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BEAM UNDERGOING ACCELERATION IN THE SYNCHROTRON.

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A. Turrin: RESONANT DEPOLARIZATION OF THE ELECTRON BEAM
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ABSTRACT. -

The possibility of accelerating vertically polarized electron beams to high energies in the Frascati Synchrotron is pointed out.

I. - The first-order spin motion equation. -

Particles trapped in the field of a circular accelerator experience - in addition to the main vertical field B_z - small oscillating horizontal magnetic fields because of their betatron oscillations and of magnetic field inhomogeneities, and therefore depolarization effects may occur in an initially polarized beam when resonance conditions between the oscillating perturbing magnetic fields and the spin precession are satisfied. ⁽³⁾⁻⁽¹⁷⁾

Consider a frame of reference (rotating rest frame) which is attached to the particle and which has one of the coordinate axes pointing in the direction of motion. From the fundamental equation for the

2.

spin motion of a particle of spin 1/2 given by Mendlowitz and Case⁽¹⁾ and by Bargman, Michel and Telegdi⁽²⁾ it follows that in this rotating rest frame the electronic spin motion is described in first-order approximation by the equation^{(15)†(17)}

$$(1) \quad \dot{\bar{S}} = \omega_p \bar{S} \times \bar{k} + \omega_{\perp} \frac{B_r}{B_0} \bar{S} \times \bar{n} + \omega_{\parallel} \frac{B_{\theta}}{B_0} \bar{S} \times \bar{u} + \frac{\omega_p}{\omega_c} \frac{z}{R\Lambda} \bar{S} \times \bar{u}$$

where \bar{S} is the polarization vector of the particle; the dot denotes differentiation with respect to the (lab.) time; \bar{u} , \bar{n} and \bar{k} are the space unit vectors (\bar{u} is the space unit vector pointing in the direction of motion); B_0 is the magnetic field at the equilibrium orbit; B_r and B_{θ} denote the oscillating horizontal magnetic fields felt by the particle; z is the vertical displacement of the particle from the equilibrium orbit; $R\Lambda$ is the mean radius (R is the equilibrium orbit radius in a quadrant); ω_c is the angular velocity of revolution;

$$(2) \quad \left| \begin{array}{l} \omega_p = \omega_c \gamma G \text{ is the angular velocity of the spin precession about} \\ \text{the direction of the guiding field } B_z \text{ (} \gamma \text{ is the ratio} \\ \text{of total energy of particle to its rest energy;} \\ G = g/2 - 1 \approx 1.16 \times 10^{-3} \text{ for electrons;} \\ g \approx 2 \text{ is the gyromagnetic ratio of the electron);} \end{array} \right.$$

$$(3) \quad \left| \begin{array}{l} \omega_{\perp} = (1 + \gamma G) \omega_c \\ \omega_{\parallel} = \frac{g}{2} \omega_c \end{array} \right. .$$

The absolute value of \bar{S} is a constant of motion, say

$$(4) \quad |\bar{S}| = 1.$$

II. - The horizontal perturbing magnetic fields and the resonance condition. -

In equation (1) the first term corresponds to the unperturbed spin motion in a steady field and all the remaining terms are small periodic perturbations.

The horizontal components B_r and B_θ are given in linear approximation by

$$(5) \quad \left| \begin{aligned} B_r &= \left(\frac{\partial B_z}{\partial x} \right)_{\substack{x=0 \\ z=0}} \cdot x \\ B_\theta &= \frac{1}{R} \left(\frac{\partial B_z}{\partial \theta} \right)_{\substack{x=0 \\ z=0}} \cdot \theta \end{aligned} \right. .$$

The Frascati Synchrotron being composed of four quadrants separated by "field-free" drift sections, either $\left(\frac{\partial B_z}{\partial x} \right)_{\substack{x=0 \\ z=0}}$ or $\frac{1}{R} \left(\frac{\partial B_z}{\partial \theta} \right)_{\substack{x=0 \\ z=0}}$ may be expressed in the form

$$(6) \quad \sum_{k=0}^{\infty} \beta_k \cos(4k\omega_c t + \varphi_k).$$

The vertical excursion z of the particle from its equilibrium orbit may be written

$$(7) \quad z \cong z_0 \cos(Q_z \omega_c t + \psi)$$

where z_0 is the amplitude of oscillation and Q_z is the number of vertical betatron oscillations/revolution.

Substituting (6) and (7) in (5) one finds that the horizontal perturbing fields B_r and B_θ seen by the electrons have frequencies given by (3) & (17)

$$(8) \quad \Omega = (4k \pm Q_z) \omega_c .$$

Resonance occurs when $\omega_p = \Omega$; then the resonances (encountered at specific energies) are

$$(9) \quad \gamma G = 4k \pm Q_z \quad (k = 0; 1; 2; \dots).$$

4.

III. - The equation of motion of the polarization vector in the magnetic field of the Frascati Synchrotron. -

For the 1 GeV Frascati Synchrotron, which has $\gamma G < 2.25$ and $Q_z \approx .9$, the only resonance of possible concern can be the following:

$$(10) \quad \gamma G = Q_z$$

(the corresponding energy being $E_{\text{res}} \approx 400$ MeV). Therefore it will suffice to restrict the further discussion to the electronic spin motion equation

$$(1') \quad \dot{\vec{S}} = \omega_p \vec{S} \times \vec{k} - \omega_1 n_0 \frac{Z_0}{R} \cos(Q_z \omega_c t + \psi) \vec{S} \times \vec{n} + \\ - \omega_p \frac{Q_z}{\Lambda} \frac{Z_0}{R} \sin(Q_z \omega_c t + \psi) \vec{S} \times \vec{u},$$

where n_0 is the field index.

Assume that during the acceleration process the spin precession frequency $\omega_p = \gamma G \omega_c$ about the main field becomes more and more close to the frequency $\Omega = Q_z \omega_c$ of the two small oscillating horizontal perturbing vectors of eq. (1')

$$\text{and} \quad \omega_1 n_0 \frac{Z_0}{R} \cos(Q_z \omega_c t + \psi) \vec{n} \\ \omega_p \frac{Q_z}{\Lambda} \frac{Z_0}{R} \sin(Q_z \omega_c t + \psi) \vec{u}.$$

In these resonant conditions it is possible to replace each oscillating vector in eq. (1') by two vectors, constant and equals in magnitude, rotating in opposite directions in the horizontal plane with angular velocity $\Omega = Q_z \omega_c$, in such a way that the sum of these two rotating vectors equals the given oscillating vector. Only the component vector rotating in the same direction as the precession of the electronic spin magnetic moment is effective in perturbing the spin motion; the other component is able to produce only a small rapid nutation of the spin axis. It is convenient therefore to assume that only this rotating component exists (for every oscillating vector in eq. (1')), the other component being unable to cause any resonant effect.

In a reference frame⁽³⁾, which rotates about the main field direction with angular velocity $\Omega = Q_z \omega_c$, the two small horizontal perturbing terms are vectors having constant direction \bar{j} and magnitude.

$$\frac{1}{2} \omega_{\perp} n_0 \frac{z_0}{R} \quad \text{and} \quad \frac{1}{2} \omega_{\perp} \frac{Q_z}{\Lambda} \frac{z_0}{R} .$$

Equation (1'), when transformed to this new rotating frame becomes

$$(11) \quad \dot{\bar{S}} = (\omega_{\perp} - \Omega) \bar{S} \times \bar{k} + \omega_{\perp} \bar{S} \times \bar{j}$$

where $\Omega = Q_z \omega_c$,

\bar{j} is a space unit vector

$$\text{and } \omega = \frac{1}{2} \frac{z_0}{R} \omega_c \left[n_0 (1 + \gamma G) + \frac{Q_z}{\Lambda} \gamma G \right] .$$

IV. - Solution of the classical equation of motion. -

Equation (11) may be expressed in terms of the three orthogonal components of the polarization vector \bar{S} :

$$(12) \quad \begin{cases} \dot{S}_x = \dot{\chi} S_y - \omega S_z \\ \dot{S}_y = -\dot{\chi} S_x \\ \dot{S}_z = \omega S_x \end{cases}$$

where $\dot{\chi} = \omega_{\perp} - \Omega$.

The system of equations (12) must be solved with the initial conditions

6.

$$(12') \quad \rho_x^d(-\infty) = 0 ; \rho_y^d(-\infty) = 0 ; \rho_z^d(-\infty) = 1.$$

One is primarily interested in the variation in time of the vertical component S_z of the polarization vector. The asymptotic solution of system (12) gives the expression of $S_z(+\infty)$ after the resonance is passed through.

By making the common simplifying assumption that $\dot{\chi}$ varies linearly with time, system (12) has been solved by several Authors: by Lobkovicz and Thorndike⁽⁷⁾ in the approximation in which only little changes in the vertical component of polarization occur at a resonance passing through; by Derbenev, Kondratenko and Skrinsky⁽¹⁶⁾ who applied perturbation methods; by Simonyan⁽¹³⁾ and Besnier⁽¹⁷⁾ numerically.

The corresponding time-dependent Schrödinger equation has been solved early by Froissart and Stora⁽³⁾ at Saclay, (and later by Ernst⁽¹⁵⁾ in connection with depolarization in AG Synchrotrons), even with the assumption that $\dot{\chi}$ is a linear function of time.

Remembering that the quantum-mechanical expectation value of any quantity follows in its time dependence exactly the classical equations of motion, it turned out to be convenient here to solve system (12) by merely transforming it to a form coincident with that of the quantum-mechanical spin motion equations already solved by Froissart and Stora. For the sake of convenience the Froissart and Stora procedure is summarized very briefly in the Appendix.

To perform the above-mentioned transformation, introduce in system (12) the two complex functions \mathcal{G} and \mathcal{E} defined by the relationships

$$(13) \quad \begin{cases} \rho_x^d + i \rho_y^d = \mathcal{G}^* (1 - \rho_z^d) \\ \rho_x^d - i \rho_y^d = \mathcal{E}^{-1} (\rho_z^d - 1) \end{cases}.$$

The star denotes complex conjugation.

It follows for S_z the expression

$$(14) \quad \rho_z^d = \frac{\mathcal{G}^* + \mathcal{E}}{\mathcal{G}^* - \mathcal{E}},$$

where \mathcal{G} and \mathcal{E} are solution of the two Riccati's equations

$$(15) \quad \begin{cases} \dot{\sigma} = \frac{\omega}{2} \sigma^2 + i\chi \dot{\sigma} + \frac{\omega}{2} \\ \dot{\varepsilon} = \frac{\omega}{2} \varepsilon^2 - i\chi \dot{\varepsilon} + \frac{\omega}{2} \end{cases}$$

with

$$(16) \quad \sigma \varepsilon = -1.$$

On introduction of two new functions f and g by the substitutions

$$(17) \quad \begin{cases} \sigma = -\frac{2}{\omega} \frac{\dot{g}}{g} \\ \varepsilon = -\frac{2}{\omega} \frac{\dot{f}}{f} \end{cases}$$

one obtains the second-order equations for f and g

$$(18') \quad \ddot{g} - i\chi \dot{g} + \left(\frac{\omega}{2}\right)^2 g = 0 \quad (\text{see eq. A.4})$$

$$(18) \quad \ddot{f} + i\chi \dot{f} + \left(\frac{\omega}{2}\right)^2 f = 0.$$

Consequently the relationship

$$(19) \quad \frac{\dot{f}\dot{g}}{fg} = -\left(\frac{\omega}{2}\right)^2$$

holds, and S_z assumes the expression

$$(20) \quad S_z = \frac{\left(\frac{2}{\omega}\right)^2 |\dot{g}|^2 - |g|^2}{\left(\frac{2}{\omega}\right)^2 |\dot{g}|^2 + |g|^2}.$$

With the normalization condition

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$$(21) \quad \left(\frac{2}{\omega}\right)^2 |\dot{g}|^2 + |g|^2 = 1$$

S_z takes the form

$$(22) \quad S_z = 1 - 2 |g|^2, \quad (\text{see eq. A.3}),$$

so that the initial conditions for g are

$$(23) \quad \begin{cases} |g(-\infty)| = 0 & (\text{see eq. A.2}) \\ |\dot{g}(-\infty)| = \frac{\omega}{2} & (\text{see eq. A.2'}). \end{cases}$$

Equations (22), (18') and (23) are just the equations encountered by Froissart and Stora as consequence of their quantum-mechanical formulation of the problem (see equations (A.3), (A.4) and (A.2) and (A.2') of the Appendix).

As expected, both the classical and the quantum-mechanical treatments lead to the same expression for S_z .

Moreover, if the two equations (18') and (18'') are multiplied by $\dot{f}/(gf)$ and $\dot{g}/(gf)$ respectively, taking into account that relationship (19) holds one obtains the equations

$$(24) \quad \begin{cases} -\frac{\ddot{g}}{\dot{g}} + \frac{\dot{f}}{f} = -i\chi \\ -\frac{\dot{f}}{f} + \frac{\dot{g}}{g} = i\chi. \end{cases}$$

Integrating once one gets the system

$$(25) \quad \begin{cases} \dot{f} = \left(\frac{\dot{f}}{g}\right)_{t=-\infty} g e^{-i\chi} & (\text{see eq. A.1'}) \\ \dot{g} = \left(\frac{\dot{g}}{f}\right)_{t=-\infty} f e^{i\chi} & (\text{see eq. A.1''}) \end{cases}$$

in the form arising from the quantum-mechanical formulation given by Froissart and Stora (see equations (A.1') and (A.1'') of the Appendix).

The linearization of

$$(13) \quad \dot{\chi} = \omega_p - \Omega = \Gamma t$$

is made⁽³⁾ by giving to Γ the value assumed by $\ddot{\chi}$ just at the resonance. It follows that Γ is expressed by

$$(13') \quad \Gamma = \frac{d}{dt}(\omega_p - \Omega) = \frac{d}{dt}(\omega_c \gamma G - \omega_c \gamma_{res} G) = \omega_c G \dot{\gamma}_{res}$$

$\Gamma^{-1/2}$ gives the half-width of the resonance, and

$$(14) \quad \frac{1}{\sqrt{\Gamma} \tau} = \frac{1}{\sqrt{2\pi G} \frac{\Delta E}{E_0}}$$

is the corresponding number of revolutions (τ is the revolution period, E_0 is the rest energy of the electron and ΔE is the energy gain/turn at the resonance energy).

The resonance half-width is ≈ 110 revolutions ($\approx .63$ MeV) for the Frascati Synchrotron.

The asymptotic expression of S_z becomes (see eq. (A.6))

$$(15) \quad S_z(+\infty) = 2 e^{-\frac{\pi}{2} \frac{\omega^2}{\Gamma}} - 1,$$

where the exponent is

$$(15') \quad \frac{\pi}{2} \frac{\omega^2}{\Gamma} = \frac{\pi^2}{4} \left(\frac{Z_0}{R}\right)^2 n_0^2 (2 + Q_z)^2 \frac{E_0}{G \Delta E}$$

(use of the approximate formula $Q_z \approx \sqrt{n_0 \Lambda}$ together with the resonance condition (10) has been made).

V. - Computation of the depolarization effect in the Frascati Synchrotron. -

From the values of the parameters of the Frascati Synchrotron the expression in the exponent (15') is $\frac{\pi}{2} \frac{\omega^2}{\Gamma} \cong .05 z_0^2$, where z_0 is measured in millimeters.

Assuming that the distribution of the vertical amplitudes is gaussian and denoting by g the vertical root mean square radius of the beam, one obtains for the mean value of the z-axis projection of the polarization vector \bar{S}

$$(16) \quad \langle S_z \rangle = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-\frac{1}{2} \left(\frac{z_0}{g} \right)^2} S_z \frac{dz_0}{g} \cong \frac{2}{\sqrt{1+.1g^2}} - 1$$

(g is given in mm).

The value of the vertical mean radius at the resonance energy is less than 2 mm, and therefore no more than 30% of the vertical polarization is lost.

From the above result the conclusion can be drawn that acceleration of vertically polarized electron beams in the Frascati Synchrotron should be promising enough to be worth trying.

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Appendix: the Froissart and Stora treatment. -

The quantum - mechanical description of resonant depolarization consists of finding the probabilities of transition of spin 1/2 particles in the state +1/2 to the state -1/2.

The Schrödinger time-dependent equation for the spin wave function leads to the system

$$(A.1') \quad \dot{f} = -i \frac{\omega}{2} g e^{-i\chi}$$

$$(A.1'') \quad \dot{g} = -i \frac{\omega}{2} f e^{i\chi}.$$

f and g are the occupation numbers for the two spin states ($|f|^2 + |g|^2 = 1$) with the initial conditions

$$(A.2) \quad |g(-\infty)| = 0 ; \quad |f(-\infty)| = 1$$

(and consequently

$$(A.2') \quad |\dot{g}(-\infty)| = \frac{\omega}{2} ; \quad |\dot{f}(-\infty)| = 0).$$

The vertical polarization, i. e. the expectation value of the spin in the z - direction at $t = +\infty$ is

$$(A.3) \quad S_z(+\infty) = 1 - 2 |g(+\infty)|^2.$$

One can obtain a second-order differential equation in g alone by combining (A.1') and (A.1''):

$$(A.4) \quad \ddot{g} - i \dot{\chi} \dot{g} + \left(\frac{\omega}{2}\right)^2 g = 0$$

$$(\dot{\chi} = \Gamma t).$$

12.

Making the change of variable $z = 1/2 i \Gamma t^2$ equation (A.4) transforms in to a confluent hypergeometric equation whose asymptotic solution (with the initial conditions $|g(-\infty)|=0$ and $|\dot{g}(-\infty)|=\frac{\omega}{2}$) is

$$(A.5) \quad |g(+\infty)| = 1 - e^{-\frac{\pi \omega^2}{2} \Gamma}$$

so that

$$(A.6) \quad \dot{g}_2(+\infty) = 2 e^{-\frac{\pi \omega^2}{2} \Gamma} - 1.$$

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