Laboratori Nazionali di Frascati

LNF-70/49

S. Ferrara and G. Rossi : A GEOMETRICAL INTERPRETATION OF SCALE INVARIANCE

Estratto da : Lett. Nuovo Cimento 4, 408 (1970)

S. FERRARA, *et al.*
29 Agosto 1970
Lettere al Nuovo Cimento
Serie I, Vol. 4, pag. 408-412

A Geometrical Interpretation of Scale Invariance.

S. FERRARA
Laboratori Nazionali del CNEN - Frascati

G. ROSSI
Istituto di Fisica dell'Università - Roma

(ricevuto il 22 Luglio 1970)

1. – In this last year strong interest has been devoted to the investigation of the scaling law for the inelastic form factors which is experimentally observed in the inelastic e-p scattering (1). In spite of the fact that many models, like vector dominance (2) and parton models (3), can reproduce this scaling law, it seems at present that this behaviour is a general feature of the theory in the sense that it may depend only on the structure of the singularities of the commutators of the hadronic electromagnetic current (4) and it is quite independent of the specific model which describes this current.

In this note we give a possible geometrical interpretation of such a behaviour starting from the observation that an external Lorentz group acts in a natural way on the functions of two complex variables leading to same interesting consequences.

Let us consider a function $W(z_1, z_2)$ defined on the complex affine plane (z_1, z_2). This space is an homogeneous space (5) with respect to the spinor Lorentz group $SL(2, C)$ and in fact it is equivalent to the quotient space $SL(2, C)/Z$, where Z is the two-dimensional group of matrices $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$. A representation of $SL(2, C)$ is then defined on these functions as follows (5):

$$T_g W(z_1, z_2) = W(\alpha z_1 + \beta z_2, \gamma z_1 + \delta z_2)\,, \tag{1}$$

where $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\alpha\delta - \beta\gamma = 1$ is an element of $SL(2, C)$.

(1) S. D. DRELL: *Lectures given at the « Ettore Majorana International Summer School »*, Erice (1969); SLAC-PUB-689 (1969).
(2) E. ETIM: private communication.
(3) S. D. DRELL, D. J. LEVY and T. M. YAN: *Phys. Rev. Lett.*, **22**, 744 (1969).
(4) H. LEUTWYLER and J. STERN: *Nucl. Phys.*, **20** B, 77 (1970).
(5) I. M. GEL'FAND, M. I. GRAEV and N. YA. VILENKIN: *Generalized Functions*, Vol. 5, Chapt. VI.

We observe that the homogeneous functions [6] of degree (n_1-1, n_2-1) play a special role in this space, in fact they form an irreducible subspace for the representation (1). Then an irreducible representation of the spinor Lorentz group is uniquely fixed by the pairs of complex numbers (n_1, n_2) whose difference is an integer and we shall call D_x $(x=(n_1, n_2))$ the irreducible space labelled by (n_1, n_2). Let us recall some basic properties of the functions belonging to D_x. From the homogeneity property, if we put

$$f(z)=W(z, 1)\,, \tag{2}$$

we have

$$W(z_1, z_2)=z_2^{n_1-1}\,\bar{z}_2^{n_2-1}\,f\left(\frac{z_1}{z_2}\right), \tag{3}$$

$$W(1, z)=z^{n_1-1}\bar{z}^{n_2-1}f(z^{-1})\,. \tag{4}$$

Furthermore the following asymptotic behaviours hold:

$$\lim_{\substack{|z_1|\to\infty\\|z_2|\to\infty\\z_1/z_2=\text{const}}} W(z_1, z_2)=z_2^{n_1-1}\,\bar{z}_2^{n_2-1}\,f\left(\frac{z_1}{z_2}\right). \tag{5a}$$

or putting $\hat{f}(z)=W(1, z)$

$$\lim_{\substack{|z_1|\to\infty\\|z_2|\to\infty\\|z_1|/|z_2|=\text{const}}} W(z_1, z_2)=z_1^{n_1-1}\,\bar{z}_1^{n_2-1}\,\hat{f}\left(\frac{z_2}{z_1}\right), \tag{5b}$$

$$\lim_{z_1/z_2\to\infty} W(z_1, z_2)\sim z_1^{n_1-1}\,\bar{z}_1^{n_2-1}W(1, 0)\,. \tag{6}$$

$$\lim_{z_2/z_1\to\infty} W(z_1, z_2)\sim z_2^{n_2-1}\,\bar{z}_2^{n_1-1}W(0, 1)\,. \tag{7}$$

Let us now consider an arbitrary function $W(z_1, z_2)$ which belongs to the space where the representation (1) acts; if this function satisfies some regularity conditions (at least it is L^2 with respect to the invariant measure on the homogeneous space [5]) it can be expanded in terms of irreducible components, *i.e.* of functions which transform irreducibly under (1). The result of the theory of the harmonic analysis on the homogeneous spaces with respect to $SL(2, C)$ gives in our case the following expansion formula:

$$W(z_1, z_2)=\frac{1}{(2\pi)^2}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}\varrho\, W(z_1, z_2; n, \varrho)\,, \tag{8}$$

where [5] $n_1=\frac{1}{2}(n+i\varrho)$, $n_2=\frac{1}{2}(-n+i\varrho)$ and $W(z_1, z_2; n, \varrho)$ is called the Mellin trans-

---

[6] A function $f(z_1, z_2)$ is called homogeenous of degree (λ, μ), where λ, μ are complex numbers differing by an integer, if for every complex number $S\neq 0$ we have $W(Sz_1, Sz_2)=S^\lambda\bar{S}^\mu W(z_1, z_2)$. We require $\lambda-\mu$ to be an integer since only then will $S^\lambda\bar{S}^\mu=|S|^{\lambda+\mu}\exp[i(\lambda-\mu)\arg S]$ be a single-valued function of S.

form of $W(z_1, z_2)$ and is defined by the equation

$$W(z_1, z_2; n, \varrho) = \frac{i}{2}\iint W(Sz_1, Sz_2) S^{-n_1} \bar{S}^{-n_2} S \, d\bar{S} . \tag{9}$$

From (9) it follows immediately that

a) $W(z_1, z_2; n, \varrho)$ is homogeneous in z_1, z_2 of degree $n_1 - 1$, $n_2 - 1$.

b) The action of the representation (1) on $W(z_1, z_2)$ induces the action of the irreducible representation (n_1, n_2) on $W(z_1, z_2; n, \varrho)$.

c) A sum rule, which is the analogue of the Plancherel theorem, holds for $W(z_1, z_2)$:

$$\int |W(z_1, z_2)|^2 \, dz_1 \, d\bar{z}_1 \, dz_2 \, d\bar{z}_2 = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\varrho \, |W(z, 1; n, \varrho)|^2 \, dz \, d\bar{z} . \tag{10}$$

Formula (9) can be used in order to analytically continue the Mellin transform for any complex value of ϱ, so eq. (8) holds also for functions which are not L^2 and we can write

$$W(z_1, z_2) = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \int_C d\varrho \, W(z_1, z_2; n, \varrho) , \tag{11}$$

where C is a suitable path in the complex ϱ-plane.

Let us now consider the Mellin transform of an homogeneous function $W(z_1, z_2) \in \in D_{\tilde{x}}(\tilde{x} = (\tilde{n}, \tilde{\varrho}))$, then it is immediate to derive

$$W(z_1, z_2, x) = (2\pi)^2 W(z_1, z_2) \, \delta_{M\tilde{M}} \, \delta(\varrho - \tilde{\varrho}) . \tag{12}$$

If we want to relate the homogeneity property to a structure of singularities of pole type it is convenient to introduce the following integral transform:

$$\begin{aligned} W^I(z_1, z_2; n, \varrho) &= \frac{1}{2\pi} \int_i d\varrho' \, \frac{W(z_1, z_2; n, \varrho')}{\varrho' - \varrho - i\varepsilon} = \\ &= \frac{i}{2} \int dS \, d\bar{S} \left(\frac{\bar{S}}{S}\right)^{n/2} W(Sz_1, Sz_2) |S|^{i\varrho} \, \theta(\log|\lambda|) \end{aligned} \tag{13}$$

(θ is the step function). We observe that

$$W(z_1, z_2; n, \varrho) = W^I(z_1, z_2; n, \varrho + i\varepsilon) - W^I(z_1, z_2; n, \varrho - i\varepsilon) . \tag{14}$$

Inserting in (11) we have

$$W(z_1, z_2) = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \left(\int_C d\varrho \, W^I(z_1, z_2; n, \varrho + i\varepsilon) - \int_C d\varrho \, W^I(z_1, z_2; n, \varrho - i\varepsilon) \right) . \tag{15}$$

If we now assume the function $W^I(z_1, z_2; n, \varrho)$ to be meromorphic in ϱ with poles at $(\varrho, n) = (\varrho_j, n_j)$, we obtain

$$W(z_1, z_2) = \sum_j W_j(z_1, z_2)\,, \tag{16}$$

i.e. the function $W(z_1, z_2)$ is a sum of a set of homogeneous functions $W_j(z_1, z_2)$ of degree $(\frac{1}{2}(n_j + i\varrho_j), \frac{1}{2}(-n_j + i\varrho_j))$. If we make the minimal hypothesis of the one-pole dominance, then $W(z_1, z_2)$ is homogeneous of degree $(\frac{1}{2}(n_0 + i\varrho_0), \frac{1}{2}(-n_0 + i\varrho_0))$, where (n_0, ϱ_0) is the irreducible representation which dominates the integral (11).

Let us assume that the W_1, W_2 structure functions which appear in inelastic e-p scattering are dominated in the high-energy region in the sense described above, respectively by the singularities at the points (n_1, ϱ_1), (n_2, ϱ_2) so we have (7)

$$W_1(q^2, \nu) = \nu^{\frac{1}{2}(n_1+i\varrho)-1}\,\bar{\nu}^{\frac{1}{2}(-n_1+i\varrho)-1}\,f_1\left(\frac{q^2}{\nu}\right), \tag{17}$$

$$W_2(q^2, \nu) = \nu^{\frac{1}{2}(n_2+i\varrho_2)-1}\,\bar{\nu}^{\frac{1}{2}(-n_2+i\varrho_2)-1}\,f_2\left(\frac{q^2}{\nu}\right). \tag{18}$$

From the experimental evidence (1) νW_2 and W_1 are scale invariant so we are led to assume $\varrho_1 = -2i$, $\varrho_2 = -i$ and we obtain

$$W_1(\nu, q^2) = f_1\left(\frac{q^2}{\nu}\right), \qquad \nu W_2(\nu, q^2) = f_2\left(\frac{q^2}{\nu}\right), \tag{19}$$

with the condition

$$\lim_{\nu/q^2\to\infty} f_2\left(\frac{q^2}{\nu}\right) = W_2(0, 1)\,. \tag{20}$$

Equation (20) relates the asymptotic behaviour of the structure function W_2 to its low-energy behaviour. This theoretical prevision can in principle be checked looking at the contribution of the representations of the type $(n, -i)$ to the form factor which describes the proton Compton scattering at energy $\nu = 2M$. These representations can be singled out using the Mellin transform defined above and we have

$$W_2^{n_1 n_2}(0, 1) = \int \mathrm{d}S\,\mathrm{d}\bar{S}\,W_2(0, S)\,S^{-n_1}\bar{S}^{-n_2} \qquad \text{with} \qquad n_1 - n_2 = n,\; n_1 + n_2 = 1\,. \tag{21}$$

Equation (21) together with eq. (20) can be read as a sum rule which relates the high-energy limit of the structure function W_2 in the deep inelastic region to the Copton scattering γ-p.

2. – In this note a geometrical description of the scaling law for the structure functions has been proposed, which uses the irreducible representations of a spinor Lorentz group acting in the complex plane (ν, q^2). The scaling law, which is deeply connected

(7) We have put $2M = 1$ so that the point 0, 1 in (20) corresponds to $Q^2 = 0$ and $\nu = 2M$.

with the irreducible components of these functions, can be derived assuming a meromorphic structure of a suitably defined integral transform. The experimentally observed high-energy behaviours can be easily obtained and an interesting consequence of these limits has been pointed out and we think that it is possible to relate the assumed « representation dominance » with the analitic structure of the hadronic electromagnetic current commutators.

* * *

We thank Prof. M. CASSANDRO for helpful discussions.