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S. Ferrara and A. F. Grillo: OPERATOR PRODUCT EXPANSIONS AND NON LINEAR REALIZATION OF THE CONFORMAL GROUP.

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ABSTRACT. -

The implications of the recently proposed non linear realization of the conformal group on operator product expansions at short distances are discussed and the role of logarithmic terms is pointed out.

Recently many authors(1, 2, 3) suggested that mass terms in a conformal invariant Lagrangian could be considered as a manifestation of a Goldston boson, the σ-field, belonging to a singlet of "chiral SU(2) $\otimes$ SU(2)" and transforming as a non linear representation of the conformal algebra.

In this letter we want to investigate, in this framework, the ensueing restrictions on operator product expansions(4), like $\phi(x) \cdot \phi(0)$, at short distances. More in detail we will consider the possibility of the occurrence of logarithmic singularities. For instance assume we have a world of pseudoscalar interacting particles. The "skeleton" is described by the Lagrangian:

(1) $$\mathcal{L}_s(x) = \partial_\mu \phi(x)\, \partial^\mu \phi(x) + \lambda\, \phi^4(x)$$

while the mass term describes the coupling to the σ-field

(2) $$\mathcal{L}_m(x) = m^2\, \phi^2(x)\, e^{2b\sigma(x)}$$

so the effective Lagrangian is

(3) $$\mathcal{L}(x) = \mathcal{L}_s(x) + \mathcal{L}_m(x) .$$

In this scheme the basic fields $\phi(x)$, $\sigma(x)$ transform under dilatation as

(4) $$\left[\phi(0), \Delta\right] = i\, l\, \phi(0) \qquad \left[\sigma(0), \Delta\right] = i\, b^{-1}\, I .$$

Note that the σ-field, under dilatation, is coupled to the identity operator. In order to write an operator product expansion we have to classify the relevant local fields (Wick-products) of the theory. We distinguish three sets of operators with different transformation law under dilatation:

(5)
$$O_n \quad : \quad \left[O_n(0), \Delta\right] = i\, l_n\, O_n(0)$$
$$\Sigma_n \quad : \quad \left[\Sigma_n(0), \Delta\right] = i\, n\, b^{-1}\, \Sigma_{n-1}(0)$$
$$Q_{nr} \quad : \quad \left[Q_{nr}(0), \Delta\right] = i\, (l_n\, Q_{nr}(0) + r\, b^{-1}\, Q_{nr-1}(0))$$

Examples of such fields are:

$$\{O_n\} = I \,;\; :\phi^2:\,,\; :\Box\sigma:\,;\; :\phi^4:\,,\; :\phi\Box\phi:\,,\; :\phi^2\Box\sigma:\,,\; \ldots$$

$$\{\Sigma_n\} = \sigma\,,\; :\sigma^2:\,,\; :\sigma^3:\,,\; \ldots$$

$$\{Q_{nr}\} = \;:\phi^2\sigma:\,,\; :\phi^2\sigma^2:\,,\; :\sigma^2\Box\sigma:\,,\; \ldots .$$

So the complete operator product expansion reads

(6) $$\phi(x)\,\phi(0) \underset{x \to 0}{\sim} \sum_{n=0}^{\infty} c_n(x^2)\, O_n + \sum_{n=1}^{\infty} b_n(x^2)\, \Sigma_n + \sum_{n,r=1}^{\infty} f_{nr}(x^2)\, Q_{nr}$$

Covariance under dilatation gives the following set of equations :

$$(x_\nu \partial^\nu + 2l)\, c_n(x^2) = l_n c_n(x^2) + f_{n1}(x^2)\, b^{-1}$$

(7) $$(x_\nu \partial^\nu + 2l)\, b_n(x^2) = (n+1)\, b^{-1}\, b_{n+1}(x^2)$$

$$(x_\nu \partial^\nu + 2l)\, f_{nr}(x^2) = l_n f_{nr}(x^2) + (n+1)\, b^{-1} f_{nr+1}(x^2)$$

The formal solution of the system is :

(8) $$\underset{x \to 0}{\phi(x)\,\phi(0)} \sim \sum_{h=0}^{\infty} :O_n\, e^{b\sigma\nabla n}: c_n(x^2)$$

where $e^{b\sigma\nabla n}$ is a formal writing for

(9) $$:O_n\, e^{b\sigma\nabla n}: c_n(x^2) = \sum_{k=0}^{\infty} \frac{b^k}{k!} :O_n\, \sigma^k : \nabla_n^k\, c_n(x^2)$$

and $\nabla n = (x_\nu \partial^\nu + 2l - l_n)$.

We observe, by inspection, that the second member of eq. (8) contains the three sets of operators appearing in eq. (6). In particular covariance under dilatation does not fix the functions $c_n(x^2)$ and we discuss now their possible behaviour for $x \to 0$. It is convenient to put :

(10) $$c_n(x^2) = c_n^s(x^2) + c_n^m(x^2)$$

when $c_n^s(x^2) = c_n^s\left(\frac{1}{x^2}\right)^{\frac{2l - l_n}{2}}$

So

(11) $$\underset{x \to 0}{\phi(x)\,\phi(0)} = \sum_{n=0}^{\infty} c_n^s(x^2)\, O_n + \sum_{n=0}^{\infty} :O_n\, e^{b\sigma\nabla n}: c_n^m(x^2)$$

where the first sum on the right-hand side is the skeleton contribution. The general form of $c_n^m(x^2)$ is assumed to be :

(12) $$c_n^m(x^2) \sim c_n^m\left(\frac{1}{x^2}\right)^{\alpha_n} \left(\lg \frac{x^2}{m^2}\right)^{\beta_n}$$

where possibly α_n, β_n depend on b. Firstly we discuss the case $\beta_n = 0$ $\alpha_n \neq \alpha_n^s = 2l - l_n$. We have :

(13) $$e^{b\sigma\nabla_n} c_n^m(x^2) = e^{b\sigma(\alpha_n^s - \alpha_n)} c_n^m(x^2).$$

This result suggests the occurence of a renormalization of the dimension due to the presence of an infinite sequence of σ-like fields in the expansion. Consider now the more interesting case $\alpha_n = \alpha_n^s$, $\beta_n \neq 0$. This is reminescent of standard perturbation theory. Using the recursion formula

(14) $$\nabla_n^k c_n^m(x^2) = 2^k \beta_n(\beta_n - 1) \ldots (\beta_n - k+1)\left(\frac{1}{x^2}\right)^{\alpha_n^s}\left(\log\frac{x^2}{m^2}\right)^{\beta_n - k} c_k^m$$

we obtain the solution

(15) $$e^{b\sigma\nabla_n} c_n^m(x^2) = c_n^m\left(\frac{1}{x^2}\right)^{\alpha_n^s}\left(\log\frac{x^2}{m^2} I + 2b\sigma\right)^{\beta_n}.$$

Regularity of the limit $b^{-1} \to 0$ implies $c_n^m = (b^{-1})^{\beta_n} c_n$ and for the leading contribution as $x \to 0$:

(16) $$e^{b\sigma\nabla_n} c_n^m(x^2) \sim c_n\left(\frac{1}{x^2}\right)^{\alpha_n^s}\left(b^{-1}\log\frac{x^2}{m^2}\right)^{\beta_n} I.$$

From expansion (12) if we call O_{n_o} the field of minimal dimension l_{n_o} we have :

(17) $$\underset{x \to 0}{\phi(x)\phi(0)} \sim \left(b^{-1}\log\frac{x^2}{m^2}\right)^{\beta_{n_o}}\left(\frac{1}{x^2}\right)^{\frac{2l - l_{n_o}}{2}} O_{n_o}(0)$$

(connected part)

We discusse in particular that the σ-fields completely disappear at the leading order so that they are just responsible of the logarithmic correction to the naïve singularity. For $b^{-1} \to 0$ this term vanishes and we recover the skeleton expansion. Finally, when β_n is integer it is evident from eq.(14) that only a finite number of

σ-fields enters in the expansion. Also the converse is true: if only a finite number of σ terms is present in the expansion the main singularity is the canonical one apart from a logarithmic correction whose power depends on the number of anomalous terms in the expansion. As an example of the above discussion consider the expansion given by eq. (6) truncated at the first order in b^{-1} :

$$(18) \qquad \phi(x)\phi(0) \underset{x\to 0}{\sim} \sum_{n=0}^{\infty} c_n(x^2) O_n + b(x^2)\sigma + \sum_{n=1}^{\infty} f_n(x^2) : O_n \sigma \; .$$

Using dilatation covariance we determine the coefficients :

$$c_n(x^2) = \left(\frac{1}{x^2}\right)^{\frac{2l - l_n}{2}} \left(c_{1_n} + c_{2_n} b^{-1} \log \frac{x^2}{m^2}\right)$$

$$(19) \qquad b(x^2) = c \left(\frac{1}{x^2}\right)^{l}$$

$$f_n(x^2) = c_{2_n} \left(\frac{1}{x^2}\right)^{\frac{2l - l_n}{2}}$$

Since the field of lowest dimension is $:\phi^2:$ we have

$$(20) \qquad \underset{x\to 0}{\phi(x)\phi(0)}_{\text{(connected part)}} \sim b^{-1} \log\left(\frac{x^2}{m^2}\right) : \phi^2(0) : \; .$$

We do not discuss the case α_m, β_n both non integers as it seems that this situation does not originate important differences from the previous ones.

We have discussed a general operator product expansion in the case of spontaneous breaking of conformal invariance. The main point is that such expansion is regular in the skeleton limit $b^{-1} \to 0$. From the above considerations we can conclude that the presence of anomalous σ-fields in the expansion is compatible both with a renormalization of the dimension and a logarithmic correction to the skeleton singularities.

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E R R A T A

Formula (3) has to be replaced with :

" (3) $$\mathcal{L}(x) = \mathcal{L}_s(x) + \mathcal{L}_m(x) + \mathcal{L}_k(x)$$

and $\mathcal{L}_k(x) = \partial_\mu \sigma(x) \partial^\mu \sigma(x)\, e^{2b\,\sigma(x)}$ is the kinetic term of the σ field. "

The expression "$\log \frac{x^2}{m^2}$" has to be replaced with "$\log m^2 x^2$" in formulae: (12), (14), (15), (16), (17), (19), and (20).