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Operator Product Expansions and Nonlinear Realization of the Conformal Group.

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Recently many authors (1-3) suggested that mass terms in a conformal invariant Lagrangian could be considered as a manifestation of a Goldstone boson, the σ-field, belonging to a singlet of « chiral $SU_2 \otimes SU_2$ » and transforming as a nonlinear representation of the conformal algebra.

In this letter we want to investigate, in this framework, the ensuing restrictions on operator product expansions (4), like $\varphi(x) \cdot \varphi(0)$, at short distances. More in detail we will consider the possibility of the occurrence of logarithmic singularities. For instance, let us assume that we have a world of pseudoscalar interacting particles.

In a Lagrangian theory, for example, the skeleton would be described by the Lagrangian

$$\mathscr{L}_s(x) = \partial_\mu \varphi(x) \partial^\mu \varphi(x) + \lambda \varphi^4(x) , \tag{1}$$

while the mass term which describes the coupling to the σ-field is

$$\mathscr{L}_m(x) = - m^2 \varphi^2(x) \exp [2b\sigma(x)] , \tag{2}$$

so the effective Lagrangian certainly contains terms like (1), (2) and, moreover, the kinetic term of the σ-field

$$\mathscr{L}_k(x) = \partial_\mu \sigma(x) \partial^\mu \sigma(x) \exp [2b\sigma(x)] . \tag{3}$$

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(1) A. SALAM and J. STRATHDEE: *Phys. Rev.*, **184**, 1760 (1969); C. J. ISHAM, A. SALAM and J. STRATHDEE: *Phys. Rev. D*, **2**, 685 (1970); C. J. ISHAM, A. SALAM and J. STRATHDEE: *Phys. Lett.*, **31** B, 300 (1970); C. J. ISHAM, A. SALAM and J. STRATHDEE: *Ann. of Phys.*, **62**, 98 (1971).
(2) B. ZUMINO: Ref. TH. 1216-CERN (1970); Rapporteur's talk at the *XV International Conference on High-Energy Physics* (Kiev, 1970).
(3) J. ELLIS: Ref. TH. 1289-CERN (1971); Paper presented at the *Coral Gables Conference on Fundamental Interactions at High Energy* (1971).
(4) K. G. WILSON: *Phys. Rev.*, **179**, 1499 (1969).

We have to remak that, usually, Lagrangians containing nonlinear σ-fields are used for obtaining in the tree graph approximation low-energy theorems; however, we note that, if conformal invariance is spontaneously broken, the σ-field has to be necessarily regarded as a fundamental field of the theory (the Goldstone boson associated to the broken simmetry), so it has to be classified between local operators appearing in a Wilson expansion. As known, in this scheme, the basic fields $\varphi(x)$, $\sigma(x)$ transform under dilatation as

$$[\varphi(0), \Delta] = il\varphi(0)\,, \qquad [\sigma(0), \Delta] = ib^{-1}I\,. \tag{4}$$

Note that the σ-field, under dilatation, is coupled to the identity operator. In order to write an operator product expansion we have to classify the relevant local fields (Wick products) of the theory. We distinguish three sets of operators with different transformation laws under dilatation:

$$\begin{cases} O_n : [O_n(0), \Delta] = il_n O_n(0)\,, \\ \Sigma_n : [\Sigma_n(0), \Delta] = inb^{-1}\Sigma_{n-1}(0)\,, \\ Q_{nr} : [Q_{nr}(0), \Delta] = i(l_n Q_{nr}(0) + rb^{-1}Q_{n\,r-1}(0))\,. \end{cases} \tag{5}$$

Examples of such fields are

$$\begin{aligned} \{O_n\} &= I,\ :\varphi^2:,\ :\Box\sigma:,\ :\varphi^4:,\ :\varphi\Box\varphi:,\ :\varphi^2\Box\sigma:, \ldots, \\ \{\Sigma_n\} &= \sigma,\ :\sigma^2:,\ :\sigma^3:, \ldots, \\ \{Q_{nr}\} &= :\varphi^2\sigma:,\ :\varphi^2\sigma^2:,\ :\sigma^2\Box\sigma: ,\ldots,\,. \end{aligned}$$

So the complete operator product expansion reads

$$\varphi(x)\cdot\varphi(0) \underset{x\to 0}{\sim} \sum_{n=0}^{\infty} c_n(x^2) O_n + \sum_{n=1}^{\infty} b_n(x^2)\Sigma_n + \sum_{n,r=1}^{\infty} f_{nr}(x^2) Q_{nr}\,. \tag{6}$$

Covariance under dilatation gives the following set of equations:

$$\begin{cases} (x_\nu \partial^\nu + 2l)\, c_n(x^2) = l_n c_n(x^2) + f_{n1}(x^2) b^{-1}\,, \\ (x_\nu \partial^\nu + 2l)\, b_n(x^2) = (n+1) b^{-1} b_{n+1}(x^2)\,, \\ (x_\nu \partial^\nu + 2l)\, f_{nr}(x^2) = l_n f_{nr}(x^2) + (n+1) b^{-1} f_{nr+1}(x^2)\,. \end{cases} \tag{7}$$

The formal solution of the system is

$$\varphi(x)\cdot\varphi(0) \underset{x\to 0}{\sim} \sum_{n=0}^{\infty} :O_n \exp[b\sigma\nabla_n]: c_n(x^2)\,, \tag{8}$$

where $\exp[b\sigma\nabla_n]$ is a formal writing for

$$:O_n \exp[b\sigma\nabla_n]: c_n(x^2) = \sum_{k=0}^{\infty} \frac{b^k}{k!} :O_n \sigma^k: \nabla_n^k c_n(x^2) \tag{9}$$

and

$$\nabla_n = x_\nu\, \partial^\nu + 2l - l_n\,,$$

We observe, by inspection, that the second member of eq. (8) contains the three sets of operators appearing in eq. (6). In particular, covariance under dilatation does not fix the functions $c_n(x^2)$ and we discuss now their possible behaviour for $x \to 0$. It is convenient to put

$$c_n(x^2) = c_n^s(x^2) + c_n^m(x^2)\,, \tag{10}$$

where

$$c_n^s(x^2) = c_n^s\left(\frac{1}{x^2}\right)^{(2l-l_n)/2}.$$

So

$$\varphi(x)\cdot\varphi(0) \underset{x\to 0}{\sim} \sum_{n=0}^{\infty} c_n^s(x^2)\,O_n + \sum_{n=0}^{\infty} :O_n \exp[b\sigma\nabla_n]:\, c_n^m(x^2)\,, \tag{11}$$

where the first sum on the right-hand side is the skeleton contribution. The coefficients $c_n^m(x^2)$ could, in principle, have any complicated singularity on the light cone. It seems there is some evidence of this fact by work on superpropagators in nonlinear theories. However, it is consistent to assume the coefficients $c_n^m(x^2)$ to have the form

$$c_n^m(x^2) \sim c_n^m\left(\frac{1}{x^2}\right)^{\alpha_n} (\log m^2x^2)^{\beta_n}\,, \tag{12}$$

where possibly α_n, β_n depend on b. Firstly, we discuss the case $\beta_n = 0$, $\alpha_n \neq \alpha_n^s = 2l - l_n$,

$$\exp[b\sigma\nabla_n]\, c_n^m(x^2) = \exp[b\sigma(\alpha_n^s - \alpha_n)]\, c_n^m(x^2)\,. \tag{13}$$

This result suggests the occurence of a renormalization of the dimension due to the presence of an infinite sequence of σ-like fields in the expansion. Consider now the more interesting case $\alpha_n = \alpha_n^s$, $\beta_n \neq 0$. This is reminescent of standard perturbation theory. Using the recursion formula

$$\nabla_n^k c_n^m(x^2) = 2^k \beta_n(\beta_n - 1)\dots(\beta_n - k + 1)\left(\frac{1}{x^2}\right)^{\alpha_n^s} (\log m^2x^2)^{\beta_n - k} c_k^m\,, \tag{14}$$

we obtain the solution

$$\exp[b\sigma\nabla_n]\, c_n^m(x^2) = c_n^m\left(\frac{1}{x^2}\right)^{\alpha_n^s} (\log m^2x^2\cdot I + 2b\sigma)^{\beta_n}\,. \tag{15}$$

Regularity of the limit $b^{-1} \to 0$ implies $c_n^m = (b^{-1})^{\beta_n} \dot{c}_n$ and for the leading contribution as $x \to 0$

$$\exp[b\sigma\nabla_n]\, c_n^m(x^2) \sim c_n\left(\frac{1}{x^2}\right)^{\alpha_n^s} (b^{-1}\log m^2x^2)^{\beta_n} I\,. \tag{16}$$

From expansion (12), if we call O_{n_0} the field of minimal dimension l_{n_0}, we have

$$\varphi(x)\cdot\varphi(0) \underset{x\to 0\ \text{(connected part)}}{\sim} (b^{-1}\log m^2x^2)^{\beta_{n_0}}\left(\frac{1}{x^2}\right)^{(2l-l_{n_0})/2} O_{n_0}(0) \tag{17}$$

We observe, in particular, that the σ-fields completely disappear at the leading order, so that they are just responsible of the logarithmic correction to the naive singularity. For $b^{-1}\to 0$ this term vanishes and we recover the skeleton expansion. Finally, when β_n is integer, it is evident from eq. (14) that only a finite number of σ-fields enters in the expansion. Also the converse is true: if only a finite number of σ-terms is present in the expansion the main singularity is the canonical one apart from a logarithmic correction whose power depends on the number of anomalous terms in the expansion. As an example of the above discussion consider the expansion given by eq. (6) truncated at the first order in b^{-1}:

$$\varphi(x)\cdot\varphi(0)\underset{x\to 0}{\sim}\sum_{n=0}^{\infty} c_n(x^2)\,O_n + b(x^2)\,\sigma + \sum_{n=1}^{\infty} f_n(x^2)\,{:}O_n\sigma{:}\;. \tag{18}$$

Using dilatation covariance we determine the coefficients

$$\begin{cases} c_n(x^2) = \left(\dfrac{1}{x^2}\right)^{(2l-l_n)/2}(c_{1n}+c_{2n}b^{-1}\log m^2x^2)\;, \\ b(x^2) \;= c\left(\dfrac{1}{x^2}\right)^{l}\;, \\ f_n(x^2) = c_{2n}\left(\dfrac{1}{x^2}\right)^{(2l-l_n)/2}\;. \end{cases} \tag{19}$$

Since the field of lowest dimension is ${:}\varphi^2{:}$, we have

$$\varphi(x)\cdot\varphi(0)\underset{x\to 0\ \text{(connected part)}}{\sim} b^{-1}\log m^2x^2\,{:}\varphi^2(0){:}\;. \tag{20}$$

We do not discuss the case α_m, β_n both nonintegers as it seems that this situation does not originate important differences from the previous ones.

We have discussed a general operator product expansion in the case of spontaneous breaking of conformal invariance. The main point is that this expansion is regular in the skeleton limit $b^{-1}\to 0$. From the above considerations we can conclude that the presence of anomalous σ-fields in the expansion is compatible both with a renormalization of the dimension and a logarithmic correction to the skeleton singularities.

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