbitrary dimensions, eliminates any hope that canonical dimensions could follow only from such an invariance.

*** We have used the representation: $\Phi(a,c;x) = \{\Gamma(c)[\Gamma(a)\,\Gamma(c-a)]^{-1}\}\, \mathrm{d}u\, \mathrm{e}^{xu}\, u^{a-1}\, (1-u)^{c-a-1}$ (see [3] p. 255, eq. (1)); and $\mathrm{e}^{ux\partial}\, O(0) = O(ux)$.

where $c_n^{\prime} = c_n^{AA}(4)^{l_n - 1/2}\,\Gamma(l_n + \frac{1}{2})$ and $I_{l_n - 1/2}$ are modified Bessel functions ‡. Another property we mention is that in general for $A \neq B$, if $l_A - l_B + l_n = -m$ (m = integer) the hypergeometric function in eq. (3) reduces to essentially a Laguerre polynomial in $x\partial$ ‡ of order m, so that a finite number of derivatives of $O_{\alpha_1 \dots \alpha_n}$ occur. The essential ingredient in deriving eq. (8) is the requisite of invariance under K_λ (on the light cone) ‡‡. We define $O^{nm}_{\alpha_1 \dots \alpha_m}(0) = \mathrm{i}^{n-m} \underset{\alpha_1 \dots \alpha_m}{\mathrm{S}} [\dots [O^{n,n}_{\alpha_1 \dots \alpha_n}(0), P_{\alpha_{n+1}}] \dots] P_{\alpha_m}]$, where $O^{n,n}_{\alpha_1 \dots \alpha_n} \equiv O_{\alpha_1 \dots \alpha_n}$ and S means symmetrization. The assumption is that the set $\{O^{nm}_{\alpha_1 \dots \alpha_m}\}$ can serve as a basis for expansion on the light cone. Writing in general

$$A(x)\,B(0) = (x^2 - \mathrm{i}\epsilon x^0)^{-1/2(l_A + l_B)} \sum_{n=0}^{\infty} (x^2 - \mathrm{i}\epsilon x^0)^{1/2(l_n - n)} \sum_{m=n}^{\infty} c^{AB}_{nm}\, x^{\alpha_1} \dots {}^{\alpha_n}\, O^{nm}_{\alpha_1 \dots \alpha_m}(0) \tag{9}$$

one can relate c^{AB}_{nm} to $c^{AB}_{nn} \equiv c^{AB}_n$, by commuting eq. (9) with $K_\lambda x^\lambda$. On the left-hand side one finds $\mathrm{i}x^2\{(l_A - l_B)\,A(x)B(0) - \mathrm{i}[A(x)\,B(0), D]\}$ where D is the dilatation operator. On the right-hand side the last summation becomes $\sum c^{AB}_{nm}\, x^\lambda x^{\alpha_1} \dots x^{\alpha_n}\, [O^{nm}_{\alpha_1 \dots \alpha_m}(0), K_\lambda]$. Comparing with eq. (9) one has $(l_A - l_B + l_n + m - n)\,c^{AB}_{nm} = b(n, m+1)\, c^{AB}_{n, m+1}$ where $b(n, m)$ is the coefficient in

$$\underset{\{\alpha,\lambda\}}{\mathrm{S}} [\partial_{\alpha_{n+1}} \dots \partial_{\alpha_m} O_{\alpha_1 \dots \alpha_n}, K_\lambda] = \mathrm{i}b(n,m) \underset{\{\alpha,\lambda\}}{\mathrm{S}} \partial_{\alpha_{n+1}} \dots \partial_{\alpha_{m-1}} O_{\alpha_1 \dots \alpha_n}\, g_{\lambda\alpha_m},$$

where $\underset{\{\alpha,\lambda\}}{\mathrm{S}}$ means symmetrization with respect to $\{\alpha_1 \dots \alpha_m, \lambda\}$. $b(n,m)$ can be calculated by a lengthy procedure making repeated use of the conformal algebra commutators: one finds $b(n,m) = (m-n)(2l_n + m - n - 1)$. The relation between $c_{n,m+1}$ and $c_{n,m}$ can then be solved through recurrency giving $c_{n,m} = c_n \Gamma(l_A - l_B + l_n + m - n)\,\Gamma(2l_n)[(m-n)!\,\Gamma(l_A - l_B + l_n)\,\Gamma(2l_n + m - n)]^{-1}$, from which eq. (3) follows, recalling the definition of $\Phi(a,c,x)$ ([3] p. 248). In writing down eq. (3) we have for simplicity assumed A and B to be scalars, the extension to other cases being straightforward. For instance, for conserved currents, $j_\mu(x) j_\nu(0)$, an expansion like in eq. (3) separately holds for each of the conformally scalar bilocal operators C_1 and C_2, in a conformally covariant decomposition as defined in Boulware et al. [11], and whose matrix elements are simply related to W_1 and νW_2 [11].

‡ $\Phi(a, 2a; x) = \Gamma(a + \frac{1}{2})(\frac{1}{4}x)^{1/2 - a}\, e^{1/2x} I_{a - 1/2}(\frac{1}{2}x)$ (see [3] p. 265, eq. (10); p. 268, eq. (36)).

‡‡ The procedures used here are essentially extensions of those discussed in CERN preprint TH-1311, by the same authors (S.F., R.G. and A.F.G.). Consistently with the assumptions, whenever less singular terms appear, they are neglected.

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