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S. Ferrara and G. Rossi : LIGHT-CONE SINGULARITIES AND LORENTZ POLES.

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ABSTRACT. -

The causal Meyer-Suura structure functions are projected into irreducible representations of the Lorentz group. A clarification of the connection between light-cone singularities and Lorentz poles is obtained : we find that in general a light-cone singularity of the type $\frac{1}{(-x^2 + i\varepsilon x_0)^\alpha}$ in the operator product of the hadronic electromagnetic current, is built up by a sequence of Lorentz poles at $\lambda_n = 1 + \alpha - n$ whose residues are polynomial of order n in the virtual photon square mass.

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I. - INTRODUCTION. -

In this paper we study the connection between scaling properties and Regge-like behaviour of the off mass-shell Compton amplitude. Recently many authors[(1)] suggested that presence of light-cone singularities in the commutator of the hadronic electromagnetic current, which are responsable of the scaling properties of the structure functions in the Bjorken limit, could imply the existence of fixed poles in the off mass-shell forward Compton amplitude. To study this phenomenon it is necessary to develop the harmonic analysis with respect to the Lorentz group in configuration space, showing explicitely how Lorentz poles contribute to build up light-cone singularities.

In ref. (2) the Wick-rotated imaginary part of the off mass--shell forward Compton amplitude has been submitted to an O(4) analysis and the connection between its Lorentz pole content and light-cone singularities has been investigated performing a Sommerfeld-Watson transform. The authors were able to relate the behaviour of the O(4) partial waves at small distances to the scaling properties of the Compton amplitude. These techniques were used to study a wide class of light-cone singularities suggested by ladder models.

In this work we discuss two kinds of expansions of the Compton amplitude; the first one, which is relevant in the Bjorken limit, is given by an integral over all possible light-cone singularities. This expansion is more transparent in momentum space where it reads as an expansion in terms of homogeneous functions of the variables q^2, ν i.e. over irreducible representations of the group of projective transformations on the complex variables q^2, ν [(3)]. The second one, relevant in the Regge limit, is obtained projecting over the irreducible representations of the Lorentz group.

We find in general that an infinite number of Lorentz poles "conspire" to build up a light-cone singularity, more precisely, a term like $\frac{1}{(-x^2 + i\varepsilon x_0)^{\alpha}}$ is related to the sequence of Lorentz poles which are located at $\lambda_n = 1 + \alpha - n$ and whose residui are polynomials of order n in the virtual photon mass. The possible non-polynomiality of the residui should be interpreted as an indication that an infinite sequence light-cone singularities contribute in Regge limit.

Finally we remark that the techniques wedevelop may also be useful to study dynamical situations suggested by some ladder models, in which these kinds of singularities are realized.

Sections II, III are devoted to study the properties of the integral transforms we are lead to introduce in order to derive the previously mentioned results. In particular the connection of the expansions in momentum and configuration space (related by Fourier-transform) is given. In sect. IV the decomposition of a light-cone singularity in terms of Lorentz poles is carried out. The proofs of the main formulas we use are collected in the appendix.

II. - CONFORMAL TRANSFORM. -

Let us consider the functions[4]

$$V_1(q^2, \nu) = \frac{1}{q^2}\left[W_1(q^2, \nu) + \frac{\nu^2}{q^2} W_2(q^2, \nu)\right] \tag{2.1}$$

$$V_2(q^2, \nu) = -\frac{1}{q^2} W_2(q^2, \nu) \tag{2.2}$$

linear combinations of the structure functions $W_1(q^2, \nu)$, $W_2(q^2, \nu)$ defined by the equations

$$\begin{aligned} W_{\mu\nu}(q, p) &= \frac{1}{(2\pi)^4}\int d^4x\, e^{-iq\cdot x} \langle p| \left[J_\mu^{el.}(x), J_\nu^{el.}(0)\right] |p\rangle = \\ &= -(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2})W_1(q^2, \nu) + (p_\mu - \frac{\nu}{q^2}q_\mu)(p_\nu - \frac{\nu}{q^2}q_\nu)W_2(q^2, \nu) \end{aligned} \tag{2.3}$$

where $J^{el.}_{\mu}(x)$ is the hadronic electromagnetic current.

For the Fourier transforms $(k = 1, 2)$

(2.4)
$$W^F_K(x^2, x\cdot p) = \int d^4q\, e^{iq\cdot x} W_K(q^2, \nu)$$
$$V^F_K(x^2, x\cdot p) = \int d^4q\, e^{iq\cdot x} V^F_K(q^2, \nu)$$

the following relations hold

(2.5)
$$W^F_1(x^2, x\cdot p) = -\Box V^F_1(x^2, x\cdot p) - P_\mu P_\nu \partial^\mu \partial^\nu V^F_2(x^2, x\cdot p)$$
$$W'^F_2(x^2, x\cdot p) = \Box V_2(x^2, x\cdot p).$$

Experimental data suggest the following asymptotic behaviour

(2.6)
$$\lim_{\substack{\nu \to \infty \\ \omega = -q^2/2\nu \text{ fixed}}} V_K(q^2, \nu) \sim \nu^{\alpha_K - 2} F_K(\omega)$$

with $\alpha_1 = 1$, $\alpha_2 = 0$.

A typical contribution $\nu^{\alpha - 2} F(\omega)$ (we shall omit the index K from now) to the structure functions in the Bjorken limit corresponds in the Fourier transform to a term of the type $[x^2]^{-\alpha} f(x\cdot p)$ near the light-cone (where we indicate with the symbol $[x^2]^{-\alpha}$ the discontinuity of $\frac{1}{(-x^2 + i0)^\alpha}$). The scaling behaviour in momentum space has an interesting geometrical interpretation. Let us in fact consider a function $V(q^2, \nu)$ defined on the (four-dimensional) complex affine plane q^2, ν. This space is homogeneous with respect to the group SL(2, C) of projective transformations i.e. it is equivalent to the quotient space SL(2, C)/Z where Z is the group of matrices of the form $\begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}$. A representation of this group is defined on these functions as follows[5]

(2.7)
$$T_g V(q^2, \nu) = V(\alpha q^2 + \beta \nu, \gamma q^2 + \delta \nu)$$

where $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $\alpha\delta - \beta\gamma = 1$.

We observe that the homogeneous functions play a special role in this space, as they form an irreducible subspace for the representation. The theory of harmonic analysis on homogeneous spaces gives the following expansion formula

$$(2.8) \qquad V(q^2, \nu) = \frac{1}{(2\pi)^2 i} \sum_{n=-\infty}^{+\infty} \int_{-i\infty}^{i\infty} d\alpha \, V(q^2, \nu; n, \alpha)$$

where $n_1 = (n+\alpha)/2$, $n_2 = (-n+\alpha)/2$ and $V(q^2, \nu; n, \alpha)$ is the Mellin transform of $V(q^2, \nu)$ defined as follows

$$(2.9) \qquad V(q^2, \nu; n, \alpha) = \frac{i}{2} \int d\eta \, d\bar{\eta} \; \eta^{-n_1} \bar{\eta}^{-n_2} \, V(\eta q^2, \eta\nu)$$

eq. (2.8) is only valid for L^2 functions, however eq. (2.9) can be used to analytically continue the Mellin transform in α so, for not square--summable functions the expansion formula reads

$$(2.10) \qquad V(q^2, \nu) = \frac{1}{(2\pi)^2 i} \sum_{n=-\infty}^{+\infty} \int_C d\alpha \, V(q^2, \nu; n, \alpha)$$

where C is a suitable path in the complex α-plane.

If we introduce the Fourier transform as

$$(2.11) \qquad V^F(x^2, u) = \int d^4q \, e^{iq\cdot x} V(q^2, \nu) \qquad u = x\cdot p$$

from the previous analysis we obtain in configuration space the expansion formula

$$(2.12) \qquad V^F(x^2, u) = \frac{1}{(2\pi)^2 i} \sum_n \int_C d\alpha \, V^F(x^2, u; n, \alpha)$$

where

$$(2.13) \qquad V^F(x^2, u) = \frac{i}{2} \int d\eta \, d\bar{\eta} \; \eta^{n_1} \bar{\eta}^{n_2 - 2} \, V^F(\eta x^2, u) .$$

If we perform the sum over n we get

(2.14) $$V^F(x^2, u) = \frac{1}{2\pi i} \int_C d\alpha \, \hat{V}^F(x^2, u; \alpha).$$

with the definition

(2.15) $$\hat{V}^F(x^2, u; \alpha) = \int_0^\infty dt \, t^{\alpha - 1} V^F(tx^2, u) .$$

Using now the causality of the Fourier transformed structure fucntions we can write

(2.16) $$V^F(x^2, u) = \frac{1}{2\pi i} \int_C d\alpha \, [x^2]^{-\alpha} f(u; \alpha)$$

where we have introduced the "conformal transform" of $V^F(x^2, u)$

(2.17) $$f(u; \alpha) = - \frac{1}{2i \sin \pi\alpha} \int_0^\infty d\sigma \, \sigma^{\alpha - 1} V^F(\sigma, u)$$

and $$[x^2]^{\alpha} = \text{disc.} \frac{1}{(-x^2 + i0)^{\alpha}} = - 2i \sin \pi\alpha \, (x^2_+)^{-\alpha} .$$

We have called the transform defined by eq. (2.17) "conformal transform" to remind the fact that a term like $[x^2]^{-\alpha} f(x \cdot p; \alpha)$ is related in the corresponding light-cone operatorial expansions to an infinite set of tensor operators $O_{\alpha_1 \dots \alpha_n}(0)$ classified according to a ladder of irreducible representations of the conformal algebra whose spectrum is given by the eigenvalues of the only non-vanishing Casimir

$$2M_{\mu\nu} M^{\mu\nu} + 2P_\lambda \cdot K^\lambda - 2D^2 + 8iD = 4n(n-\alpha-1) + 2\alpha^2 - 8$$

and $M_{\mu\nu}$, P_λ, K_λ, D are the generators of the conformal algebra[6].

In momentum space we can write the expansion

(2.18) $$V(q^2, \nu) = \frac{1}{2\pi i} \int_C d\alpha \, \nu^{\alpha - 2} F(\omega; \alpha)$$

where the scaling function is expressed in terms of the conformal transform $f(u;\alpha)$ as

(2.19) $$F(\omega,\alpha) = \frac{1}{(2\pi)^2}\, 2^{-\alpha}\, e^{-i\frac{\pi}{2}\alpha}\, \frac{\sin\pi\alpha}{\Gamma(\alpha)} \int_0^{\infty} du\, e^{i\omega u}\, u^{1-\alpha} f(u;\alpha)$$

To obtain the complete diagonalization of the expansions (2.16), (2.18) we define the Mellin transform with respect to the u variable of $f(u;\alpha)$

(2.20) $$f(\alpha,\tau) = \int_0^{\infty} du\, u^{\tau-1} f(u;\alpha)$$

so in configuration space we have

(2.21) $$V(x^2,u) = \frac{1}{(2\pi i)^2} \int_C d\alpha \int_{C-i\infty}^{C+i\infty} d\tau\, [x^2]^{-\alpha}\, u^{-\tau} f(\alpha,\tau)$$

and in momentum space

(2.22) $$V(q^2,\nu) = \frac{1}{(2\pi)^4} \int_C d\alpha \int_{C-i\infty}^{C+i\infty} 2^{2-2\alpha}\, e^{-i\frac{\pi}{2}\alpha}\, e^{-i\frac{\pi}{2}\tau} \times$$
$$\times \frac{\sin\pi\alpha}{\Gamma(\alpha)}\, \Gamma(2-\alpha-\tau)(-q^2)^{\alpha-2}\, \omega^{\tau} f(\alpha,\tau)$$

where we used the relation

(2.23) $$F(\omega,\alpha) = \frac{1}{(2\pi)^3}\, 2^{-\alpha}\, e^{-i\frac{\pi}{2}\alpha}\, \frac{\sin\pi\alpha}{\Gamma(\alpha)}\, \omega^{\alpha-2} \times$$
$$\times \int_{C-i\infty}^{C+i\infty} d\tau\, e^{-i\frac{\pi}{2}\tau}\, \Gamma(2-\alpha-\tau)\, \omega^{\tau} f(\alpha,\tau).$$

III. - LORENTZ AND WEYL TRANSFORMS. -

In this section we will expand directly the amplitude in configuration space into irreducible representations of the Lorentz group. The complete diagonalization will be obtained in this case by means of the Weyl transform which is defined as the product of the Lorentz and Mellin transform in the x^2 variable. These transforms obviously commute. We start projecting out the dependence of $\tilde{V}^F(x^2, \cosh \zeta_x) = V^F(x^2, x\cdot p)$ from $\cosh \zeta_x = (x\cdot p)/\sqrt{x^2}$ performing the usual Lorentz transform(7)

$$(3.1) \qquad \int_0^\infty \tilde{V}^F(x^2, \cosh \zeta_x)\, \mathcal{D}_\lambda (\cosh \zeta_x) \sinh^2 \zeta_x \, d\zeta_x = \tilde{V}^F_\lambda (x^2)$$

as defined by eq. (A. 1).

The Plancherel theorem gives

$$(3.2) \qquad \tilde{V}^F(x^2, \cosh \zeta_x) = \frac{i}{\pi} \int_{-i\infty}^{i\infty} \lambda^2 d\lambda \; \tilde{V}^F_\lambda (x^2)\, \mathcal{D}_{-\lambda} (\cosh \zeta_x)$$

where the path has to be suitably shifted for not L^2 functions. Possible behaviours of the Lorentz transform $\tilde{V}^F_\lambda(x^2)$ were studied in ref. (2) expecially in connection with simple structure suggested by ladder models. The corresponding expansion in momentum space is obtained, computing the Lorentz transform of the Fourier kernel $e^{-iq\cdot x}$ (see eq. (A. 5)), by means of the formula

$$(3.3) \qquad \tilde{V}_\lambda (q^2) = \int dR \, R^3 \, \frac{K_\lambda (R\sqrt{-q^2})}{R\sqrt{-q^2}} \tilde{V}^F_\lambda (R^2)$$

where $K_\lambda (x)$ is a modified Bessel function of third kind (Hankel function); and $R = \sqrt{x^2}$. Inversion formula reads

$$(3.4) \qquad V(q^2, \nu) = 4\pi i \int_C \lambda^2 d\lambda \; \tilde{V}_\lambda (q^2)\, \mathcal{Q}_\lambda (\nu, q^2)$$

where $\mathcal{a}_\lambda(\nu, q^2)$ is a second kind matrix-element on the Lorentz group defined by eq. (A. 7). We note that the partial wave $\tilde{V}_\lambda(q^2)$ defined by eq. (3. 3) could have in principle λ singularities originated by the Hankel function. This phenomenon is more transparent performing the complete diagonalization by means of the Mellin transform in the variable x^2. We define the Weyl transform as

$$\tilde{V}^F_{\lambda\rho} = \iint \tilde{V}^F(x^2, \cosh\xi_x)\mathcal{D}_\lambda(\cosh\xi_x)\sinh^2\xi_x\, d\xi_x (x^2)^{\rho-1} dx^2 \tag{3. 5}$$

according to eq. (A. 9). We call it Weyl transform as it performs the diagonalization with respect to irreducible representations of the Weyl group. The inversion fromula is given by

$$\tilde{V}^F(x^2, \cosh\xi_x) = \frac{1}{2\pi^2}\int \lambda^2 d\lambda \int d\rho (x^2)^{-\rho}\mathcal{D}_\lambda(\cosh\xi_x)\,\tilde{V}^F_{\lambda\rho} \tag{3. 6}$$

and in momentum space using eq. (A. 12)

$$\begin{aligned} V(q^2, \nu) = \frac{4}{\pi} \int d\lambda \int d\rho\; \lambda^2\, 2^{-2\rho}\,\Gamma\left(\frac{\lambda}{2} - \rho + \frac{3}{2}\right)\Gamma\left(-\frac{\lambda}{2} - \rho + \frac{3}{2}\right) \times \\ \times\; (-q^2)^{\rho-2}\mathcal{a}_\lambda(\nu, q^2)\,\tilde{V}^F_{\lambda\rho} \end{aligned} \tag{3. 7}$$

We observe that the path of integration in the λ and ρ planes are along the imaginary axis for L^2 functions and they have to be suitable shifted for not-L^2 functions.

The Weyl transform is simply connected to the conformal transform introduced in the previous section. To see this we start by rewriting eq. (2. 21) as

$$V(x^2, u) = \frac{2}{(2\pi i)^2}\iint d\alpha\, d\rho \left[x^2\right]^{-\rho}(\cosh\xi_x)^{2\alpha-2\rho} f(\alpha, 2\rho - 2\alpha) \tag{3. 8}$$

after the change of variable $\tau = 2\rho - 2\alpha$. This connection clarifies the kinematical origin of the Lorentz-pole content of a light-cone contribution; this will be shown explicitely in the next section.

IV. - DECOMPOSITION OF A LIGHT-CONE SINGULARITY INTO LORENTZ POLES CONTRIBUTIONS. -

In this section we want to investigate the connection between the two integral representations for the causal structure functions $V^F(x^2, u)$ (2. 21), (3. 6) (and their related momentum space versions (2. 23), (3. 7)). If we remind the structure of the eq. (3. 8) we see that the transformation function which relates the two expansions is nothing but the Lorentz transform of the power $(\cosh \gamma_x)^{2\rho - 2\alpha}$. Its Lorentz transform is given by eq. (A. 13). So we have in terms of irreducible Lorentz representations

$$\tilde{V}^F(x^2, \cosh \gamma_x) = \frac{-1}{(2\pi i)^3} \iiint d\alpha \, d\rho \, d\lambda \; \lambda^2 \; \frac{2^{2\rho - 2\alpha - 1}}{\Gamma(2\rho - 2\alpha)} \quad \times$$

$$\times \; f(\alpha, 2\rho - 2\alpha) \Gamma(\tfrac{\lambda}{2} - \tfrac{1}{2} + \rho - \alpha) \Gamma(-\tfrac{\lambda}{2} - \tfrac{1}{2} + \rho - \alpha) \quad \times \tag{4. 1}$$

$$\times \; [x^2]^{-\rho} \mathcal{D}_\lambda (\cosh \gamma_x)$$

and in momentum space by means of Fourier-transform

$$V(q^2, \nu) = \frac{2}{(2\pi)^2 i} \iiint d\alpha \, d\rho \, d\lambda \; \lambda^2 \; \frac{2^{-2\alpha}}{\Gamma(2\rho - 2\alpha)} \quad \times$$

$$\times \; f(\alpha, 2\rho - 2\alpha)(-2i \sin \pi\alpha) \Gamma(\tfrac{\lambda}{2} - \tfrac{1}{2} + \rho - \alpha) \Gamma(-\tfrac{\lambda}{2} - \tfrac{1}{2} + \rho - \alpha) \times \tag{4. 2}$$

$$\times \; \Gamma(\tfrac{\lambda}{2} - \rho + \tfrac{3}{2}) \Gamma(-\tfrac{\lambda}{2} - \rho + \tfrac{3}{2}) (-q^2)^{\rho - 2} \mathcal{A}_\lambda (\nu, q^2)$$

Compearing eq. (3. 7) and (4. 2) we have the connection between the two previously introduced conformal and Weyl transforms

$$\tilde{V}^F_{\lambda\rho} = \frac{1}{8\pi i} \int d\alpha \; \frac{2^{2\rho - 2\alpha}}{\Gamma(2\rho - 2\alpha)} \Gamma(\tfrac{\lambda}{2} - \tfrac{1}{2} + \rho - \alpha) \Gamma(-\tfrac{\lambda}{2} - \tfrac{1}{2} + \rho - \alpha) \; \times$$

$$\times \; f(\alpha, 2\rho - 2\alpha)(-2i \sin \pi\alpha) \tag{4. 3}$$

This result clearly means that a Weyl contribution is in general built up by an infinite sequence of light-cone singularities. In order to study the matching between light-cone singularities and Lorentz poles we exchange the integration order in the integral representation (4.2) and perform explicitely the λ -integration by means of Cauchy theorem. We get

(4.4)
$$
\begin{aligned}
V(q^2,\nu) = & \sum_{n=0}^{\infty} \frac{1}{\pi} \iint d\alpha \, d\rho \, \frac{(-1)^{n-1}}{(n-1)!} \frac{2^{-2\alpha}}{\Gamma(2\rho-2\alpha)} (-2i\sin\pi\alpha) \times \\
& \times f(\alpha, 2\rho-2\alpha)\, \Gamma(2\rho-2\alpha-1+n)\, \Gamma(-\alpha+1+n)\, \Gamma(\alpha-2\rho+2-n) \times \\
& \times (2\rho-2\alpha-1+2n)^2 (-q^2)^{\rho-2}\, a_{2\rho-2\alpha-1+2n}(\nu, q^2) + \\
& + \sum_{n=0}^{\infty} \frac{1}{\pi} \iint d\alpha \, d\rho \, \frac{(-1)^{n-1}}{(n-1)!} \frac{2^{-2\alpha}}{\Gamma(2\rho-2\alpha)} (-2i\sin\pi\alpha) \times \\
& \times f(\alpha, 2\rho-2\alpha)\, \Gamma(-\alpha+1+n)\, \Gamma(-\alpha+2\rho-2-n)\, \Gamma(-2\rho+3+n) \times \\
& \times (-2\rho+3+2n)^2 (-q^2)^{\rho-2}\, a_{-2\rho+3+2n}(\nu, q^2)
\end{aligned}
$$

where we have taken the contribution of the Lorentz poles at $\lambda = 2\rho - 2\alpha - 1 + 2n$, $\lambda = -2\rho + 3 + 2n$ and we closed the interpretation path toward the right half plane where the functions $a_\lambda(\nu, q^2)$ go to zero.

To see the Lorentz pole content of a light-cone contribution we assume that the ρ integration can be performed by an appropriate deformation of the integration path in such a way that the Mellin transform is analytic in the corresponding region of the ρ plane. This corresponds, in Regge pole language, to consider the Compton amplitude with true Regge poles subtracted.[1]

Performing the integral we get

(4.5)
$$
\begin{aligned}
V(q^2,\nu) = & \sum_{n,m=0}^{\infty} 4\int d\alpha \, \frac{(-1)^{n-1}}{(n-1)!} \frac{(-1)^{m-1}}{(m-1)!} \frac{2^{-2\alpha}\sin\pi\alpha}{\Gamma(-\alpha+2-n+m)} \times \\
& \times f(\alpha - \alpha+2-n+m)\, \Gamma(-\alpha+1+n)\, \Gamma(-\alpha+1+m)\, (-\alpha+1+n+m)^2 \times
\end{aligned}
$$

$$\times \; (-q^2)^{\frac{\alpha+2-n+m}{2}-2} \; a_{-\alpha+1+n+m}(\nu, q^2) \quad +$$

(4.5)

$$+ \sum_{n,m=0}^{\infty} 4\int d\alpha \; \frac{(-1)^{n-1}}{(n-1)!} \; \frac{(-1)^{m-1}}{(m-1)!} \; \frac{2^{-2\alpha}\sin\pi\alpha}{\Gamma(-\alpha+2+n-m)} \quad \times$$

$$\times \; f(\alpha_1-\alpha+2+n-m)\,\Gamma(-\alpha+1+n)\,\Gamma(-\alpha+1+m)\,(-\alpha+1+n+m)^2 \quad \times$$

$$\times \; (-q^2)^{\frac{\alpha+2+n-m}{2}-2} \; a_{-\alpha+1+n+m}(\nu, q^2)$$

where we have taken the contributions of the poles at $\rho = \frac{-\alpha+2-n+m}{2}$ $\rho = \frac{2+\alpha+n-m}{2}$ in the first and second integral of (4.5) respectively. The integral has been closed in such a way that the background goes to zero moving the integration path at infinity.

Formula (4.5) can be rewritten as

$$V(q^2,\nu) = \sum_{n,m=0}^{\infty} 4\int d\alpha \; \Big(\frac{f(\alpha_1-\alpha+2-n+m)}{\Gamma(-\alpha+2-n+m)} \; (-q^2)^{\frac{\alpha+2-n+m}{2}-2} \quad +$$

(4.6)

$$+ \; n \rightleftarrows m\Big) \; \frac{(-1)^{n-1}}{(n-1)!} \; \frac{(-1)^{m-1}}{(m-1)!} \; 2^{-2\alpha} \; \sin\pi\alpha \; \Gamma(-\alpha+1+m) \quad \times$$

$$\times \; \Gamma(-\alpha+1+m)\,(-\alpha+1+n+m)^2 \; a_{-\alpha+1+n+m}(\nu, q^2)$$

To see the behaviour of a light-cone singularity we pick up a contribution to the integrand in (4.6). In the Regge limit we have

$$V_{\alpha}(q^2,\nu) \underset{\nu\to\infty}{=} \sum_{n,m=0}^{\infty} -4\;\Big(\frac{f(\alpha_1-\alpha+2-n+m)}{\Gamma(-\alpha+2-n+m)}(-q^2)^m + n \rightleftarrows m\Big) \quad \times$$

(4.7)

$$\times \; \frac{(-1)^{n-1}}{(n-1)!} \; \frac{(-1)^{m-1}}{(m-1)!} \; 2^{-2\alpha} \; \sin\pi\alpha \; \Gamma(-\alpha+1+n)\,\Gamma(-\alpha+1+m) \quad \times$$

$$\times \; (-\alpha+1+n+m)\; e^{i\frac{\pi}{2}(2-\alpha+n+m)} \; \nu^{\alpha-2-n-m}$$

when we used the asymptotic behaviour of the functions $a_{\lambda}(\nu, q^2)$ defined in eq. (A.7). In particular we observe that the Γ functions exactly cancel each other for α = negative integer, corresponding to a case of a derivative of a δ-singularity on the light-cone. In particular the leading Regge pole corresponding to the α-light-cone singularity goes as

$$(4.7) \qquad V_{\alpha}(q^2, \nu) \underset{\nu \to \infty}{\sim} \text{const} \cdot \nu^{\alpha - 2}$$

with residue independent of q^2. In general the residue of the n-th non leading poles is a polynomial in q^2 of order n. This results establish that a light-cone singularity $(x_+^2)^{-\alpha}$ in the current commutator corresponds to a sequence of Regge poles at $J_n = \alpha - n$ whose residues are polynomial of order n in the photon square mass.

APPENDIX. -

In this appendix we recall some formulae which we need in the text. We start by recalling thet the causal structure function $V^F(x^2, x\cdot p) = \tilde{V}^F(x^2, \cosh \zeta_x)$ ($\cos \zeta_x = \frac{x\cdot p}{\sqrt{x^2}}$) can be considered, for fixed x^2, as a bicovariant function[8] defined over SL(2, C) (the universal covering group of the Lorentz group). This means that it is a function $\tilde{V}(x^2, a)$, $a \in SL(2,C)$, which satisfies the covariance relation $\tilde{V}(x^2, a) = \tilde{V}(x^2, h_1 a h_2)$ for $h_1, h_2 \in SU(2)$. Its Lorentz transform is given by the formula

$$\int_{SL(2,C)} \tilde{V}^F(x^2, a) \mathcal{D}^{0\lambda}_{0000}(a)\, d^6a = 4\pi^3 \int_0^{\infty} \tilde{V}^F(x^2, \cosh \zeta_x) \times \tag{A.1}$$

$$\times \quad \mathcal{D}_\lambda (\cosh \zeta_x) \sinh^2 \zeta_x \, d\zeta_x = 4\pi^3 \tilde{V}^F_\lambda (x^2)$$

where

$$\mathcal{D}_\lambda (\cosh \zeta_x) = d^{0\lambda}_{000}(\cos h\zeta_x) = \frac{\sinh \lambda \zeta_x}{\lambda \sinh \zeta_x}$$

is a matrix-element of an irreducible representation of the type $(0, \lambda)$. This function is called elementary-spherical harmonic of SL(2, C). Plancherel theorem gives

$$\tilde{V}^F(x^2, \cos h\zeta_x) = \frac{i}{\pi} \int_{-i\infty}^{i\infty} \lambda^2 d\lambda \, \tilde{V}^F_\lambda(x^2) \mathcal{D}_{-\lambda} (\cos h\zeta_x) \tag{A.3}$$

for functions L^2 over SL(2, C). For not L^2 functions the integration path must suitably shifted.

Computation of the Lorentz transform of the inverse Fourier--transform

$$V(q^2, \nu) = \int e^{-iq\cdot x} \, V^F(x^2, x\cdot p)\, d^4x \tag{A.4}$$

requires the knowledge of the Lorentz transform of the Fourier kernel $e^{-iq\cdot x}$. This has been evaluated in ref. (9) and we get

$$\int e^{-iq\cdot x} \mathcal{D}^{0\lambda}_{0000}(a) d^6a = 2\pi^2 \frac{1}{R\sqrt{-q^2}} K \quad (R\sqrt{-q^2}) \mathcal{D}_\lambda (\nu, q^2) \tag{A.5}$$

where $K_\lambda(x)$ is a modified Bessel function of third kind[10] and

(A. 6) $$\mathcal{D}_\lambda(\nu, q^2) = d^{0\lambda}_{000}(\nu/\sqrt{q^2})$$

is a matrix-element of a $(0, \lambda)$ representation of SL(2, C) continued to imaginary values of $\cosh \zeta_q$ by means of the formulae

(A. 7)
$$\mathcal{D}_\lambda(\nu, q^2) = \mathcal{A}_\lambda(\nu, q^2) - \mathcal{A}_{-\lambda}(\nu, q^2) \quad \text{and}$$
$$\mathcal{A}_{-\lambda}(\nu, q^2) = \frac{1}{\lambda} \frac{1}{\sqrt{\nu^2 - q^2}} (\nu + \sqrt{\nu^2 - q^2})^\lambda (-q^2)^{\frac{1-\lambda}{2}} e^{i\frac{\pi}{2}(1-\lambda)}$$

At this point we have to make a remark : in principle the projection of the Fourier kernel $e^{-iq \cdot x}$, which acts from a homogeneous space of the kind SL(2, C)/SU(2) ($x^2 > 0$) to one of the kind SL(2, C)/SU(1, 1) ($q^2 < 0$) receives contributions also from irreducible representations of the type (M, 0). Newertheless, as explained in ref. (11) the Plancherel measure in the inversion formula in momentum space has a support $\Omega_{x,q}$ which is the intersection of the supports Ω_x, Ω_q of the Plancherel measures on the two different homogeneous spaces so

$$\Omega_{x,q} = \Omega_x \cap \Omega_q = (0, \lambda) \cap (0, \lambda) + (M, 0) = (0, \lambda)$$

We then obtain the Lorentz expansion in q-space in the form

(A. 8) $$V(q^2, \nu) = 2\pi i \int \lambda^2 d\lambda\, \tilde{V}_\lambda(q^2) \mathcal{D}_\lambda(\nu, q^2)$$

where

$$\tilde{V}_\lambda(q^2) = \int dR\, R^3 \frac{K_\lambda(R\sqrt{-q^2})}{R\sqrt{-q^2}} \tilde{V}^F_\lambda(R^2)$$

and we have used the formula

$$\theta(x^2)\, \theta(x^0)\, d^4x = \frac{1}{\pi} R^3\, dR\, d^3X$$

where d^3X is the invariant measure over SL(2, C)/SU(2) and is defined by the formula $$d^3u\, d^3X = d^6a$$

being d^3u, d^6a the Haar measures over SU(2), SL(2, C) respectively.

We observe that, to obtain complete diagonalization of the Lorentz expansion, we have to perform the Mellin transform in the variable $R = \sqrt{x^2}$. If we define the Weyl transform as

$$4\pi^3 \tilde{V}^F_{\lambda\rho} = \int \tilde{V}^F(x^2, \cos h\zeta_x) \mathcal{D}_\lambda(a)\, d^6a (x^2)^{\rho-1} dx^2 \tag{A. 9}$$

we get

$$V^F(x^2, x\cdot p) = \frac{1}{2\pi^2} \int_{-i\infty}^{i\infty} \lambda^2 d\lambda \int_{C-i\infty}^{C+i\infty} d\rho (x^2)^{-\rho} \mathcal{D}_\lambda(\cos h\zeta_x) \tilde{V}^F_{\lambda\rho} \tag{A. 10}$$

and the corresponding expansion in momentum space is

$$V(q^2, \nu) = \frac{2}{\pi} \int_{-i\infty}^{i\infty} d\lambda \int_{C-i\infty}^{C+i\infty} d\rho\, \lambda^2 \tilde{V}^F_{\lambda\rho}\, 2^{-2\rho} (-q^2)^{\rho-2} \times$$

$$\times \Gamma\left(\frac{\lambda}{2} - \rho + \frac{3}{2}\right) \Gamma\left(-\frac{\lambda}{2} - \rho + \frac{3}{2}\right) \mathcal{D}_\lambda(\nu, q^2). \tag{A. 11}$$

To derive the last equation we performed the Mellin trasnform of the function $K_\lambda(x)$ [12]

$$\int \frac{dx}{x} x^{-2\rho+3} K_\lambda(x) = 2^{-2\rho+1} \Gamma\left(\frac{\lambda}{2} - \rho + \frac{3}{2}\right) \Gamma\left(-\frac{\lambda}{2} - \rho + \frac{3}{2}\right). \tag{A. 12}$$

Finally we perform the Weyl transform of a light-cone singularity contribution. This is necessary to relate the expansions introduced in sect. II and III. We get[13]

$$\int \left(\frac{1}{x^2}\right)^\alpha f(x\cdot p, \alpha) \mathcal{D}_\lambda(a) d^6a (x^2)^{\rho-1} dx^2 = 2 \int dx\, x^{-2\alpha+2\rho-1} \times$$

$$\times f(x, \alpha) \int (\cos h\zeta_x)^{2\alpha-2\rho} \mathcal{D}_\lambda(a) d^6a = \pi^3 f(\alpha, -2\alpha+2\rho) \times \tag{A. 13}$$

$$\times \frac{2^{2\rho-2\alpha}}{\Gamma(2\rho-2\alpha)} \Gamma\left(\frac{\lambda}{2} - \frac{1}{2} + \rho - \alpha\right) \Gamma\left(-\frac{\lambda}{2} - \frac{1}{2} + \rho - \alpha\right).$$

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