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S. Ferrara and G. Rossi: SCALING LAWS, LIGHT CONE BEHAVIOUR AND HARMONIC ANALYSIS ON SL(2, C). -

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1. - INTRODUCTION. -

Recently strong attention has been devoted to the study of light-cone expansion[1÷4] of products of local operators, mainly because of its relevance in high energy and high momentum transfer electroproduction processes. A better understanding of the $x^2 \simeq 0$ limit - whose counterpart in the physical momentum space is the so-called Bjorken limit[5]- could also provide informations on basic properties of the interacting fields involved, such as their effective canonical dimensions.

In a previous work[6] a geometrical interpretation of the "scalong laws" was proposed, based on a suitably definied expansion of the structure functions[5] over irreducible representations of the SL(2, C) group, which seems the most natural way to obtain asymptotic behaviours in the $\omega = -q^2/2\nu$ fixed limit. The assumption of "scale invariance" reads as a dominance of a well defined representation in this expansion and a q^2-power breaking of the simple scaling behaviour follows immediately in our scheme. This breaking is the most natural suggested also from field theoretical perturbative expansions[4].

A ω-fixed sum rule, analogous to the t-fixed finite energy sum

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rule in the Regge-pole theory, can be derived, which relates the low energy region (barionic resonances production) to the scaling law region: it allows to express the scaling functions $F_1(\omega)$ and $F_2(\omega)$ as an average over the low q^2 and ν region of the structure functions.

In this paper we first recall the main steps of our analysis and derive the above-mentioned sum rule (sec. II), then (sec. III) we show how our group-theoretical approach is connected to the light-cone expansion in the configuration space. The connection is only group-theoretical and in fact it is based on theorems on harmonic analysis in homogeneous spaces. Finally we outline the relation between the singularity structure of the matrix elements of field products, as implied by our analysis, and the operator expansion near the light-cone, suggested by some authors(4).

2. - GROUP-THEORETICAL INTERPRETATION OF THE BJORKEN LIMIT AND A SUM RULE. -

In this section we want to show how it is possible to recover the Bjorken limit of the structure functions, starting from the observation that the group SL(2, C) acts in a natural way on functions of two complex variables(7).

Let us consider a function $W(z_1, z_2)$, defined on the complex affine plane (z_1, z_2). This space is an homogeneous space with respect to the spinor Lorentz group SL(2, C) and infact it is equivalent to the quotient space SL(2, C)/Z, where Z is the group of matrices $\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}$. A representation of SL(2, C) is defined on these functions as follows:

$$T_g W(z_1, z_2) = W(\alpha z_1 + \beta z_2, \gamma z_1 + \delta z_2), \tag{2.1}$$

where

$$g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \qquad \text{with} \qquad \alpha\delta - \beta\gamma = 1$$

is an element of SL(2, C).

We observe that the homogeneous functions(8) of degree $(n_1 - 1, n_2 - 1)$ play a special role in this space, as they form an irreducible subspace for the representation (2.1). Then an irreducible representation of SL(2, C) is uniquely fixed by the pairs of complex numbers (n_1, n_2), whose difference is an integer(8) and we shall call D_{n_1, n_2} the corresponding invariant subspace.

Let us recall some basic properties of the functions belonging to D_{n_1, n_2}. From the homogeneity properties, if we put $w(z) = W(z, 1)$ and $\hat{w}(z) = W(1, z)$, we have:

$$W(z_1, z_2) = z_2^{n_1 - 1} \bar{z}_2^{n_2 - 1} \quad w(\xi) \tag{2.2a}$$

$$W(z_1, z_2) = z_1^{n_1 - 1} \bar{z}_2^{n_2 - 1} \quad w(\xi^{-1}) \tag{2.2b}$$

and

$$w(\xi) = \xi^{n_1 - 1} \bar{\xi}^{n_2 - 1} \quad w(\xi^{-1}), \tag{2.3}$$

where

$$\xi = \frac{z_1}{z_2} .$$

Furthermore the following asymptotic behaviours hold:

$$\lim_{\substack{|z_1|, |z_2| \to \infty \\ |\xi| \text{ fixed}}} W(z_1, z_2) = z_2^{n_1 - 1} \bar{z}_2^{n_2 - 1} \quad w(\xi) \tag{2.4a}$$

$$\lim_{\substack{|z_1|, |z_2| \to \infty \\ |\xi| \text{ fixed}}} W(z_1, z_2) = z_1^{n_1 - 1} \bar{z}_2^{n_2 - 1} \quad \hat{w}(\xi^{-1}) \tag{2.4b}$$

and

$$\lim_{|\xi^{-1}| \to \infty} W(z_1, z_2) = z_2^{n_1 - 1} \bar{z}_2^{n_2 - 1} \quad w(0) \tag{2.5a}$$

$$\lim_{|\xi| \to \infty} W(z_1, z_2) = z_1^{n_1 - 1} \bar{z}_1^{n_2 - 1} \quad \hat{w}(0), \tag{2.5b}$$

if respectively $W(0, 1)$ or $W(1, 0)$ are finite. Note that if we identify z_1 with q^2 and z_2 with ν, where q^2 and $\nu = (pq)$ (p is the four momentum of the initial nucleon) are the squared mass and the energy in the laboratory frame of the virtual photon, eqs. (2.4) refer to the Bjorken limit, while eq. (2.5a) and eq. (2.5b) to the Regge limit and to the old Bjorken limit respectively.

Let us now consider a function $W(z_1, z_2)$, belonging to the space where the representation (2.1)acts; if this function satisfies certain regularity conditions (at least if it is square summable respect to the SL(2,C) invariant measure on the homogeneous space), it can be expanded in terms of irreducible components, i.e. of functions which transform irreducibly under (2.1). The theory of harmonic analysis on homogeneous spaces gives in this case the following expansion formula:

(2.6) $$W(z_1, z_2) = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\varrho \, W(z_1, z_2; n, \varrho),$$

where $n_1 = \frac{1}{2}(n+i\varrho)$, $n_2 = \frac{1}{2}(-n+i\varrho)$ and $W(z_1, z_2; n, \varrho)$ is the Mellin transform of $W(z_1, z_2)$ defined by the equation:

(2.7) $$W(z_1, z_2; n, \varrho) = \frac{i}{2} \int d\zeta d\bar{\zeta} \, \zeta^{-n_1} \bar{\zeta}^{-n_2} W(\zeta z_1, \zeta z_2),$$

in which the integral is extended over the whole complex ζ-plane.

From (2.7) it follows immediately that

a) $W(z_1, z_2; n, \varrho)$ is homogeneous of degree (n_1-1, n_2-1);

b) the action of the representation (2.1) on $W(z_1, z_2)$ induces the action of the irreducible representation labelled by (n_1-1, n_2-1) on $W(z_1, z_2; n, \varrho)$;

c) the Plancherel theorem for $W(z_1, z_2)$ holds in the form:

(2.8) $$\frac{i}{2} \int \left| W(z_1, z_2) \right|^2 dz_1 \, d\bar{z}_1 \, dz_2 \, d\bar{z}_2 = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} d\varrho \left| W(z, 1; n, \varrho) \right|^2 dz \, d\bar{z}$$

We can use the eq. (2.7) to analitically continue the Mellin transform for any complex value of ϱ, so the expansion (2.6) can be valid also for functions which are not square summable in the form:

(2.9) $$W(z_1, z_2) = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \int_{C_\varrho} d\varrho \, W(z_1, z_2; n, \varrho) ,$$

where C_ϱ is a suitable path in the complex ϱ-plane. In particular if $W(z_1, z_2)$ is an homogeneous function of degree $(\tilde{n}_1 - 1, \tilde{n}_2 - 1)$, hence belonging to $D_{\tilde{n}_1, \tilde{n}_2}$, we have:

$$W(z_1, z_2; n, \varrho) = 2\pi \, \delta_{n, \tilde{n}} \int_{-\infty}^{+\infty} d\sigma \, e^{i\sigma(\tilde{\varrho} - \varrho)} ,$$

where $\tilde{n}_1 = \frac{1}{2}(\tilde{n} + i\tilde{\varrho})$ and $\tilde{n}_2 = \frac{1}{2}(-\tilde{n} + i\tilde{\varrho})$.

To relate the homogeneity properties of $W(z_1, z_2)$ to a pole type singularity structure it can be useful to introduce an integral transform of $W(z_1, z_2; n, \varrho)$, defined as

(2.10) $$\tilde{W}^{(\pm)}(z_1, z_2; n, \varrho) = \frac{1}{2\pi} \int_{C^{(\pm)}} d\varrho' \frac{W(z_1, z_2; n, \varrho')}{\varrho' - \varrho} ,$$

where in the complex ϱ'-plane $C^{(+)}$ $(C^{(-)})$ is a path lying above (below) the singularities of $W(z_1, z_2; n, \varrho')$ and passing through the point $\mathrm{Im}\,\varrho - i\varepsilon$ $(\mathrm{Im}\,\varrho + i\varepsilon)$. For an homogeneous function belonging to $D_{\tilde{n}_1, \tilde{n}_2}$ one has

(2.11) $$\tilde{W}^{(+)}(z_1, z_2; n, \varrho) = \tilde{W}^{(-)}(z_1, z_2; n, \varrho) = \delta_{n, \tilde{n}} \frac{1}{\tilde{\varrho} - \varrho} .$$

Although in simple cases both $\tilde{W}^{(+)}$ and $\tilde{W}^{(-)}$ exist and coincide, this is not true in general: it can well happen that $\tilde{W}^{(+)}$ or $\tilde{W}^{(-)}$ or even both do not exist, the integral representation (2.10) not being valid for any value of ϱ.

Inserting eq. (2.7) into eq. (2.10), if it is allowed to interchang the $\zeta, \bar{\zeta}$-integrations with the ϱ'-integration, one easily gets:

(2.12) $$\tilde{W}^{(\pm)}(z_1, z_2; n, \varrho) = \pm \frac{1}{2} \int d\zeta \, d\bar{\zeta} \, \zeta^{-1/2(n+i\varrho)} \bar{\zeta}^{-1/2(-n+i\varrho)} \times$$
$$\times W(\zeta z_1, \bar{\zeta} z_2) \, \theta(\pm(|\zeta| - 1)),$$

where $\theta(x)$ is the step function ($\theta(x)=1$ for $x>0$, $\theta(x)=0$ for $x<0$).

Let us now consider the functions[9, 10]

(2.13a) $$V_1(q^2,\nu)=\frac{1}{q^2}\left(W_1(q^2,\nu)+\frac{\nu^2}{q^2}W_2(q^2,\nu^2)\right)$$

(2.13b) $$V_2(q^2,\nu)=-\frac{1}{q^2}W_2(q^2,\nu),$$

linear combinations of the structure functions W_1 and W_2, defined by the equations

(2.14) $$W_{\mu\nu}(q^2,\nu)=\frac{1}{(2\pi)^4}\int d^4x\, e^{iqx}\langle p|\,J_\mu^{em}(x)\;J_\nu^{em}(0)\,|p\rangle$$

(2.15) $$W_{\mu\nu}(q^2,\nu)=-\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)W_1(q^2,\nu)+ \\ +\left(p_\mu-\frac{\nu}{q^2}q_\mu\right)\left(p_\nu-\frac{\nu}{q^2}q_\nu\right)W_2(q^2,\nu),$$

where $J_\mu^{em}(x)$ is the electromagnetic hadronic current [11]. For the Fourier transforms:

(2.16) $$W_k^F(x^2,px)=\int d^4q\, e^{-iqx}\,W_k(q^2,\nu), \qquad k=1,2$$

and

(2.17) $$V_k^F(x^2,px)=\int d^4q\, e^{-iqx}\,V_k(q^2,\nu), \qquad k=1,2$$

the following relations hold[12]

(2.18a) $$W_1^F(x^2,px)=-\Box V_1(x^2,px)-p_\mu p_\nu\,\partial_\mu\partial_\nu\,V_2(x^2,px)$$

(2.18b) $$W_2^F(x^2,px)=\Box V_2(x^2,px).$$

The $V_k(q^2, \nu)$ are free of kinematical singularities and are Fourier transforms of causal functions[13]. In the Bjorken limit $|q^2| \to \infty$, $\nu \to \infty$ with $\omega = -q^2/2\nu$ fixed (hereafter referred to as B-limit) the experimental data[14] seem to suggest the asymptotic behaviours

$$(2.19) \qquad \lim_{B} V_k(q^2, \nu) = \nu^{\alpha_k} F_k(\omega)$$

with $\alpha_1 = -1$ and $\alpha_2 = -2$.

From the previous analysis we can expand the functions V_k in SL(2, C) irreducible components:

$$(2.20) \qquad V_k(q^2, \nu) = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \int d\rho \, V_k(q^2, \nu; n, \rho);$$

so, assuming that in the Bjorken limit a discrete (finite or infinite) class of representations (all with $i\rho_k - 2 = \alpha_k$) dominates the integral (2.20), eq. (2.19) directly follows.

Under the hypothesis of the existence of $\tilde{V}^{(+)}$ (or $\tilde{V}^{(-)}$) in the form expressed by the equation (2.12), it is possible to derive a kind of continuum momentum sum rule, which allows to express the scaling function F_k as an integral over the "low energy" (low q^2 and ν) region of the structure functions, V_k. Let us start from

$$\lim_{B} \sum_{n=-\infty}^{+\infty} \int_{C(+)} \frac{d\rho'}{\rho' - \rho} \int d\zeta \, d\bar{\zeta} \, \zeta^{-1/2(n+i\rho')} \bar{\zeta}^{-1/2(-n+i\rho')} \times$$

$$(2.21) \qquad \times V_k(\zeta q^2, \zeta \nu) = \nu^{\alpha_k} F_k(\omega) \sum_{n=-\infty}^{+\infty} \int_{C(+)} \frac{d\rho'}{\rho' - \rho} \int d\zeta \, d\bar{\zeta} \times$$

$$\times \zeta^{\alpha_k - 1/2(n+i\rho')} \bar{\zeta}^{-1/2(-n+i\rho')} \theta(|\zeta \nu| - \nu_{th})$$

where the θ-function in the l. h. s. has been introduced to properly take into account the support properties of $F_k(\omega)$ and $\nu_{th} = m_\pi (2 + m_\pi)/2(1-\omega)$ (m_π is the pion mass); after a simple calculation one obtains ($\omega = -q^2/2\nu \neq 1, \omega > 0$)

$$(2.22) \qquad \int_0^\infty d\eta \, V_k(-2\eta\omega, \eta) \, \eta^{1+\beta} = -\frac{F_k(\omega)}{\alpha_k + \beta + 2} (\nu_{th})^{\alpha_k + \beta + 2},$$

where $\beta = \mathrm{Im}\,\varsigma$ can be any real number with the only limitation $\alpha_k+\beta+2<0$.

Note that eq. (2.22) could also have been obtained starting from $\tilde{V}_k^{(-)}$, so the existence of $\tilde{V}_k^{(+)}$ or $\tilde{V}_k^{(-)}$ is sufficient to derive this sum rule.

Equation (2.22) can be easily written as a finite energy sum rule, using the asymptotic behaviour (2.19). If $\bar{\nu}$ is a value of ν above which $V_k(q^2,\nu)$ can be well approximated by $\nu^{\alpha_k} F_k(\omega)$, one gets

$$(2.23)\qquad \int_{\nu_{th}}^{\bar{\nu}} d\eta\, V_k(-2\eta\omega,\eta)\,\eta^{1+\beta} = -\frac{F_k(\omega)}{\alpha_k+\beta+2}\left((\nu_{th})^{\alpha_k+\beta+2}-(\bar{\nu})^{\alpha_k+\beta+2}\right)$$

3. - HARMONIC ANALYSIS IN THE CONFIGURATION SPACE AND CONNECTION WITH THE LIGHT-CONE EXPANSIONS. -

In this section we want to discuss what our analysis in the momentum space implies for the Fourier transforms of the functions $V_k(q^2,\nu)$.

Recalling the definition (2.17).

$$(3.1)\qquad V_k^F(x^2,\mu) = \int d^4q\, e^{-iqx}\, V_k(q^2,\nu), \qquad \mu = px$$

and the expansion theorem (2.6), we have

$$(3.2)\qquad \begin{aligned} V_k(x^2,\mu) &= \frac{1}{(2\pi)^2}\int d^4q\, e^{-iqx}\sum_n \int_{C_\varsigma} d\varsigma\, V_k(q^2,\nu;\mu,\varsigma) = \\ &= \frac{i}{2}\frac{1}{(2\pi)^2}\int d^4q\, e^{-iqx}\sum_n \int_{C_\varsigma} d\varsigma \int d\zeta\, d\bar{\zeta}\, \zeta^{-n_1}\bar{\zeta}^{-n_2}\, V_k(\zeta q^2,\zeta\nu). \end{aligned}$$

The substitution $q \to \sqrt{\zeta}\,q$ in eq. (3.1) gives

$$(3.3)\qquad \int d^4q\, e^{-iqx}\, V_k(\zeta q^2,\zeta\nu) = \frac{1}{\zeta^2}\, V_k^F\!\left(\frac{x^2}{\zeta},\mu\right),$$

so interchanging the integration orders in eq. (3.2), one gets

(3.4) $$V_k^F(x^2,\mu) = \frac{1}{(2\pi)^2} \sum_n \int d\rho \, V_k^F(x^2,\mu;n,\rho),$$

where

(3.5) $$V_k^F(x^2,\mu;n,\rho) = \frac{i}{2} \int d\zeta \, d\bar{\zeta} \, \zeta^{n_1} \bar{\zeta}^{n_2-2} V_k^F(\zeta x^2,\mu).$$

If we are allowed to perform the sum over n, the expansion formula in the x-space reads

(3.6) $$V_k^F(x^2,\mu) = \frac{i}{2\pi} \int_{C_\lambda} d\lambda \, \hat{V}_k^F(x^2,\mu;\lambda)$$

where $\lambda = 1 - i\rho$ and

(3.7) $$\hat{V}_k^F(x^2,\mu;\lambda) = \int_0^\infty d\sigma \, \sigma^{-\lambda} V_k^F(\sigma x^2,\mu)$$

is an homogeneous function of x^2 of degree $\lambda - 1$. Using the notations introduced in the Appendix and the definition

(3.8) $$(\sigma_\pm^{-\lambda}, V)(\mu) = \pm \int_0^{\pm\infty} d\sigma \, |\sigma|^{-\lambda} V(\sigma,\mu),$$

the eq. (3.7) can be rewritten in the more expressive form

(3.9) $$\hat{V}_k^F(x^2,\mu;\lambda) = (x_+^2)^{\lambda-1}(\sigma_+^{-\lambda}, V_k)(\mu) + (x_-^2)^{\lambda-1}(\sigma_-^{-\lambda}, V_k)(\mu)$$

This representation holds for all the values of λ, in which $(x_\pm^2)^{\lambda-1}$ and $(\sigma_+^{-\lambda}, V_k)(\mu)$ are not singular. Eq. (3.9) shows that the harmonic analysis in the momentum space induces in the configuration space an expression respect to the variable x^2 in terms of representations of the unidimensional translation group.

Let us now discuss briefly the properties of the functions $(\sigma_\pm^{-\lambda}, V_k)(\mu)$. Since we are dealing with matrix elements of the product of two local operators (currents), we must have

$$(3.10)\qquad (\sigma_+^{-\lambda}, V_k)(\mu) = = (\theta(x_o)e^{i\pi(\lambda-1)} + \theta(-x_o)e^{-i\pi(\lambda-1)})(\sigma_-^{-\lambda}, V_k)(\mu)$$

which gives the correct behaviour

$$(3.11)\qquad \hat{V}_k^F(x^2,\mu;\lambda) = (-x^2+i\varepsilon x_o)^{\lambda-1}(\sigma_-^{-\lambda}, V_k)(\mu)$$

with $(\sigma_-^{-\lambda}, V_k)(\mu)$ real. The reality of $(\sigma_-^{-\lambda}, V_k)(\mu)$ follows from the requirement that the discontinuity of V_k^F is only different from zero for $x^2 > 0$; infact we get in this case :

$$(3.12)\qquad \text{disc}\, \hat{V}_k^F(x^2,\mu;\lambda) = = 2i\,\text{sen}\,\pi(\lambda-1)\varepsilon(x_o)(x_+^2)^{\lambda-1}(\sigma_-^{-\lambda}, V_k)(\mu).$$

Since the distribution $(x_+^2)^{\lambda-1}$ is a meromorphic function of λ with only poles, which is regular for $\lambda = n$, $n = 1, 2, \ldots$ (see Appendix), we have :

$$(3.13)\qquad \text{disc}\, \hat{V}_k^F(x^2,\mu;n) = 0 \qquad \text{for } n = 1,2,\ldots$$

because of the factor $\text{sen}\,\pi(\lambda-1)$. Again this equation holds only is also $(\sigma_-^{-\lambda}, V_k)(\mu)$ is regular. Actually we may expect $(\sigma_-^{-\lambda}, V_k)(\mu)$ to have "fixed poles" in λ which originate from the derivative type relations between $W_k^F(x^2,\mu)$ and $V_k^F(x^2,\mu)$. Infact, because of the presence of the d'Alambertian operator in the equations (2.18), to the $(x^2)^{\lambda-1}$ power in $\hat{V}_k^F(x^2,\mu;\lambda)$ will correspond in $\hat{W}_k^F(x^2,\mu;\lambda)$(15) a term with $(x^2)^{\lambda-2}$, so that a factor $1/(\lambda-1)$ is embodied in the definition of $(\sigma_-^{-\lambda}, V_k)(\mu)$(16). This generates a pole type singularity in $(\sigma_-^{-\lambda}, V_k)(\mu)$ unless a zero is present in the $\hat{W}_k^F(x^2,\mu;\lambda)$ partial wave. As we shall see, the existence of a simple pole at $\lambda = 1$ in $(\sigma_-^{-\lambda}, V_k)(\mu)$ is necessary in order not to get zero for $\nu^2 V_k(q^2,\nu)$ in the Bjorken limit.

To complete our analysis we have to relate the homogeneous SL(2,C) components of $V_k(q^2,\nu)$ with the corresponding "partial waves" of its Fourier transform, as given by the eq. (3.6). This can be easily done generalizing a procedure due to R. A. Brandt(17) to cover the cases in which λ is not an integer. We start from the equation

$$(3.14)\qquad \hat{V}_k(q^2,\nu;\lambda) = \frac{1}{(2\pi)^4}\int d^4x\, e^{iqx}\,\hat{V}_k^F(x^2,\mu;\lambda),$$

where

$$\hat{V}_k(q^2, \nu ; \mu) = \frac{1}{2\pi^4} \frac{i}{2} \sum_{n=-\infty}^{+\infty} \int d\zeta \, d\bar{\zeta} \; \zeta^{-n_1} \bar{\zeta}^{-n_2} \quad \times$$

(3.15)

$$\times \, V_k(\zeta q^2, \zeta \nu) = \int_0^{\infty} dt \; t^{\lambda} \; V_k(tq^2, t\nu)$$

is an homogeneous function of degree $-\lambda - 1$, which verifies :

(3.16) $$\hat{V}_k(q^2, \nu ; \lambda) = \nu^{-(\lambda+1)} \hat{V}_k(\frac{q^2}{\nu}, 1; \lambda) \equiv \nu^{-(\lambda+1)} F_k(\omega;\lambda).$$

Using this property we can evaluate the integral (3.15) in the Bjorken limit and we have :

$$F_k(\omega;\lambda) = \frac{\nu^{\lambda+1}}{(2\pi)^4} \int d^4x \, e^{iqx} \hat{V}_k^F(x^2, \mu ; \lambda) =$$

(3.17)

$$= \lim_B \frac{\nu^{\lambda+1}}{(2\pi)^4} \int d^4x \, e^{iqx} \hat{V}_k^F(x^2, \mu ; \lambda) .$$

Note that in the electroproduction case the Fourier transform of

(3.18) $$\left[\hat{V}_k^F(x^2, \mu ; \lambda) \right]^{*} = V_k^F(x^2, \mu ; \lambda) - \mathrm{disc} \; \hat{V}_k^F(x^2, \mu ; \lambda)$$

must be zero, since it would get contributions from intermediate states of energy (in the laboratory frame) $1 - q_o$ (10), where q_o is necessarily positive, being the energy loss of the scattered electron, so we can write

$$F_k(\omega ; \lambda) = \frac{2\, i \, \mathrm{sen}\, \pi (\lambda - 1)}{(2\pi)^4} \lim_B \nu^{\lambda+1} \quad \times$$

(3.19)

$$\times \int d^4x \, e^{iqx} \, \varepsilon(x_o) \, (x_+^2)^{\lambda - 1} (\sigma_-^{-\lambda}, V_k)(\mu) .$$

With the position

$$x_o = \varphi + \tau , \qquad x_3 = \varphi - \tau ,$$

if we fix the four momentum q_μ to be

$$q_\mu = (\nu, 0, 0, -(\nu^2 - q^2)^{1/2}),$$

the scalar product qx becomes in the Bjorken limit :

$$qx \xrightarrow[B]{} 2\varphi\nu + \omega(\varphi - \tau),$$

so in the laboratory frame, we have

$$F_k(\omega;\lambda) = \frac{2i\,\mathrm{sen}\,\pi(\lambda-1)}{(2\pi)^4} \lim_{B} \nu^{\lambda+1} \int d\varphi\, d\tau\, d^2\vec{x} \quad \times$$

$$\times \quad e^{i\omega(\varphi-\tau)} e^{2i\varphi\nu} \,\varepsilon(\varphi+\tau)(4\varphi\tau-|\vec{x}|^2)^{\lambda-1} \quad \times$$

$$\times \quad \theta(4\varphi\tau-|\vec{x}|^2)(\sigma_-^{-\lambda}, V_k)(\varphi+\tau).$$

Observing with R.A. Brandt[16] that in the Bjorken limit only the points with $\varphi \sim 0$ will contribute to the integral, because of the rapidly oscillating factor $\exp(2i\varphi\nu)$ and using for the Fourier transforms the formulas given in the Appendix, we obtain :

$$(3.20) \qquad F_k(\omega,\lambda) = \frac{\pi^2}{(2\pi)^4} \frac{2^\lambda e^{i\frac{\pi}{2}\lambda}}{\Gamma(-\lambda+1)} \int d\tau\, e^{i\omega\tau} \tau^\lambda (\sigma_-^{-\lambda}, V_k)(\tau)$$

Note that in the case $k=2$ for the dominant wave $\lambda = 1$ (remember that to $\alpha = u$ corresponds $\lambda = -u-1$) the eq. (3.20) would give :

$$F_2(\omega, 1) = 0,$$

unless $(\sigma_-^{-\lambda}, V_2)(\mu)$ has a pole at $\lambda = 1$; so we are forced to assume :

$$(\sigma_-^{-\lambda}, V_2)(\mu) \underset{\lambda\simeq 1}{\simeq} \frac{\mathrm{Res}_{\lambda=1}\,(\sigma_-^{-\lambda}, V_2)(\mu)}{\lambda-1}.$$

Under this hypothesis, using for $s=2$ the Taylor expansion of $(-x^2+i\varepsilon x_o)^{\lambda+s}$ near $\lambda = -2$ (see Appendix) :

$$(-x^2+i\varepsilon x_o)^{-2+s} = (-x^2+i\varepsilon x_o)^{-2+s} + (\lambda-2) \times$$

$$\times \quad (-x^2+i\varepsilon x_o)^{-2+s} \log(-x^2+i\varepsilon x_o) + \cdots,$$

we get

$$\mathrm{disc}\, \hat{V}_k^F(x^2,\mu;1) = 2\pi i\, \varepsilon(x_o)\, \theta(x^2) \,\mathrm{Res}_{\lambda=1}\,(\sigma_-^{-\lambda}, V_2)(\mu)$$

and hence

$$(3.21b) \qquad F_2(\omega;1) = -\frac{2\pi^2 i}{(2\pi)^4} \int d\tau\, e^{-i\omega\tau} \tau \,\mathrm{Res}_{\lambda=1}\,(\sigma_-^{-\lambda}, V_2)(\tau).$$

In the case $k=1$ the dominant wave in the Bjorken limit correspond to $\lambda = 0$ and we have

$$(3.21a) \qquad F_1(\omega, 0) = \frac{\pi^2}{(2\pi)^4} \int d\tau \, e^{-i\omega\tau} \, (\sigma^0_-, V_1)(\tau) .$$

We conclude illustrating the relation between the general analysis of the structure functions in the x-space, carried out in this section, and the light-cone expansions of products of local operators, suggeted by renormalizable perturbation theories[1, 4]. Let us refer to eqs. (3.6) and (3.11) and compare them with the expressions obtained for the corresponding matrix elements in terms of complete sets of local tensor operators, each with a well-defined relation between spin and dimensionality[4]. The two expansions look very similar and the difference is in the physical meaning attached to the exponents of the x^2-powers. For instance in the Bjorken limit, in our language, the leading x^2-singularity near the light-cone is fixed by the indices which label the dominant SL(2, C) representation in the integral (2.20), while in the Brandt-Preparata[4] approach the leading x^2-singularity is brought by the string of tensor operators whose scalar member has the smallest allowed dimensionality in the expansion.

For instance neglecting all spin complications, we have, by definition for two local scalar operators $A(x)$, $B(0)$:

$$\langle p | A(x) B(0) | p \rangle \equiv V^F(x^2, \mu) = \int d^4 q \, e^{-iqx} \, V(q^2, \nu).$$

Comparing the light-cone expansion for $x^2 \simeq 0$ with eq. (3.6), we obtain :

$$\frac{i}{2\pi} \int_{C_\lambda} d\lambda \, (-x^2 + i\varepsilon x_0)^{\lambda - 1} \, (\sigma_-^{-\lambda}, V)(\mu) \xrightarrow[x^2 \simeq 0]{}$$

$$\sum_i G^i(-x^2 + i\varepsilon x_0) \sum_n C^i_n \langle p | O^{(i)}_{\mu_1 \ldots \mu_n}(0) | p \rangle x^{\mu_1} \ldots x^{\mu_n} ,$$

where the C^i_n are adimensional constants and $G^i(x^2)$ in an homogeneous function of x^2 of degree $(d^A + d^B - d^i_0)/2$, d^A, d^B and d^i_0 being respectively the dimensions of the operators $A(x)$, $B(0)$ and $O^i(0)$; for $O^{(i)}_{\mu_1 \ldots \mu_n}(0)$ we have $d^i_n = d^i_0 + n$. The value of λ which labels the dominant SL(2, C) representation in the Bjorken limit is then given by

$$\lambda = \frac{d^A + d^B - d_0}{2} - 1 ,$$

where $d_0 = \min_i (d^i_0)$.

APPENDIX. -

In this Appendix[18] we summarize some useful properties and formulas concerning the generalized functions which are relevant in the light-cone analysis. It is of some interest to give these properties for the general case of a pseudoeuclidean space O(p, q) of dimension $n = p+q$, because the singularities of these functions strongly depend on p and q.

Let us start with the analogous of the Dirac δ-function. If we put :

$$x^2 = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2 ,$$

it is possible to define two different sets of distributions, $\delta_1^{(k)}(x^2)$ and $\delta_2^{(k)}(x^2)$ (where (k) means the K-th derivative), related by the property

$$\text{(A. 1)} \qquad \delta_2^{(k)}(x^2) = (-1)^k \, \delta_1^{(k)}(-x^2) .$$

If n is odd as well as if n is even and $k < \frac{1}{2}n - 1$ $\delta_1^{(k)}(x^2) = \delta_2^{(k)}(x^2)$, while if n is even and $k \geq \frac{1}{2}n - 1$ the difference is generalized function concentrated on the vertex of the $x^2 = 0$ surface, which can be written[12]:

$$\text{(A. 2)} \qquad \delta_1^{(k)}(x^2) - \delta_2^{(k)}(x^2) = c_{pqk} \, \Box^{k - \frac{1}{2}n + 1} \, \delta(x) ,$$

where the coeffcients c_{pqk} are numerical constants we are not interested to specify further and $\delta(x)$ is a shorthand notation for $\delta(x_1)\,\delta(x_2)\,\dots\,\delta(x_{p+q})$.

Introducing the generalized functions $(x_+^2)^\lambda$ and $(x_-^2)^\lambda$, defined as :

$$\text{(A. 3)} \qquad (x_+^2)^\lambda = \begin{cases} (x^2)^\lambda & x^2 > 0 \\ 0 & x^2 < 0 \end{cases} \qquad \text{and} \qquad (x_-^2)^\lambda = \begin{cases} 0 & x^2 > 0 \\ |x^2|^\lambda & x^2 < 0 \end{cases}$$

it can be shown that they are meromorphic functions of λ with the two possible set of poles

a) $\lambda = -1, -2, \dots, -k, \dots$ where k is a positive integer

b) $\lambda = -\frac{n}{2}, -\frac{n}{2} - 1, \dots, -\frac{n}{2} - k, \dots$ where k is a nonnegative integer.

Referring, for instance, to $(x_+^2)^\lambda$ we have three possible cases:

I - The singularity $\lambda = -k$ is in the set a) but not in b). This is always the case of n is odd, while if n is even this happens only if $\lambda > -n/2$. In this case we have:

(A. 4) $$\operatorname*{Res}_{\lambda=-k} (x_+^2)^\lambda = \frac{(-1)^{k-1}}{(k-1)!} \delta_1^{(k-1)}(x^2).$$

II - The singular point λ belongs to the set b) and not to the set a). This happens for $\lambda = -{}^n/_2 - k$ and n odd. For p odd and q even the residuum of $(x_+^2)^\lambda$ at $\lambda = -{}^n/_2 - k$ is given by:

(A. 5) $$\operatorname*{Res}_{\lambda=-\frac{n}{2}-k} (x_+^2)^\lambda = \frac{(-1)^{q/2}\pi^{n/2}}{4^k k!\,\Gamma(\frac{n}{2}+k)} \Box^k \delta(x),$$

while for p even and q odd $(x_+^2)^\lambda$ is regular.

III - The singularity is both in a) and in b). This happens for n even and $\lambda = -{}^n/2 - k$. For p and q both even we have:

(A. 6) $$\operatorname*{Res}_{\lambda=-\frac{n}{2}-k} (x_+^2)^\lambda = \frac{(-1)^{\frac{n}{2}+k-1}}{\Gamma(\frac{n}{2}+k)} \delta_1^{(\frac{n}{2}+k-1)}(x^2) + \frac{(-1)^{q/2}\pi^{n/2}}{4^k k!\,\Gamma(\frac{n}{2}+k)} \Box^k \delta(x).$$

For p and q odd $(x_+^2)^\lambda$ has double poles at $\lambda = -{}^n/_2 - k$ and its Laurent expansion can be written:

(A. 7) $$(x_+^2)^\lambda \underset{\lambda \simeq -\frac{n}{2}-k}{\simeq} \frac{C_{-2}^{(k)}(x)}{(\lambda+\frac{n}{2}+k)^2} + \frac{C_{-1}^{(k)}(x)}{\lambda+\frac{n}{2}+k} + \text{regular terms}$$

where

(A. 8) $$C_{-2}^{(k)}(x) = \frac{(-1)^{\frac{1}{2}(q+1)}\pi^{\frac{n}{2}-1}}{4^k k!\,\Gamma(\frac{n}{2}+k)} \Box^k \delta(x)$$

(A. 9) $$C_{-1}^{(k)}(x) = \frac{(-1)^{\frac{n}{2}+k-1}}{\Gamma(\frac{n}{2}+k)} \delta_1^{(\frac{n}{2}+k-1)}(x^2) +$$

$$+ \frac{(-1)^{\frac{1}{2}(q+1)} \pi^{\frac{n}{2}-1} \left[\psi(\frac{p}{2}) - \psi(\frac{n}{2}) \right]}{4^k k! \Gamma(\frac{n}{2}+k)} \Box^k \delta(x) \; ; \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \; .$$

All we said for $(x_+^2)^\lambda$ is valid also for $(x_-^2)^\lambda$ provided p and q are interchanged and we replace $\delta_1^{(k)}(x^2)$ by $\delta_1^{(k)}(-x^2) =$ $= (-1)^k \delta_2^{(k)}(x^2)$.

Since the generalized functions which are of physical interest are boundary values on the real axis of generalized functions of complex arguments, let us study the properties of the functions $(x^2 + i\varepsilon)^\lambda$ and $(x^2 - i\varepsilon)^\lambda$. They are related to $(x_+^2)^\lambda$ and $(x_-^2)^\lambda$ by the equations :

(A. 10) $$(x^2 + i\varepsilon)^\lambda = (x_+^2)^\lambda + e^{i\pi\lambda} (x_-^2)^\lambda$$

(A. 11) $$(x^2 - i\varepsilon)^\lambda = (x_-^2)^\lambda + e^{-i\pi\lambda} (x_-^2)^\lambda$$

Note that for non negative integral values of λ $(x^2 + i\varepsilon)^\lambda$, $(x^2 - i\varepsilon)^\lambda$ and $(x^2)^\lambda$ coincide. For $\lambda \to -k$ (k is a positive integer) we have :

(A. 12) $$\lim_{\lambda \to -k} \left[(x^2 + i\varepsilon)^\lambda - (x^2 - i\varepsilon)^\lambda \right] = 2\pi i (-1)^k \operatorname{Res}_{\lambda = -k} (x_-^2)^\lambda \; .$$

Let us specialize this equation in the physical case p = 1 and q = 3. For $\lambda = -1$ we obtain :

(A. 13) $$\lim_{\lambda \to -1} ((x^2 + i\varepsilon)^\lambda - (x^2 - i\varepsilon)^\lambda) = -2\pi i \delta(x^2) .$$

For $\lambda = 0$ using the Taylor expansion of $(x^2 + i\varepsilon)^{\lambda + k}$ near $\lambda = -\frac{n}{2}$:

(A. 14) $$(x^2 + i\varepsilon)^{\lambda + k} \underset{\lambda \simeq \frac{n}{2}}{\simeq} (x^2 + i\varepsilon)^{-\frac{n}{2} + k} + \\ + (\lambda + \frac{n}{2}) (x^2 + i\varepsilon)^{-\frac{n}{2} + k} \log (x^2 + i\varepsilon)$$

in the case $k = \frac{n}{2}$, we deduce :

(A. 15) $$\log (x^2 + i\varepsilon) - \log (x^2 - i\varepsilon) = 2\pi i \,\theta(-x^2) .$$

The corresponding formulas for $(-x^2+i\varepsilon)^\lambda$ and $(-x^2-i\varepsilon)^\lambda$ can be obtained by simply interchanging x_+^2 with x_-^2.

We conclude giving the Fourier transforms of $(x^2+i\varepsilon)^\lambda$ and $(x^2-i\varepsilon)^\lambda$:

$$\int d^n x\, e^{i\sigma x}(x^2 \pm i\varepsilon)^\lambda =$$

(A. 16)
$$= \frac{e^{\mp i\frac{\pi}{2}q}\, 2^{2\lambda+n}\, \pi^{\frac{n}{2}}\, \Gamma(\frac{n}{2}+\lambda)}{\Gamma(-\lambda)}\, (\sigma^2 \mp i\varepsilon)^{-\lambda-\frac{1}{2}n} .$$

In the unidimensional case we have :

(A. 17) $$\int dx\, e^{i\sigma x}(x \pm i\varepsilon)^\lambda = \frac{2\pi\, e^{\pm i\frac{\pi}{2}\lambda}}{\Gamma(-\lambda)}\, (\sigma_{\mp})^{-\lambda-1}$$

and $(\lambda \neq -1, -2, \ldots)$

(A. 18) $$\int dx\, e^{i\sigma x}(x_\pm)^\lambda = \pm i\, e^{\pm i\frac{\pi}{2}\lambda}\, \Gamma(\lambda+1)(\sigma \pm i\varepsilon)^{-\lambda-1} .$$

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(8) - A function $f(z_1, z_2)$ is called homogeneous of degree (λ, μ) where λ and μ are complex numbers differing by an integer, if for every complex number $\zeta \neq 0$, we have:

$$f(\zeta z_1, \zeta z_2) = \zeta^{\lambda} \bar{\zeta}^{\mu} f(z_1, z_2).$$

We require $\lambda - \mu$ to be an integer since only in this case

$$\zeta^{\lambda} \bar{\zeta}^{\mu} = |\zeta|^{\lambda+\mu} \exp\left[i(\lambda-\mu) \arg \zeta \right]$$

will be a single valued function of ζ.
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$$\hat{W}_k^F(x^2, \mu; \lambda) = \int_0^{\infty} d\sigma \, \sigma^{-\lambda} \, W_k^F(\sigma x^2, \mu)$$

(16) - The fixed pole in the partial wave $V_k^F(x^2, \mu; \lambda)$ is due to the vector character of the electromagnetic current. This is analogous to what happens in Regge-pole theory.
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