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Scaling Laws, Light-Cone Behaviour and Harmonic Analysis on $SL_{2,C}$.

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Summary. — The electroproduction structure functions, defined on a suitable homogeneous space with respect to the spinor group $SL_{2,C}$, are subjected to harmonic analysis. In this way the Bjorken limit and the scaling law can be recovered by simple assumptions and a sum rule is obtained. The connection of this analysis with a generalized light-cone expansion is derived.

1. – Introduction.

Recently much attention has been devoted to the study of light-cone expansions [1-4] of products of local operators, mainly because of their relevance in high-energy and high-momentum-transfer electroproduction processes. A better understanding of the $x^2 \simeq 0$ limit—whose counterpart in the physical momentum space is the so-called Bjorken limit [5]—could also provide informa-

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(1) K. WILSON: *Phys. Rev.*, **179**, 1499 (1969).
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(3) W. ZIMMERMANN: *Comm. Math. Phys.*, **6**, 161 (1967).
(4) R. A. BRANDT and G. PREPARATA: preprint CERN TH-1208.
(5) J. D. BJORKEN: *Phys. Rev.*, **179**, 1547 (1969).

tion on basic properties of the interacting fields involved, such as their effective canonical dimensions.

In a previous work (6), a geometrical interpretation of the «scaling laws» was proposed, based on a suitably defined expansion of the structure functions (5) over irreducible representations of the $SL_{2,C}$ group, which seems the most natural way to obtain asymptotic behaviours in the $\omega = -q^2/2\nu$ fixed limit. The assumption of «scale invariance» reads as a dominance of a well-defined representation in this expansion and a q^2-power breaking of the simple scaling behaviour follows immediately in our scheme. This breaking is the most natural suggested also from field-theoretical perturbative expansions (4).

A ω-fixed sum rule, analogous to the t-fixed finite-energy sum rule in the Regge-pole theory, can be derived, which relates the low-energy region (baryonic-resonance production) to the scaling law region; it allows us to express the scaling functions $F_1(\omega)$ and $F_2(\omega)$ as an integral over the low q^2 and ν region of the structure functions.

In this paper we first recall the main steps of our analysis and derive the above-mentioned sum rule (Sect. **2**), then (Sect. **3**) we show how our group-theoretical approach is connected to the light-cone expansion in the configuration space. The connection is only group-theoretical and in fact it is based on theorems on harmonic analysis in homogeneous spaces. Finally, we outline the relation between the singularity structure of the matrix elements of field products, as implied by our analysis, and the operator expansion near the light-cone, suggested by some authors (4).

2. – Group-theoretical interpretation of the Bjorken limit and a sum rule.

In this Section we want to show how it is possible to recover the Bjorken limit of the structure functions, starting from the observation that the group $SL_{2,C}$ acts in a natural way on the functions of two complex variables (7).

Let us consider a function $W(z_1, z_2)$, defined on the complex affine plane (z_1, z_2). This space is a homogeneous space with respect to the spinor Lorentz group $SL_{2,C}$ and, in fact, it is equivalent to the quotient space $SL_{2,C}/Z$, where Z is the group of matrices $\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}$. A representation of $SL_{2,C}$ is defined on these functions as follows:

$$T_g\, W(z_1, z_2) = W(\alpha z_1 + \beta z_2,\ \gamma z_1 + \delta z_2)\,, \tag{2.1}$$

(6) S. FERRARA and G. ROSSI: *Lett. Nuovo Cimento*, **4**, 408 (1970); Università di Roma, Nota interna n. 286 (revised version).

(7) I. M. GEL'FAND, M. I. GRAEV and N. YA. VILENKIN: *Generalized Functions*, Vol. **5** (New York, 1964).

where

$$g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \quad \text{with } \alpha\delta-\beta\gamma=1$$

is an element of $SL_{2,C}$.

We observe that the homogeneous functions (8) of degree (n_1-1, n_2-1) play a special role in this space, as they form an irreducible subspace for the representation (2.1). Then an irreducible representation of $SL_{2,C}$ is uniquely fixed by the pairs of complex numbers (n_1, n_2), whose difference is an integer (8) and we shall call D_{n_1,n_2} the corresponding invariant subspace.

Let us recall some basic properties of the functions belonging to D_{n_1,n_2}. From the homogeneity properties, if we put $w(z)=W(z,1)$ and $\hat{w}(z)=W(1,z)$, we obtain

$$W(z_1, z_2)=z_2^{n_1-1}\bar{z}_2^{n_2-1}w(\xi)\,, \tag{2.2a}$$

$$W(z_1, z_2)=z_2^{n_1-1}\bar{z}_2^{n_2-1}w(\xi^{-1}) \tag{2.2b}$$

and

$$w(\xi)=\xi^{n_1-1}\bar{\xi}^{n_2-1}w(\xi^{-1})\,, \tag{2.3}$$

where

$$\xi=\frac{z_1}{z_2}\,.$$

Furthermore, the following asymptotic behaviours hold:

$$\lim_{\substack{|z_1|,|z_2|\to\infty\\ |\xi|\,\text{fixed}}} W(z_1, z_2)=z_2^{n_1-1}\bar{z}_2^{n_2-1}w(\xi)\,, \tag{2.4a}$$

$$\lim_{\substack{|z_1|,|z_2|\to\infty\\ |\xi|\,\text{fixed}}} W(z_1, z_2)=z_1^{n_1-1}\bar{z}_2^{n_2-1}\hat{w}(\xi) \tag{2.4b}$$

and

$$\lim_{|\xi^{-1}|\to\infty} W(z_1\ z_2)=z_2^{n_1-1}\bar{z}_2^{n_2-1}w(0)\,, \tag{2.5a}$$

$$\lim_{|\xi|\to\infty} W(z_1\ z_2)=z_1^{n_1-1}\bar{z}_1^{n_2-1}\hat{w}(0)\,, \tag{2.5b}$$

if respectively $W(0,1)$ or $W(1,0)$ are finite. Note that if we identify z_1 with q^2 and z_2 with ν, where q^2 and $\nu=(pq)$ (p is the four-momentum of the initial nucleon) are the squared mass and the energy in the laboratory frame of the virtual photon, eqs. (2.4) refer to the Bjorken limit, while eq. (2.5a) and eq. (2.5b) refer to the Regge limit and to the old Bjorken limit respectively.

(8) A function $f(z_1, z_2)$ is called homogeneous of degree (λ, μ), where λ and μ are complex numbers differing by an integer, if for every complex number $\zeta\neq 0$, we have $f(\zeta z_1, \zeta z_2)=\zeta^\lambda\bar{\zeta}^\mu f(z_1, z_2)$. We require $\lambda-\mu$ to be an integer, since only in this case $\zeta^\lambda\bar{\zeta}^\mu=|\zeta|^{\lambda+\mu}\exp[i(\lambda-\mu)\arg\zeta]$ will be a single-valued function of ζ.

Let us now consider a function $W(z_1, z_2)$, belonging to the space where the representation (2.1) acts; if this function satisfies certain regularity conditions (at least if it is square summable with respect to the $SL_{2,\sigma}$ invariant measure on the homogeneous space), it can be expanded in terms of irreducible components, *i.e.* of functions which transform irreducibly under (2.1). The theory of harmonic analysis on homogeneous spaces gives in this case the following expansion formula:

$$W(z_1, z_2) = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}\varrho\, W(z_1, z_2; n, \varrho), \tag{2.6}$$

where $n_1 = \frac{1}{2}(n + i\varrho)$, $n_2 = \frac{1}{2}(-n + i\varrho)$ and $W(z_1, z_2; n, \varrho)$ is the Mellin transform of $W(z_1, z_2)$ defined by the equation

$$W(z_1, z_2; n, \varrho) = \frac{i}{2} \int \mathrm{d}\zeta\, \mathrm{d}\bar{\zeta}\, \zeta^{-n_1} \bar{\zeta}^{-n_2} W(\zeta z_1, \zeta z_2)\,, \tag{2.7}$$

in which the integral is extended over the whole complex ζ-plane.

From (2.7) it follows immediately that:

a) $W(z_1, z_2; n, \varrho)$ is homogeneous of degree $(n_1 - 1, n_2 - 1)$;

b) the action of the representation (2.1) on $W(z_1, z_2)$ induces the action of the irreducible representation labelled by $(n_1 - 1, n_2 - 1)$ on $W(z_1, z_2; n, \varrho)$;

c) the Plancherel theorem for $W(z_1, z_2)$ holds in the form

$$\frac{i}{2} \int |W(z_1, z_2)|^2 \mathrm{d}z_1 \mathrm{d}\bar{z}_1 \mathrm{d}z_2 \mathrm{d}\bar{z}_2 = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \int |W(z, 1; n, \varrho)|^2 \mathrm{d}\varrho\, \mathrm{d}z\, \mathrm{d}\bar{z}\,. \tag{2.8}$$

We can use the eq. (2.7) to analytically continue the Mellin transform for any complex value of ϱ, so the expansion (2.6) can be valid also for functions which are not square summable in the form

$$W(z_1, z_2) = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \int_{C_\varrho} \mathrm{d}\varrho\, W(z_1, z_2; n, \varrho)\,, \tag{2.9}$$

where C_ϱ is a suitable path in the complex ϱ-plane. In particular, if $W(z_1, z_2)$ is a homogeneous function of degree $(\tilde{n}_1 - 1, \tilde{n}_2 - 1)$, hence belonging to $D_{\tilde{n}_1, \tilde{n}_2}$, we have

$$W(z_1, z_2; n, \varrho) = 2\pi \delta_{n,\tilde{n}} \int_{-\infty}^{+\infty} \mathrm{d}\sigma \exp\left[i\sigma(\tilde{\varrho} - \varrho)\right],$$

where $\tilde{n}_1 = \frac{1}{2}(\tilde{n} + i\tilde{\varrho})$ and $\tilde{n}_2 = \frac{1}{2}(-\tilde{n} + i\tilde{\varrho})$.

To relate the homogeneity properties of $W(z_1, z_2)$ to a pole-type singularity structure it can be useful to introduce an integral transform of $W(z_1, z_2; n, \varrho)$, defined as

$$\tilde{W}^{(\pm)}(z_1, z_2; n, \varrho) = \frac{1}{2\pi}\int_{C^{(\pm)}} \mathrm{d}\varrho' \, \frac{W(z_1, z_2; n, \varrho')}{\varrho' - \varrho}, \tag{2.10}$$

where in the complex ϱ'-plane $C^{(+)}$ ($C^{(-)}$) is a path lying above (below) the singularities of $W(z_1, z_2; n, \varrho')$ and passing through the point $\operatorname{Im}\varrho - i\varepsilon$ ($\operatorname{Im}\varrho + i\varepsilon$). For a homogeneous function belonging to $D_{\tilde{n}_1,\tilde{n}_2}$ one has

$$\tilde{W}^{(+)}(z_1, z_2; n, \varrho) = \tilde{W}^{(-)}(z_1, z_2; n, \varrho) = \delta_{n,\tilde{n}} \frac{1}{\tilde{\varrho} - \varrho}. \tag{2.11}$$

Although in simple cases both $\tilde{W}^{(+)}$ and $\tilde{W}^{(-)}$ exist and coincide, this is not true in general; it can well happen that $\tilde{W}^{(+)}$ or $\tilde{W}^{(-)}$ or even both do not exist, the integral representation (2.10) not being valid for any value of ϱ.

Inserting eq. (2.7) into eq. (2.10), if one is allowed to interchange the ζ, $\bar{\zeta}$ integrations with the ϱ'-integration, one easily gets

$$\tilde{W}^{(\pm)}(z_1, z_2; n, \varrho) = \pm \tfrac{1}{2}\int \mathrm{d}\zeta\, \mathrm{d}\bar{\zeta}\, \zeta^{-\frac{1}{2}(n+i\varrho)} \bar{\zeta}^{-\frac{1}{2}(-n+i\varrho)} W(\zeta z_1, \zeta z_2)\, \theta\big(\pm(|\zeta| - 1)\big), \tag{2.12}$$

where $\theta(x)$ is the step function ($\theta(x) = 1$ for $x > 0$, $\theta(x) = 0$ for $x < 0$).

Let us now consider the functions [9,10]

$$V_1(q^2, \nu) = \frac{1}{q^2}\left(W_1(q^2, \nu) + \frac{\nu^2}{q^2} W_2(q^2, \nu)\right), \tag{2.13a}$$

$$V_2(q^2, \nu) = -\frac{1}{q^2} W_2(q^2, \nu), \tag{2.13b}$$

linear combinations of the structure functions W_1 and W_2, defined by the equations

$$W_{\mu\nu}(q^2, \nu) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4x \exp[iqx] \langle p|J_\mu^{\mathrm{em}}(x) J_\nu^{\mathrm{em}}(0)|p\rangle, \tag{2.14}$$

$$W_{\mu\nu}(q^2, \nu) = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) W_1(q^2, \nu) + \left(p_\mu - \frac{\nu}{q^2} q_\mu\right)\left(p_\nu - \frac{\nu}{q^2} q_\nu\right) W_2(q^2, \nu), \tag{2.15}$$

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(9) H. LEUTWILER and J. STERN: *Nucl. Phys.*, **20** B, 771 (1970).

(10) We adopt units in which the nucleon mass is set equal to 1.

where $J^{\text{em}}_\mu(x)$ is the electromagnetic hadronic current [11]. For the Fourier transforms

$$W^F_k(x^2, px) = \int d^4q \exp[-iqx]\, W_k(q^2, \nu)\,, \qquad k = 1, 2\,, \tag{2.16}$$

and

$$V^F_k(x^2, px) \int d^4q \exp[-iqx]\, V_k(q, \nu), \qquad k = 1, 2, \tag{2.17}$$

the following relations hold [12]:

$$W^F_1(x^2, px) = -\Box V^F_1(x^2, px) - p_\mu p_\nu \partial_\mu \partial_\nu V^F_2(x^2, px)\,, \tag{2.18a}$$

$$W^F_2(x^2, px) = \Box V^F_2(x^2, px)\,. \tag{2.18b}$$

The $V^F_k(q^2, \nu)$ are free of kinematical singularities and are Fourier transforms of causal functions [13]. In the Bjorken limit $|q^2| \to \infty$, $\nu \to \infty$ with $\omega = -q^2/2\nu$ fixed (hereafter referred to as B-limit) the experimental data [14] seem to suggest the asymptotic behaviours

$$\lim_{\text{B}} V_k(q^2, \nu) = \nu^{\alpha_k} F_k(\omega)\,, \tag{2.19}$$

with $\alpha_1 = -1$ and $\alpha_2 = -2$.

From the previous analysis we can expand the functions V_k in $SL_{2,C}$ irreducible components

$$V_k(q^2, \nu) = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \int d\varrho\, V_k(q^2, \nu; n, \varrho)\,, \tag{2.20}$$

so, if we assume that in the Bjorken limit a discrete (finite or infinite) class of representations (all with $i\varrho_k - 2 = \alpha_k$) dominates the integral (2.20), eq. (2.19) directly follows.

Under the hypothesis of the existence of $\tilde{V}^{(+)}$ (or $\tilde{V}^{(-)}$) in the form expressed by eq. (2.12), it is possible to derive a kind of continuum-momentum sum

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(11) For the expression of the cross-section in terms of $W_1(q, \nu)$ and $W_2(q^2, \nu)$ see, for instance, S. D. DRELL and J. D. WALECKA: *Ann. of Phys.*, **28**, 18 (1964).

(12) $\Box$ is the d'Alembertian operator: $\Box = g^{\mu\nu} \partial_\mu \partial_\nu$ where $g^{\mu\nu}$ is the metric tensor of the space.

(13) J. W. MEYER and H. SUURA: *Phys. Rev.*, **160**, 1366 (1967).

(14) F. D. BLOOM, D. H. COWARD, H. DESTAEBLER, J. DREES, G. MILLER, L. W. MO, R. E. TAYLOR, M. BREIDENBACH, J. I. FRIEDMAN, G. C. HARTMANN and H. W. KENDALL: *Phys. Rev. Lett.*, **23**, 930, 935 (1969). For a complete review of the experimental situation, see: R. E. TAYLOR: *Proceedings of the IV International Symposium on Electron and Photon Interactions at High Energies*, edited by D. W. BRAKEN (Daresbury, 1969); R. WILSON: rapporteur's talk presented at the *XV International Conference on High-Energy Physics* (Kiev, 1970).

rule, which allows us to express the scaling function F_k as an integral over the « low-energy » (low q^2 and ν) region of the structure functions V_k. Let us start from

$$\lim_{\mathrm{B}} \sum_{n=-\infty}^{+\infty} \int_{\sigma^{(+)}} \frac{\mathrm{d}\varrho'}{\varrho'-\varrho} \int \mathrm{d}\zeta\, \mathrm{d}\bar{\zeta}\, \zeta^{-\frac{1}{2}(n+i\varrho')} \bar{\zeta}^{-\frac{1}{2}(-n+i\varrho')} V_k(\zeta q^2, \zeta\nu) = \\ = \nu^{\alpha_k} F_k(\omega) \sum_{n=-\infty}^{+\infty} \int_{\sigma^{(+)}} \frac{\mathrm{d}\varrho'}{\varrho'-\varrho} \int \mathrm{d}\zeta\, \mathrm{d}\bar{\zeta}\, \zeta^{\alpha_k-\frac{1}{2}(n+i\varrho')} \bar{\zeta}^{-\frac{1}{2}(-n+i\varrho')} \theta\big(|\zeta\nu| - \nu_{\mathrm{th}}\big) , \tag{2.21}$$

where the θ-function in the l.h.s. has been introduced to take properly into account the support properties of $F_k(\omega)$ and $\nu_{\mathrm{th}} = m_\pi(2+m_\pi)/2(1-\omega)$ (m_π is the pion mass); after a simple calculation one obtains ($\omega = -q^2/2 \neq 1$, $\omega > 0$)

$$\int_0^\infty \mathrm{d}\eta\, V_k(-2\eta\omega, \eta)\, \eta^{1+\beta} = -\frac{F_k(\omega)}{\alpha_k+\beta+2} (\nu_{\mathrm{th}})^{\alpha_k+\beta+2} , \tag{2.22}$$

where $\beta = \operatorname{Im} \varrho$ can be any real number with the only limitation $\alpha_k + \beta + 2 < 0$.

Note that eq. (2.22) could also have been obtained starting from $\tilde{V}_k^{(-)}$, so the existence of $\tilde{V}_k^{(+)}$ or of $\tilde{V}_k^{(-)}$ is sufficient to derive this sum rule.

Equation (2.22) can be easily written as a finite-energy sum rule, using the asymptotic behaviour (2.19). If $\bar{\nu}$ is a value of ν above which $V_k(q^2, \nu)$ can be well approximated by $\nu^{\alpha_k} F_k(\omega)$, one gets

$$\int_{\nu_{\mathrm{th}}}^{\bar{\nu}} \mathrm{d}\eta\, V_k(-2\eta\omega, \eta)\eta^{1+\beta} = -\frac{F_k(\omega)}{\alpha_k+\beta+2} \left((\nu_{\mathrm{th}})^{\alpha_k+\beta+2} - (\bar{\nu})^{\alpha_k+\beta+2}\right) . \tag{2.23}$$

3. – Harmonic analysis in the configuration space and connection with the light-cone expansions.

In this Section we want to discuss what our analysis in the momentum space implies for the Fourier transforms of the functions $V_k(q^2, \nu)$.

Recalling the definition (2.17)

$$V_k^F(x^2, \mu) = \int \mathrm{d}^4 q \exp[-iqx]\, V_k(q^2, \nu) , \qquad \mu = px , \tag{3.1}$$

and the expansion theorem (2.6), we have

$$V_k(x^2, \mu) = \frac{1}{(2\pi)^2} \int \mathrm{d}^4 q \exp[-iqx] \sum_n \int_{\sigma_\varrho} \mathrm{d}\varrho\, V_k(q^2, \nu; n, \varrho) = \\ = \frac{i}{2} \frac{1}{(2\pi)^2} \int \mathrm{d}^4 q \exp[-iqx] \sum_n \int_{\sigma_\varrho} \mathrm{d}\varrho \int \mathrm{d}\zeta\, \mathrm{d}\bar{\zeta}\, \zeta^{-n_1} \bar{\zeta}^{-n_2} V_k(\zeta q^2, \zeta\nu) . \tag{3.2}$$

The substitution $q \to \sqrt{\zeta} q$ in eq. (3.1) gives

$$\int \mathrm{d}^4 q \exp[-iqx] V_k(\zeta q^2, \zeta\nu) = \frac{1}{\zeta^2} V_k^F\left(\frac{x^2}{\zeta}, \mu\right). \tag{3.3}$$

So, interchanging the integration orders in eq. (3.2), one gets

$$V_k^F(x^2, \mu) = \frac{1}{(2\pi)^2} \sum_n \int \mathrm{d}\varrho\, V_k^F(x^2, \mu; n, \varrho), \tag{3.4}$$

where

$$V_k^F(x^2, \mu; n, \varrho) = \frac{i}{2} \int \mathrm{d}\zeta\, \mathrm{d}\bar{\zeta}\, \zeta^{n_1} \bar{\zeta}^{n_2 - 2} V_k^F(\zeta x^2, \mu). \tag{3.5}$$

If we are allowed to perform the sum over n, the expansion formula in the x-space becomes

$$V_k^F(x^2, \mu) = \frac{i}{2\pi} \int_{\sigma_\lambda} \mathrm{d}\lambda\, \hat{V}_k^F(x^2, \mu; \lambda), \tag{3.6}$$

where $\lambda = 1 - i\varrho$ and

$$\hat{V}_k^F(x^2, \mu; \lambda) = \int_0^\infty \mathrm{d}\sigma\, \sigma^{-\lambda} V_k^F(\sigma x^2, \mu) \tag{3.7}$$

is a homogeneous function of x^2 of degree $\lambda - 1$. Using the notations introduced in the Appendix and the definition

$$(\sigma_\pm^{-\lambda}, V)(\mu) = \pm \int_0^{\pm\infty} \mathrm{d}\sigma |\sigma|^{-\lambda} V(\sigma, \mu), \tag{3.8}$$

we can rewrite eq. (3.7) in the more expressive form

$$\hat{V}_k^F(x^2, \mu; \lambda) = (x_+^2)^{\lambda-1} (\sigma_+^{-\lambda}, V_k)(\mu) + (x_-^2)^{\lambda-1} (\sigma_-^{-\lambda}, V_k)(\mu). \tag{3.9}$$

This representation holds for all the values of λ in which $(x_\pm^2)^{\lambda-1}$ and $(\sigma_\pm^{-\lambda}, V_k)(\mu)$ are not singular. Equation (3.9) shows that the harmonic analysis in the momentum space induces in the configuration space an expression with respect to the variable x^2 in terms of representations of the unidimensional dilatation group.

Let us now discuss briefly the properties of the functions $(\sigma_\pm^{-\lambda}, V_k)(\mu)$. Since we are dealing with matrix elements of the product of two local operators (currents), we must have

$$(\sigma_+^{-\lambda}, V_k)(\mu) = \theta(x_0) \exp[i\pi(\lambda - 1)] + \theta(-x_0) \exp[-i\pi(\lambda - 1)](\sigma_-^{-\lambda}, V_k)(\mu), \tag{3.10}$$

which gives the correct behaviour

$$\hat{V}^F_k(x^2, \mu; \lambda) = (-x^2 + i\varepsilon x_0)^{\lambda-1}(\sigma_-^{-\lambda}, V_k)(\mu) \tag{3.11}$$

with $(\sigma_-^{-\lambda}, V_k)(\mu)$ real. The reality of $(\sigma_-^{-\lambda}, V_k)(\mu)$ follows from the requirement that the discontinuity of V^F_k is only different from zero for $x^2 > 0$; in fact we get in this case

$$\text{disc}\, \hat{V}^F_k(x^2, \mu; \lambda) = 2i \sin \pi(\lambda - 1)\varepsilon(x_0)(x^2_+)^{\lambda-1}(\sigma_-^{-\lambda}, V_k)(\mu)\,, \tag{3.12}$$

where $\varepsilon(x_0) = +1$ for $x_0 > 0$, $\varepsilon(x_0) = -1$ for $x_0 < 0$.

Since the distribution $(x^2_+)^{\lambda-1}$ is a meromorphic function of λ with only poles, which is regular for $\lambda = n$, $n = 1, 2, \ldots$ (see Appendix), we have

$$\text{disc}\, \hat{V}^F_k(x^2, \mu; n) = 0 \qquad \text{for } n = 1, 2, \ldots\,, \tag{3.13}$$

because of the factor $\sin(\lambda - 1)$. Again this equation holds only if also $(\sigma_-^{-\lambda}, V_k)(\mu)$ is regular. Actually we may expect $(\sigma_-^{-\lambda}, V_k)(\mu)$ to have «fixed poles» in λ which originate from the derivative-type relations between $W^F_k(x^2, \mu)$ and $V^F_k(x^2, \mu)$. In fact, because of the presence of the d'Alembertian operator in the eqs. (2.18), to the $(x^2)^{\lambda-1}$ power in $\hat{V}^F_k(x^2, \mu; \lambda)$ it will correspond in $\hat{W}^F_k(x^2, \mu; \lambda)$ (15) a term with $(x^2)^{\lambda-2}$, so that a factor $1/(\lambda - 1)$ is embodied in the definition of $(\sigma_-^{-\lambda}, V_k)(\mu)$ (16). This generates a pole-type singularity in $(\sigma_-^{-\lambda}, V_k)(\mu)$ unless a zero is present in the $\hat{W}^F_k(x^2, \mu; \lambda)$ partial wave. As we shall see, the existence of a simple pole at $\lambda = 1$ in $(\sigma_-^{-\lambda}, V_k)(\mu)$ is necessary in order not to get zero for $\nu^2 V_2(q^2, \nu)$ in the Bjorken limit.

To complete our analysis we have to relate the homogeneous $SL_{2,C}$ components of $V_k(q^2, \nu)$ with the corresponding «partial waves» of its Fourier transform, as given by eq. (3.6). This can be easily done generalizing a procedure due to BRANDT (17) to cover the cases in which λ is not an integer. We start from the equation

$$\hat{V}_k(q^2, \nu; \lambda) = \frac{1}{(2\pi)^4}\int d^4x \exp[iqx]\hat{V}^F_k(x^2, \mu; \lambda)\,, \tag{3.14}$$

where

$$\hat{V}_k(q^2, \nu; \lambda) = \frac{1}{2\pi}\frac{i}{2}\sum_{n=-\infty}^{+\infty}\int d\zeta\, d\bar{\zeta}\, \zeta^{-n_1}\bar{\zeta}^{-n_2} V_k(\zeta q^2, \zeta\nu) = \int_0^\infty dt\, t^\lambda V_k(tq^2, t\nu) \tag{3.15}$$

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(15) $\hat{W}^F_k(x^2, \mu; \lambda)$ is defined analogously to $\hat{V}^F_k(x^2, \mu; \lambda)$ as $\hat{W}^F_k(x^2, \mu; \lambda) = \int_0^\infty d\sigma\sigma^{-\lambda} \cdot W^F_k(\sigma x^2, \mu)$.

(16) The fixed pole in the partial wave $\hat{V}^F_k(x^2, \mu; \lambda)$ is due to the vector character of the electromagnetic current. This is analogous to what happens in Regge-pole theory.

(17) R. A. BRANDT: *Phys. Rev. D*, **1**, 2808 (1970).

is an homogeneous function of degree $-\lambda-1$, which satisfies

$$\hat{V}_k(q^2, \nu; \lambda) = \nu^{-(\lambda+1)}\hat{V}_k\left(\frac{q^2}{\nu}, 1; \lambda\right) \equiv \nu^{-(\lambda+1)}F_k(\omega; \lambda)\,. \tag{3.16}$$

Using this property we can evaluate the integral (3.15) in the Bjorken limit and we have

$$F_k(\omega; \lambda) = \frac{\nu^{\lambda+1}}{(2\pi)^4}\int \mathrm{d}^4x \exp[iqx]\hat{V}_k^F(x^2, \mu; \lambda) = \lim_{\mathrm{B}} \frac{\nu^{\lambda+1}}{(2\pi)^4}\int \mathrm{d}^4x \exp[iqx]\hat{V}_k^F(x^2, \mu; \lambda)\,. \tag{3.17}$$

Note that in the electroproduction case the Fourier transform of

$$[\hat{V}_k^F(x^2, \mu; \lambda)]^* = V_k^F(x^2, \mu; \lambda) - \mathrm{disc}\,\hat{V}_k^F(x^2, \mu; \lambda) \tag{3.18}$$

must be zero, since it would get contributions from intermediate states of energy (in the laboratory frame) $1-q_0$ [10], where q_0 is necessarily positive, being the energy loss of the scattered electron; so we can write

$$F_k(\omega; \lambda) = \frac{2i\sin\pi(\lambda-1)}{(2\pi)^4}\lim_{\mathrm{B}}\nu^{\lambda+1}\int \mathrm{d}^4x \exp[iqx]\,\varepsilon(x_0)(x_+^2)^{\lambda-1}(\sigma_-^{-\lambda}, V_k)(\mu)\,. \tag{3.19}$$

With the position

$$x_0 = \varphi + \tau\,, \qquad x_3 = \varphi - \tau\,,$$

if we fix the four-momentum q_μ to be

$$q_\mu = (\nu, 0, 0, -(\nu^2 - q^2)^{\frac{1}{2}})\,,$$

the scalar product qx becomes in the Bjorken limit

$$qx \underset{\mathrm{B}}{\rightarrow} 2\varphi\nu + \omega(\varphi - \tau)\,,$$

so in the laboratory frame we have

$$F_k(\omega; \lambda) = \frac{2i\sin\pi(\lambda-1)}{(2\pi)^4}\lim_{\mathrm{B}}\nu^{\lambda+1}\int \mathrm{d}\varphi\,\mathrm{d}\tau\,\mathrm{d}^2\boldsymbol{x}\cdot$$
$$\cdot\exp[i\omega(\varphi-\tau)]\exp[2i\varphi\nu]\,\varepsilon(\varphi+\tau)\left(4\varphi\tau - |\boldsymbol{x}|^2\right)^{\lambda-1}\theta\left(4\varphi\tau - |\boldsymbol{x}|^2\right)(\sigma_-^{-\lambda}, V_k)(\varphi+\tau)\,.$$

Observing with BRANDT [16] that in the Bjorken limit only the points with $\varphi \sim 0$ will contribute to the integral, because of the rapidly oscillating factor

$\exp[2i\varphi\nu]$ and using for the Fourier transforms the formulae given in the Appendix, we obtain

$$F_k(\omega,\lambda) = \frac{\pi^2}{(2\pi)^4}\frac{2^\lambda \exp[i(\pi/2)\lambda]}{\Gamma(-\lambda+1)}\int d\tau \exp[i\omega\tau]\,\tau^\lambda(\sigma_-^{-\lambda}, V_k)(\mu)\,. \tag{3.20}$$

Note that in the case $k=2$ for the dominant wave $\lambda=1$ (remember that to $\alpha=u$ corresponds $\lambda=-u-1$) eq. (3.20) would give

$$F_2(\omega, 1) = 0\,,$$

unless $(\sigma_-^{-\lambda}, V_2)(\mu)$ has a pole at $\lambda=1$; so we are forced to assume

$$(\sigma_-^{-\lambda}, V_2)(\mu) \underset{\lambda=1}{\simeq} \frac{\operatorname{res}_{\lambda=1}(\sigma_-^{-\lambda}, V_2)(\mu)}{\lambda-1}\,.$$

Under this hypothesis, using for $s=2$ the Taylor expansion of $(-x^2+i\varepsilon x_0)^{\gamma+s}$ near $\gamma=-2$ (see Appendix)

$$(-x^2+i\varepsilon x_0)^{\gamma+s} = (-x^2+i\varepsilon x_0)^{\gamma+s} + (\gamma+2)\,(-x^2+i\varepsilon x_0)^{-2+s}\log(-x^2+i\varepsilon x_0) + \dots,$$

we get

$$\operatorname{disc} \hat{V}_k^F(x^2, \mu; 1) = 2\pi i\varepsilon(x_0)\theta(x^2) \operatorname{res}_{\lambda=1}(\sigma_-^{-\lambda}, V_2)(\mu)$$

and hence

$$F_2(\omega; 1) = -\frac{2\pi^2 i}{(2\pi)^4}\int d\tau \exp[-i\omega\tau]\,\tau \operatorname{res}_{\lambda=1}(\sigma_-^{-\lambda}, V_2)(\mu)\,. \tag{3.21a}$$

In the case $k=1$ the dominant wave in the Bjorken limit corresponds to $\lambda=0$ and we have

$$F_1(\omega, 0) = \frac{\pi^2}{(2\pi)^4}\int d\tau \exp[-i\omega\tau](\sigma_-^0, V_1)(\mu)\,. \tag{3.21b}$$

We conclude by illustrating the relation between the general analysis of the structure functions in the x-space, carried out in this Section, and the light-cone expansions of products of local operators, suggested by renormalizable perturbation theories (1,4). Let us refer to eqs. (3.6) and (3.11) and compare them with the expressions obtained for the corresponding matrix elements in terms of complete sets of local tensor operators, each with a well-defined relation between spin and dimensionality (4). The two expansions look very similar and the difference is in the physical meaning attached to the exponents of the x^2-powers. For instance, in the Bjorken limit, in our language, the leading x^2-singularity near the light-cone is fixed by the indices wich label the dominant

$SL_{2,C}$ representation in the integral (2.20), while in the Brandt-Preparata ([4]) approach the leading x^2-singularity is brought by the string of tensor operators whose scalar member has the smallest allowed dimensionality in the expansion.

For instance, neglecting all spin complications, we have, by definition for two local scalar operators $A(x)$, $B(0)$,

$$\langle p|A(x)B(0)|p\rangle \equiv V^F(x^2, \mu) = \int \mathrm{d}^4 q \exp[-iqx]\, V(q^2, \nu)\,.$$

Comparing the light-cone expansion for $x^2 \simeq 0$ with eq. (3.6), we obtain

$$\frac{i}{2\pi}\int_{\sigma_l} \mathrm{d}\lambda (-x^2 + i\varepsilon x_0)^{\lambda-1} (\sigma_-^{-\lambda}, V)(\mu) \to \xrightarrow[x^2\simeq 0]{} \sum_i G^i(-x^2 + i\varepsilon x_0) \sum_n C_n^i \langle p|O^{(i)}_{\mu_1\ldots\mu_n}(0)|p\rangle\, x^{\mu_1}\ldots x^{\mu_n}\,,$$

where the C_n^i are adimensional constants and $G^i(x^2)$ is a homogeneous function of x^2 of degree $(d^A + d^B - d_0^i)/2$, d^A, d^B and d_0^i being, respectively, the dimensions of the operators $A(x)$, $B(0)$ and $O^i(0)$; for $O^{(i)}_{\mu_1\ldots\mu_n}(0)$ we have $d_n^i = d_0^i + n$. The value of λ which labels the dominant $SL_{2,C}$ representation in the Bjorken limit is then given by

$$\lambda = \frac{d^A + d^B - d_0}{2} - 1\,,$$

where $d_0 = \min_i (d_0^i)$.

APPENDIX

In this Appendix ([18]) we summarize some useful properties and formulae concerning the generalized functions which are relevant in the light-cone analysis. It is of some interest to give these properties for the general case of a pseudo-Euclidean space $O_{p,q}$ of dimension $n = p + q$, because the singularities of these functions strongly depend on p and q.

Let us start with the analogue of the Dirac δ-function. If we put

$$x^2 = x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2\,,$$

it is possible to define two different sets of distributions, $\delta_1^{(k)}(x^2)$ and $\delta_2^{(k)}(x^2)$ (where (k) means the k-th derivative), related by the property

$$\delta_2^{(k)}(x^2) = (-1)^k\, \delta_1^{(k)}(-x^2)\,. \tag{A.1}$$

([18]) The proofs of the theorems stated in the Appendix can be found in I. M. GEL'FAND and G. E. SHILOV: *Generalized Functions*, Vol. 1 (New York, 1964).

If n is odd as well as if n is even and $k < \frac{1}{2}n - 1$, $\delta_1^{(k)}(x^2) = \delta_2^{(k)}(x^2)$, while if n is even and $k \geqslant \frac{1}{2}n - 1$ the difference is a generalized function concentrated on the vertex of the $x^2 = 0$ surface, which can be written [12]

$$\delta_1^{(k)}(x^2) - \delta_2^{(k)}(x^2) = c_{pqk}\, \Box^{k-\frac{1}{2}n+1}\delta(x)\,, \tag{A.2}$$

where the coefficients c_{pqk} are numerical constants we are not interested to specify further and $\delta(x)$ is a short-hand notation for $\delta(x_1)\,\delta(x_2)\,...\,\delta(x_{p+q})$.

Introducing the generalized functions $(x_+^2)^\lambda$ and $(x_-^2)^\lambda$, defined as

$$(x_+^2)^\lambda = \begin{cases} (x^2)^\lambda\,, & x^2 > 0\,, \\ 0\,, & x^2 < 0\,, \end{cases} \quad \text{and} \quad (x_-^2)^\lambda = \begin{cases} 0\,, & x^2 > 0\,, \\ |x^2|^\lambda\,, & x^2 < 0\,, \end{cases} \tag{A.3}$$

we can show that they are meromorphic functions of λ with the two possible set of poles:

a) $\lambda = -1, -2, ..., -k$, where k is a positive integer;

b) $\lambda = -n/2, -n/2 - 1, ..., -n/2 - k$, where k is a nonnegative integer.

Referring, for instance, to $(x_+^2)^\lambda$ we have three possible cases:

1) The singularity $\lambda = -k$ is in the set *a*) but not in *b*). This is always the case if n is odd, while if n is even this happens only if $\lambda > -n/2$. In this case we have

$$\operatorname*{res}_{\lambda=-k} (x_+^2)^\lambda = \frac{(-1)^{k-1}}{(k-1)!}\,\delta_1^{(k-1)}(x^2)\,. \tag{A.4}$$

2) The singular point λ belongs to the set *b*) and not to the set *a*). This happens for $\lambda = -n/2 - k$ and n odd. For p odd and q even the residue of $(x_+^2)^\lambda$ at $\lambda = -n/2 - k$ is given by

$$\operatorname*{res}_{\lambda=-n/2-k} (x_+^2)^\lambda = \frac{(-1)^{q/2}\pi^{n/2}}{4^k k!\,\Gamma(n/2+k)}\,\Box^k\delta(x)\,, \tag{A.5}$$

while for p even and q odd $(x_+^2)^\lambda$ is regular.

3) The singularity is both in *a*) and in *b*). This happens for n even and $\lambda = -n/2 - k$. For p and q both even we have

$$\operatorname*{res}_{\lambda=-n/2-k} (x_+^2)^\lambda = \frac{(-1)^{n/2+k-1}}{\Gamma(n/2+k)}\,\delta_1^{(n/2+k-1)}(x^2) + \frac{(-1)^{q/2}\pi^{n/2}}{4^k k!\,\Gamma(n/2+k)}\,\Box^k\delta(x)\,. \tag{A.6}$$

For p and q odd $(x_+^2)^\lambda$ has double poles at $\lambda = -n/2 - k$ and its Laurent expansion can be written as

$$(x_+^2)^\lambda \underset{\lambda\simeq-n/2-k}{\simeq} \frac{C_{-2}^{(k)}(x)}{(\lambda+n/2+k)^2} + \frac{C_{-1}^{(k)}(x)}{\lambda+n/2+k} + \text{regular terms}\,, \tag{A.7}$$

where

$$C_{-2}^{(k)}(x) = \frac{(-1)^{\frac{1}{2}(q+1)}\pi^{n/2-1}}{4^k k!\,\Gamma(n/2+k)}\Box^k\delta(x)\,, \tag{A.8}$$

$$C_{-1}^{(k)}(x) = \frac{(-1)^{(n/2+k-1)}}{\Gamma(n/2+k)}\,\delta_1^{(n/2+k-1)}(x^2) + \\ + \frac{(-1)^{\frac{1}{2}(q+1)}\pi^{n/2-1}[\psi(p/2)-\psi(n/2)]}{4^k k!\,\Gamma(n/2+k)}\Box^k\delta(x)\,, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}\,. \tag{A.9}$$

All we said for $(x_+^2)^\lambda$ is valid also for $(x_-^2)^\lambda$ provided p and q are interchanged and we replace $\delta_1^{(k)}(x^2)$ by $\delta_1^{(k)}(-x^2) = (-1)^k\delta_2^{(k)}(x^2)$.

Since the generalized functions which are of physical interest are boundary values on the real axis of generalized functions of complex arguments, let us study the properties of the functions $(x^2+i\varepsilon)^\lambda$ and $(x^2-i\varepsilon)^\lambda$. They are related to $(x_+^2)^\lambda$ and $(x_-^2)^\lambda$ by the equations

$$(x^2+i\varepsilon)^\lambda = (x_+^2)^\lambda + \exp[i\pi\lambda](x_-^2)^\lambda\,, \tag{A.10}$$

$$(x^2-i\varepsilon)^\lambda = (x_-^2)^\lambda + \exp[-i\pi\lambda](x_-^2)^\lambda\,. \tag{A.11}$$

Note that for nonnegative integral values of λ $(x^2+i\varepsilon)^\lambda$, $(x^2-i\varepsilon)^\lambda$ and $(x^2)^\lambda$ coincide. For $\lambda\to -k$ (k is a positive integer) we have

$$\lim_{\lambda\to -k}\,[(x^2+i\varepsilon)^\lambda - (x^2-i\varepsilon)^\lambda] = 2\pi i(-1)^k \operatorname*{res}_{\lambda=-k}(x_-^2)^\lambda\,. \tag{A.12}$$

Let us specialize this equation in the physical case $p=1$ and $q=3$. For $\lambda=-1$ we obtain

$$\lim_{\lambda\to -1}\big((x^2+i\varepsilon)^\lambda - (x^2-i\varepsilon)^\lambda\big) = -2\pi i\,\delta(x^2)\,. \tag{A.13}$$

For $\lambda=0$ using the Taylor expansion of $(x^2+i\varepsilon)^{\lambda+k}$ near $\lambda=-n/2$

$$(x^2+i\varepsilon)^{\lambda+k} \underset{\lambda\simeq n/2}{\simeq} (x^2+i\varepsilon)^{-n/2+k} + \left(\lambda+\frac{n}{2}\right)(x^2+i\varepsilon)^{-n/2+k}\log(x^2+i\varepsilon) \tag{A.14}$$

in the case $k=n/2$, we deduce

$$\log(x^2+i\varepsilon) - \log(x^2-i\varepsilon) = 2\pi i\theta(-x^2)\,. \tag{A.15}$$

The corresponding formulae for $(-x^2+i\varepsilon)^\lambda$ and $(-x^2-i\varepsilon)^\lambda$ can be obtained by simply interchanging x_+^2 with x_-^2.

We conclude by giving the Fourier transforms of $(x^2+i\varepsilon)^\lambda$ and $(x^2-i\varepsilon)^\lambda$:

$$\int d^n x \exp[i\sigma x](x^2\pm i\varepsilon)^\lambda = \\ = \frac{\exp[\mp i(\pi/2)q]\,2^{2\lambda+n}\pi^{n/2}\Gamma(n/2+\lambda)}{\Gamma(-\lambda)}(\sigma^2\mp i\varepsilon)^{-\lambda-(1/2)n}\,. \tag{A.16}$$

In the unidimensional case we have

$$\int \mathrm{d}x \exp[i\sigma x](x \pm i\varepsilon)^{\lambda} = \frac{2\pi \exp[\pm i(\pi/2)\lambda]}{\Gamma(-\lambda)} (\sigma_{\mp})^{-\lambda-1} \tag{A.17}$$

and $(\lambda \neq -1, -2, \ldots)$

$$\int \mathrm{d}x \exp[i\sigma x](x_{\pm})^{\lambda} = \pm i \exp[\pm i(\pi/2)\lambda]\,\Gamma(\lambda+1)(\sigma \pm i\varepsilon)^{-\lambda-1}\,. \tag{A.18}$$

● RIASSUNTO

Si effettua l'analisi armonica delle funzioni di struttura dell'elettroproduzione, dopo averle definite come funzioni su un opportuno spazio omogeneo rispetto al gruppo spinoriale $SL_{2,C}$. In questo modo si possono derivare, con semplici ipotesi, il limite di Bjorken e la legge di scala e si può ottenere una regola di somma. Infine si connette questa analisi ad un'espansione generalizzata sul cono luce.

Законы подобия, поведение на световом конусе и гармонический анализ на $SL_{2,C}$.

Резюме (*). — Проводится гармонический анализ функций структуры электророждения, которые определены на соответствующем однородном пространстве по отношению к спинорной группе $SL_{2,C}$. Таким образом можно вывести, с помощью простых предположений, предел Бьёркена и закон подобия, а также можно получить правило сумм. Выводится связь этого анализа с обобщенным разложением на световом конусе.

(*) *Переведено редакцией.*