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Projection Formulae for $O(2, 1)$-Invariant Off-Shell Functions

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Projection formulae, which allow us to perform a generalized Laplace transform for a class of $O(2, 1)$-invariant off-shell functions, are derived in a rigorous way. The usefulness of this transformation for the physical case of the two spinless particle scattering amplitudes at fixed space-like momentum transfer described by ladder graphs is shown.

1. INTRODUCTION

There has been an increasing interest in the last year on the group-theoretical approach to the high energy scattering amplitude [1]. In this formalism the amplitude is defined as a function on the little group of the reaction momentum transfer Q. When $Q^2 = W < 0$ the little group is $O(2, 1)$ and, if the amplitude is square-integrable on the group manifold, it can be analyzed in terms of a class of unitary irreducible representations whose matrix elements, according to a theorem given by Bargmann [2], span the Hilbert space of the square-integrable functions. The coefficients of this expansion are the crossed partial wave amplitudes: they depend on two number l, τ which label the irreducible representations of $O(2, 1)$. The projection method can be extended to a set of representations (not necessarily unitary) which correspond to values of l inside a strip of the complex plane; this more general transformation is called the "Laplace transform" on the group and is a necessary mathematical tool in the group-theoretical description of the Regge theory. In fact if an amplitude contains some Regge pole contributions it is not L^2 on the group and it cannot be expressed only by means of unitary irreducible representations; some additional representations, also not unitary, are needed.

And so it is, speaking of the crossed partial wave analysis of the amplitude in a kinematical sense. The situation is more complicated dealing with some dynamical models, where off-shell functions must be considered. Actually, because of covariance conditions, the W-fixed off-shell amplitude (we do not write explicitly its dependence on the invariant masses) depends on the scalar product on the unitary hyperboloids in a three-dimensional Minkowsky space. These hyperboloids are equivalent to the quotient spaces $O(2, 1)/O(2)$ (two-sheet hyperboloid) and $O(2, 1)/O(1, 1)$ (one-sheet hyperboloid) and so the amplitude, as a function defined on $O(2, 1)$ is constant over the subgroup $O(2)$ or $O(1, 1)$. If only two sheet hyperboloids are involved no problems arise but when the one-sheet hyperboloid must be considered (as necessarily happens in the off-shell extension) the amplitude is never L^2 on $O(2, 1)$ because of the noncompactness of $O(1, 1)$. Due to this fact the projection method cannot be performed in a simple way.

The aim of this work is a rigorous derivation of the projection formulae of $O(2, 1)$-invariant off-shell functions and to give a wide set of sufficient conditions for their applicability. These techniques were employed some years ago in a heuristic way by Sertorio and Toller [3] in order to investigate an approximate field theoretical model, the fixed momentum transfer ladder Bethe–Salpeter equation.

In Section 2 we introduce some group-theoretical properties of the unitary hyperboloids and in Section 3 we define the Laplace transform for a class of $O(2, 1)$-invariant functions and we give out the projection formulae. Convergence and analyticity properties are treated in detail. In Section 4 we apply this method to the physical case of the ladder graphs.

2. Representations of $O(2, 1)$ on the Unitary Hyperboloids and Notations

In this section we recall some group-theoretical properties of the two and one-sheet hyperboloids in a three-dimensional Minkowsky space [4].

In general if we have some space S and a transformation group G on it, S is said to be homogeneous with respect to G if G acts as a transitive group of motion on it. This means that for every pair of points $x, x' \in S$ an element of G exists such that $x_g = x'$, x_g being the point x transformed under G. If x is an arbitrary point of S (different from zero) the subset H of G for which $x_h = x$ is a subgroup H of G which is called the little group of x. It is immediate to observe that there exists a mutually single-valued correspondence between S and the space of right cosets G/H. If an invariant measure $dx = dx_g$ exists on S, we can associate to it the following unitary representation D_g of G acting on the Hilbert space $L^2(S, dx)$:

$$D_g f(x) = f(x_g) \tag{2.1}$$

If G has good mathematical properties (type I group), for each function $f(x) \in L^2$ it is possible to define a corresponding vector-valued function f^σ which transforms according to the irreducible representation D_g^σ of G. This function satisfies the following Plancherel formula:

$$\int_S |f(x)|^2 \, dx = \int_\Omega \|f^\sigma\|^2 \, d\sigma, \tag{2.2}$$

where $d\sigma$ is the Plancherel measure (defined up to equivalence) and Ω is its support.

We now identify the group G with the group $O(2, 1)$ (connected part implemented by the space-time inversion) and the space S with the two kinds of hyperboloids. Under the action of $O(2, 1)$ the elements $\hat{\mathbf{p}} = \mathbf{p}/p$ of these spaces transform as follows:

$$\hat{\mathbf{p}}_g = g^{-1}\hat{\mathbf{p}} \quad \text{and} \quad \hat{\mathbf{p}}_t = -\hat{\mathbf{p}}, \tag{2.3}$$

where $g \in O(2, 1)$ and t is the space–time inversion. These elements are in correspondence with the elements of $O(2, 1)/O(2)$ or $O(2, 1)/O(1, 1)$. We now choose the following parametrization of the hyperboloids $p^2 = \rho(\rho = \pm 1)$:

$$\begin{aligned} \hat{\mathrm{p}}_0 &= \sqrt{z^2 - \rho} \cos \varphi, \\ \hat{\mathrm{p}}_1 &= \sqrt{z^2 - \rho} \sin \varphi, \\ \hat{\mathrm{p}}_2 &= z, \end{aligned} \tag{2.4}$$

with $|z| \geqslant \rho$, $0 \leqslant \varphi < 2\pi$.

The invariant measure on these spaces is

$$d\hat{\mathbf{p}} = \delta(p^2 - \rho) \, d^2\mathbf{p} = dz \, d\varphi. \tag{2.5}$$

We associate to these homogeneous spaces the following unitary representation D_g^ρ of $O(2, 1)$:

$$D_g^\rho f^\rho(\hat{\mathbf{p}}) = f^\rho(\hat{\mathbf{p}}_g), \tag{2.6}$$

where $f^\rho(\hat{\mathbf{p}})$ is L^2 with respect to $d\hat{\mathbf{p}}$ on $p^2 = \rho$. The Plancherel formula is

$$\int_{p^2=\rho} |f^\rho(\hat{\mathbf{p}})|^2 \, d\hat{\mathbf{p}} = \int_{\Omega^\rho} \|f^{\rho\sigma}\|^2 \, d^\rho\sigma. \tag{2.7}$$

The supports Ω^ρ are given by the following representations[1]:

Ω^1—principal series (one-valued type), (2.8)

Ω^{-1}—principal series (one-valued type) + discrete series (one-valued type). (2.9)

[1] This result can be obtained by direct calculations considering the restriction of the representation of $SL(2, C)$ acting on the homogeneous space $SL(2, C)/SU(1, 1)$ to its subgroup $SU(1, 1)$ (see Ref. [4], Chap. VI, Section 4) and using the decomposition technique given by A. Sciarrino and M. Toller [*J. Math. Phys.* **8**, No. 6 (1967), 1252].

In the dynamical models, where the crossed partial wave analysis of the off-shell amplitude is performed, it is sufficient to consider the decomposition on those representations which contribute to the reconstruction of the on-shell amplitude, therefore we are interested in the set of unitary representations given by

$$\Omega^1 \cap \Omega^{-1} = \Omega^1, \text{ i.e., the principal series (one-valued type).} \tag{2.10}$$

We now introduce a basis in the spaces irreducible with respect to the representations of the principal series. We consider the hyperbolic harmonics associated with the spaces $p^2 = \rho$. They are, by definition,[2] the eigenfunctions of the two Casimir operators $L^2 = L_0^2 - L_1^2 - L_2^2$ and $T = D_t^\rho$ and of the infinitesimal generator L_0. In our variables they are the regular solutions of the following set of equations:

$$\left(\frac{\partial}{\partial z}(z^2 - \rho)\frac{\partial}{\partial z} + \frac{1}{\rho z^2 - 1}\frac{\partial^2}{\partial \varphi^2}\right) D^\rho_{l\tau m}(\hat{\mathbf{p}}) = l(l+1)\, D^\rho_{l\tau m}(\hat{\mathbf{p}}), \tag{2.11}$$

$$TD^\rho_{l\tau m}(\hat{\mathbf{p}}) = \tau D^\rho_{l\tau m}(\hat{\mathbf{p}}), \tag{2.12}$$

$$-i\frac{\partial}{\partial \varphi} D^\rho_{l\tau m}(\hat{\mathbf{p}}) = m D^\rho_{l\tau m}(\hat{\mathbf{p}}). \tag{2.13}$$

From explicit calculation it turns out that

$$D^1_{l\tau m}(\hat{\mathbf{p}}) = C^1_{l\tau m}[\theta(z-1)\, P_{lm}(z) + \tau e^{im\pi}\theta(-z-1)\, P_{lm}(-z)]\, e^{im\varphi} \tag{2.14}$$

and

$$D^{-1}_{l\tau m}(\hat{\mathbf{p}}) = C^{-1}_{l\tau m}[P_{lm}(iz) + \tau e^{im\pi} P_{lm}(-iz)]\, e^{im\varphi}, \tag{2.15}$$

where $C^\rho_{l\tau m}$ are some constants which can be chosen in a suitable way. The functions $D^\rho_{l\tau m}(\hat{\mathbf{p}})$ can be analytically continued (if the constants $C^\rho_{l\tau m}$ are chosen in a analytic way) in the whole complex l plane and they still define a basis for the following irreducible representations of $O(2, 1)$:

Unitary

Principal series: $\operatorname{Re} l = -\frac{1}{2}$.

Complementary series: $\operatorname{Im} l = 0 - 1 < \operatorname{Re} l < 0$.

Nonunitary

Any other complex l (l not an integer).

Representations with index l and $-l - 1$ are equivalent.

[2] Let G be a given Lie group and H a given subgroup of G. Let L_i be a realization of the generators of its associated Lie Algebra on the homogeneous space G/H. We call a harmonic component on G/H any solution of the equation $\Delta f = \lambda f$ where Δ is the Laplace operator on G/H.

We want to remark [5] that the functions $D^{\rho}_{l\tau m}(\hat{\mathbf{p}}) \in L^q \cap L^\infty$ for

$$q > (\tfrac{1}{2} - |\operatorname{Re} l + \tfrac{1}{2}|)^{-1} - 1 < \operatorname{Re} l < 0.$$

In Section 3 we will use these functions in order to define the Laplace transform of a $O(2, 1)$-invariant function $K^{\rho\rho'}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')$.

3. The Laplace Transform

We start by demonstrating a theorem which is necessary in order to introduce the concept of a Laplace transform on a unitary hyperboloid.

We put $\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}' = Z$, then we have the following

Theorem.[3] *If $K^{\rho\rho'}(Z) \in L^q$ and $1 \leqslant q < 2$, its convolution with the hyperbolic harmonics is given in the strip $q^{-1} < \operatorname{Re} l < -1 + q^{-1}$* by

$$\int_{p'^2=\rho'} K^{\rho\rho'}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')\, D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\, d\hat{\mathbf{p}}' = K^{\rho\rho'}_{l\tau} D^{\rho}_{l\tau m}(\hat{\mathbf{p}}) \tag{3.1}$$

and the function $K^{\rho\rho'}_{l\tau}$, which we call the Laplace transform of $K^{\rho\rho'}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')$, is analytic in the strip and it is given by the following projection formula [3]:

$$K^{\rho\rho'}_{l\tau} = \int_{-\infty}^{+\infty} K^{\rho\rho'}(Z)\, f^{\rho\rho'}_{l\tau}(Z)\, dZ. \tag{3.2}$$

In order to demonstrate this theorem we first establish the following Lemma:

Lemma. *The convolution integral*

$$\int_{p'^2=\rho'} K^{\rho\rho'}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')\, D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\, d\hat{\mathbf{p}}'$$

uniformly converges (with respect to $\hat{\mathbf{p}}$) and is bounded in l in the strip

$$q^{-1} < \operatorname{Re} l < -1 + q^{-1}.$$

In fact $D^{\rho}_{l\tau m}(\hat{\mathbf{p}}) \in L^{q'} \cap L^\infty$ where $q'^{-1} + q^{-1} = 1$, and using the Hölder inequality we derive the following result (see Appendix A):

$$\int_{p'^2=\rho'} |K^{\rho\rho'}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')\, D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')|\, d\hat{\mathbf{p}}' \leqslant C^{\rho\rho'}_{l\tau m} \,\|K^{\rho\rho'}(Z)\|_q \,\|D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\|_{q'} \tag{3.3}$$

in the strip $q^{-1} < \operatorname{Re} l < -1 + q^{-1}$.

[3] This theorem can be regarded as an implement of some results obtained by R. L. Lipsan [*Bull. Amer, Math, Soc.* 7 (1957), 652].

Because of properties of the hyperbolic harmonics[4] it can be shown that the quantity $C^{\rho\rho'}_{l\tau m} \| D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\|_{q'}$ is bounded with respect to l in the strip and, therefore, the lemma is proved. As an immediate consequence of the lemma we have the following corollary:

COROLLARY. *The convolution integral is an analytic function of l in the strip $q^{-1} < \operatorname{Re} l < -1 + q^{-1}$.*

In fact the functions $D^{\rho}_{l\tau m}(\hat{\mathbf{p}})$ are analytic in the strip, therefore

$$D^{\rho}_{l\tau m}(\hat{\mathbf{p}}) = \frac{1}{2\pi i} \int dl' \, \frac{D^{\rho}_{l\tau m}(\hat{\mathbf{p}})}{l' - l}, \tag{3.4}$$

and by means of an exchange of order of integration we have

$$\int_{p'^2=\rho'} K^{\rho\rho'}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')\, D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\, d\hat{\mathbf{p}}' = \frac{1}{2\pi i} \int dl' \frac{1}{l' - l} \int K^{\rho\rho'}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')\, D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\, d\hat{\mathbf{p}}'. \tag{3.5}$$

In fact the second integral converges also if the integrand is substituted by its modulus.

Now it is a straightforward calculation, which is given in Appendix B, to show that, under our hypothesis and from the above proved lemma, formula (3.1) holds for every function $K^{\rho\rho'}(Z)$ belonging to $L^q \cap C^2 (1 \leqslant q < 2)$

In order to extend this result to an arbitrary L^q function $(1 \leqslant q < 2)$ note that $L^q \cap C^2$ is dense in L^q, therefore a sequence $K^{\rho\rho'}_N(Z) \in L^q \cap C^2$ exists such that

$$\lim_{N\to\infty} \| K^{\rho\rho'}_N(Z) - K^{\rho\rho'}(Z)\|_q = 0; \tag{3.6}$$

then from (3.3) we have

$$\left| \int_{p'^2=\rho'} K^{\rho\rho'}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')\, D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\, d\hat{\mathbf{p}}' - K^{\rho\rho'}_{Nl\tau} D^{\rho}_{l\tau m}(\hat{\mathbf{p}}) \right|$$
$$\leqslant \| K^{\rho\rho'}(Z) - K^{\rho\rho'}_N(Z)\|_q \, \| D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\|_{q'} . \tag{3.7}$$

If we perform the limit $N \to \infty$ we obtain

$$\int_{p'^2=\rho'} K^{\rho\rho'}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')\, D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\, d\hat{\mathbf{p}}' = K^{\rho\rho'}_{l\tau} D^{\rho}_{l\tau m}(\hat{\mathbf{p}}), \tag{3.8}$$

where

$$K^{\rho\rho'}_{l\tau} = \lim_{N\to\infty} K^{\rho\rho'}_{Nl\tau} . \tag{3.9}$$

[4] See Ref. [7], Chap. VI,

From (3.3) and (3.8) it turns out that

$$| K_{l\tau}^{\rho\rho'} | \leqslant C_{l\tau m}^{\rho\rho'} \| K^{\rho\rho'}(Z)\|_q \| D_{l\tau m}^{\rho'}(\hat{\mathbf{p}}')\|_q (\| D_{l\tau m}^{\rho}(\hat{\mathbf{p}})\|_\infty)^{-1}. \tag{3.10}$$

The projection formula (3.2) can be derived, under our hypothesis, following the procedure given in the Appendix of Ref. [3].

4. Physical Applications

The off-shell functions and the crossed partial wave analysis enter, in physical problems, in some well-known dynamical models for the scattering amplitude like Bethe–Salpeter and multiperipheral integral equations. In these models invariant functions $K^{\rho\rho'}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')$ (we do not indicate explicitely the variables which are not affected by the projection) act as kernels of integral transformations in the space of fixed-momentum transfer two-particle off-shell states $f^{\rho}(\hat{\mathbf{p}})$ or as mapping of physical on-shell states $f^{1}(\hat{\mathbf{p}})$ into the off-shell ones. We consider both the cases at once.

The problem we are interested in is therefore the projection of integral transformations like:

$$g^{\rho}(\hat{\mathbf{p}}) = \sum_{\rho'} \int_{p'^2=\rho'} K^{\rho\rho'}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') f^{\rho'}(\hat{\mathbf{p}}') \, d\hat{\mathbf{p}}'. \tag{4.1}$$

For our purposes it suffices to consider functions that belong to the set $L^1 \cap L^2$ which is dense in L^2. We have for their projection:

$$f_{l\tau m}^{\rho} = \int_{p^2=\rho} f^{\rho}(\hat{\mathbf{p}}) \, D_{l\tau m}^{\rho}(\hat{\mathbf{p}}) \, d\hat{\mathbf{p}}, \tag{4.2}$$

and the integral absolutely converges in the strip $-1 < \operatorname{Re} l < 0$. We now evaluate the projection $g_{l\tau m}^{\rho}$ of the integral transformation (4.1) where

$$K^{\rho\rho'}(Z) \in L^q(1 \leqslant q < 2):$$

$$g_{l\tau m}^{\rho} = \int_{p^2=\rho} g^{\rho}(\hat{\mathbf{p}}) \, D_{l\tau m}^{\rho}(\hat{\mathbf{p}}) \, d\hat{\mathbf{p}} = \sum_{\rho'} \int_{p^2=\rho} d\hat{\mathbf{p}} \, D_{l\tau m}^{\rho}(\hat{\mathbf{p}}) \int_{p'^2=\rho'} d\hat{\mathbf{p}}' \, K^{\rho\rho'}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') f^{\rho'}(\hat{\mathbf{p}}') \tag{4.3}$$

in the strip $q^{-1} < \operatorname{Re} l < -1 + q^{-1}$, using the Hölder inequality and by the Lemma of the previous section, we obtain:

$$g_{l\tau m}^{\rho} = \sum_{\rho'} \int_{p'^2=\rho'} d\hat{\mathbf{p}}' f^{\rho}(\hat{\mathbf{p}}') \int_{p^2=\rho} d\hat{\mathbf{p}} \, K^{\rho\rho'}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') \, D_{l\tau m}^{\rho'}(\hat{\mathbf{p}}) = \sum_{\rho'} K_{l\tau}^{\rho\rho'} f_{l\tau m}^{\rho'}. \tag{4.4}$$

Now we consider, as an example, the amplitude associated with the ladder irreducible graph which describes the scattering of two scalar bosons via an unstable spinless particle, we have

$$K^{\rho\rho'}(\hat{\mathrm{p}} \cdot \hat{\mathrm{p}}') = \frac{g_0}{a + b\hat{\mathrm{p}} \cdot \hat{\mathrm{p}}' - i\gamma} \qquad (\gamma > 0) \tag{4.5}$$

in this case $K^{\rho\rho'}(Z) \in L^q$ with $q = 1 + \delta(\delta > 0)$, therefore our result holds in every open strip contained in $-1 < \mathrm{Re}\, l < 0$. The Laplace transform can be evaluated as in the work of Sertorio and Toller by the Cauchymethod. The Laplace transform of the iterated graphs is the product of the Laplace transform of the single graphs. To see this it is sufficient to perform an allowed exchange of the order of integration.

Appendix A

In order to demonstrate the inequality (3.3) we distinguish the two cases $\rho = \pm 1$. When $\rho = 1$ the proof is immediate, in fact, using directly the Hölder inequality:

$$\int_{p'^2=\rho'} | K^{1\rho'}(\hat{\mathrm{p}} \cdot \hat{\mathrm{p}}') D^{\rho'}_{l\tau m}(\hat{\mathrm{p}}')| \, d\hat{\mathrm{p}}' \leqslant \Big(\int_{p^2=\rho} | K^{1\rho'}(\hat{\mathrm{p}} \cdot \hat{\mathrm{p}}')|^q \, d\hat{\mathrm{p}}'\Big)^{1/q} \Big(\int_{p'^2=\rho'} | D^{\rho'}_{l\tau m}(\hat{\mathrm{p}}')|^{q'} \, d\hat{\mathrm{p}}'\Big)^{1/q'} \tag{A.1}$$

where $q^{-1} + q'^{-1} = 1$.

From the $O(2, 1)$ invariance of the measure $d\hat{\mathrm{p}}$ we have

$$\int_{p'^2=\rho'} | K^{1\rho'}(\hat{\mathrm{p}} \cdot \hat{\mathrm{p}}')|^q \, d\hat{\mathrm{p}}' = \int_{p'^2=\rho'} | K^{1\rho'}(Z)|^q \, d\hat{\mathrm{p}}' = 2\pi(\| K^{1\rho'}(Z)\|_q)^q \tag{A.2}$$

and therefore

$$\int_{p'^2=\rho'} | K^{1\rho'}(\hat{\mathrm{p}} \cdot \hat{\mathrm{p}}') D^{\rho'}_{l\tau m}(\hat{\mathrm{p}}')| \, d\hat{\mathrm{p}}' \leqslant (2\pi)^{1/q} \| K^{1\rho'}(Z)\|_q \, \| D^{\rho'}_{l\tau m}(\hat{\mathrm{p}}')\|_{q'} . \tag{A.3}$$

Remembering that $D^{\rho}_{l\tau m}(\hat{\mathrm{p}}) \in L^{q'} \cap L^\infty$, where $q' > (\frac{1}{2} - | \mathrm{Re}\, l - \frac{1}{2} |)^{-1}$, we obtain that (A.3) holds in the strip $q^{-1} < \mathrm{Re}\, l < -1 + q^{-1}$.

For the case $\rho = -1$ the procedure is different and in fact the integral (A.2) always diverges. In the inequality,

$$\Big| \int_{p'^2=\rho} K^{-1\rho'}(\hat{\mathrm{p}} \cdot \hat{\mathrm{p}}') D^{\rho'}_{l\tau m}(\hat{\mathrm{p}}') \, d\hat{\mathrm{p}}' \Big| \leqslant \int_{p'^2=\rho'} | K^{-1\rho'}(\hat{\mathrm{p}} \cdot \hat{\mathrm{p}}')| \, | D^{\rho'}_{l\tau m}(\hat{\mathrm{p}}')| \, d\hat{\mathrm{p}}', \tag{A.4}$$

we observe that the integral in the second member cannot depend on φ', so we put $\varphi = 0$ and we perform a transformation assuming as new independent variables z and $Z = \hat{\mathfrak{p}} \cdot \hat{\mathfrak{p}}'$.

The measure element is written as

$$d\hat{\mathfrak{p}} = dz\, d\varphi = \frac{\theta(x)}{\sqrt{x}}\, dZ\, dz, \tag{A.5}$$

where $x = -\rho' + 2zz'Z + z'^2 - \rho' z^2 - Z^2$ and θ is the step function.

If we put

$$B^{\rho'}_{l\tau m}(Z, z) = \int_{|z'| \geqslant \rho'} dz' \mid D^{\rho'}_{l\tau m}(z') \mid \frac{\theta(x)}{\sqrt{x}}, \tag{A.6}$$

the expression (A.4) becomes

$$\left| \int_{p'^2 = \rho'} K^{-1\rho'}(\hat{\mathfrak{p}} \cdot \hat{\mathfrak{p}}')\, D^{\rho'}_{l\tau m}(\hat{\mathfrak{p}}')\, d\hat{\mathfrak{p}}' \right| \leqslant \int_{-\infty}^{+\infty} dZ \mid K^{-1\rho'}(Z) \mid B^{\rho'}_{l\tau m}(Z, z) \tag{A.7}$$

using some standard methods it is not difficult to derive the following inequality:

$$\int_{-\infty}^{+\infty} (B^{\rho'}_{l\tau m}(Z, z))^{q'}\, dZ < (\tilde{C}_{q,l\tau m})^{q} \parallel D^{\rho'}_{l\tau m}(z') \parallel^{q'}_{q'} = (C_{q',l\tau m} \parallel D^{\rho'}_{l\tau m}(\hat{\mathfrak{p}}') \parallel_{q'})^{q'}, \tag{A.8}$$

where $q' > (\frac{1}{2} - \mid \mathrm{Re}\, l + \frac{1}{2} \mid)^{-1}$ and $C_{q',l\tau m} = (2\pi)^{-1/q'}\, \tilde{C}_{q',l\tau m}$.

Therefore we finally obtain, using the Hölder inequality:

$$\int \mid K^{-1\rho'}(\hat{\mathfrak{p}} \cdot \hat{\mathfrak{p}}')\, D^{\rho'}_{l\tau m}(\hat{\mathfrak{p}}')\, d\hat{\mathfrak{p}}' \leqslant C_{q,l\tau m} \parallel K^{-1\rho'}(Z) \parallel_q \parallel D^{\rho'}_{l\tau m}(\hat{\mathfrak{p}}') \parallel_{q'} \tag{A.9}$$

($q^{-1} + q'^{-1} = 1$) in the strip $q^{-1} < \mathrm{Re}\, l < -1 + q^{-1}$ and the inequality (3.3) is completely proved.

Appendix B

In this Appendix we give a proof of the formula (3.1) and we prove that it is sufficient to consider functions $K^{\rho\rho'}(Z) \in L^q \cap C^2$. To do this we must show that the functions

$$(KD)^{\rho\rho'}_{l\tau m}(\hat{\mathfrak{p}}) = \int_{p'^2 = \rho'} K^{\rho\rho'}(\hat{\mathfrak{p}} \cdot \hat{\mathfrak{p}}')\, D^{\rho'}_{l\tau m}(\hat{\mathfrak{p}}')\, d\hat{\mathfrak{p}}' \tag{B.1}$$

satisfy the set of equations (2.11), (2.12), and (2.13) which define the hyperbolic harmonic $D^{\rho}_{l\tau m}(\hat{\mathbf{p}})$ with the same boundary conditions.

The derivation of (2.12) from (B.1) trivial. In order to satisfy Eq. (2.13) we have, integrating by parts:

$$m \int_{p'^2=\rho'} K^{\rho\rho'}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\, D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\, d\hat{\mathbf{p}}'$$
$$= \int_{p'^2=\rho'} K^{\rho\rho'}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\left(-i\frac{\partial}{\partial\varphi'} D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\right) d\hat{\mathbf{p}}'$$
$$= \int_{|z'|\geqslant\rho'} K^{\rho\rho'}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\, D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\, dz'\Big|_0^{2\pi} + \int -i\frac{\partial}{\partial\varphi} K^{\rho\rho'}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\, D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\, d\hat{\mathbf{p}}'. \tag{B.2}$$

Taking in account the periodicity of the first integral and of its uniform boundedness with respect to $\hat{\mathbf{p}}$ (see Eq. 3.3) we still obtain the final answer. The verification of (2.11) is not trivial, as it would happen for the corresponding operator of $O(3)$. In fact a transformation of $O(2, 1)$ which carries L_0 in L_1(or L_2) does not exist. Then we must derive the action of L_1(or L_2) on the convolution integral (B.1). To do this it is convenient to parametrize the elements of the hyperboloids as follows:

$$(\hat{\mathbf{p}}_0, \hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2) \equiv (\sqrt{z^2+\rho}\cosh\zeta, \sqrt{z^2+\rho}\sinh\zeta, z).$$

The invariant measure becomes

$$d\hat{\mathbf{p}} = dz\, d\zeta. \tag{B.3}$$

The differential operator L_2 assumes the form

$$L_2 = 2\frac{\partial}{\partial\zeta}. \tag{B.4}$$

Therefore we have

$$\int_{p'^2=\rho'} K^{\rho\rho'}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\frac{\partial}{\partial\zeta} D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\, d\hat{\mathbf{p}}'$$
$$= \int_{|z'|\geqslant\rho'} K^{\rho\rho'}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\, D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\, dz'\Big|_{-\infty}^{+\infty} + \int \frac{\partial}{\partial\zeta} K^{\rho\rho'}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\, D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}')\, d\hat{\mathbf{p}}'. \tag{B.5}$$

If we take in account that $K^{\rho\rho'}(Z) \in L^q \cap C^2$ and that

$$D^{\rho'}_{l\tau m}(\hat{\mathbf{p}}') \to 0 \quad \text{as} \quad |\zeta| \to \infty \quad (-1 < \operatorname{Re} l < 0)$$

we have that the first term of the second member of (B.5) vanishes and we obtain the desired result. Obviously the same is true for L_1: in fact, in order to carry L_2 in L_1 it is sufficient to apply a transformation of $O(2)$ (which is a subgroup of $O(2, 1)$) of an angle $\pi/2$.

From the above conclusion we finally obtain that the operator

$$L^2 = L_0^2 - L_1^2 - L_2^2$$

commutes with the "convolution operator" on the hyperboloids and then (B.2) satisfies (2.11) too. In order to obtain the final result we find now the general solution of the set of equations (2.11), (2.12), and (2.13). From (2.13) we have

$$(KD)_{l\tau m}^{\rho\rho'}(\hat{\mathbf{p}}) = \chi_{l\tau m}^{\rho\rho'}(z)\, e^{im\varphi}, \tag{B.6}$$

and inserting in (2.11) we have

$$\left(\frac{d}{dz}(z^2 - \rho)\frac{d}{dz} + \frac{m^2}{1 - \rho z^2}\right)\chi_{l\tau m}^{\rho\rho'}(z) = l(l + 1)\,\chi_{l\tau m}^{\rho\rho'}\,; \tag{B.7}$$

furthermore from (2.12) the following relation must be verified:

$$\chi_{l\tau m}^{\rho\rho'}(z) = \tau e^{im\pi}\chi_{l\tau m}^{\rho\rho'}(-z). \tag{B.8}$$

The functions $\chi_{l\tau m}^{\rho\rho'}(z)$ are solutions of second order differential equation of totally fuchsian type (namely the Legendre equation) thus they must be spanned by two linearly independent solutions of (B.7). We consider before the case $\rho = 1$. If we choose as independent solutions the Legendre functions of the first and second kind $P_{lm}(z)$, $Q_{lm}(z)$ we obtain:

$$\begin{aligned}\chi_{l\tau m}^{1\rho'}(z) = F_{l\tau m}^{1\rho'}\{&\theta(z - 1)[P_{lm}(z) + a_{lm}Q_{lm}(z)]\\ &+ \tau e^{im\pi}\theta(-z - 1)[P_{lm}(-z) + a_{lm}Q_{lm}(-z)]\},\end{aligned} \tag{B.9}$$

where the constants $F_{l\tau m}^{1\rho'}$ are arbitrarily chosen. From the inequality (3.3) we have that the function $\chi_{l\tau m}^{1\rho'}(z)$ must be bounded in $z = \pm 1$ so $Q_{lm} = 0$. The values of the constants $F_{l\tau m}^{1\rho'}$ will be discussed later. In the case $\rho = -1$ it is convenient to choose as a set of independent solutions the functions $P_{lm}(z)$ and $P_{lm}(-z)$, so we have

$$\chi_{l\tau m}^{-1\rho'}(z) = F_{l\tau m}^{-1\rho'}\{P_{lm}(iz) + b_{\tau lm}P_{lm}(-iz)\}, \tag{B.10}$$

but since (B.8) has to be verified we univocally obtain $b_{\tau lm} = \tau e^{im\pi}$. The constants $F_{l\tau m}^{\rho\rho'}$ can be put in the form $K_{l\tau m}^{\rho\rho'}\, C_{l\tau m}^{\rho}$ and we have

$$(KD)_{l\tau m}^{\rho\rho'}(\hat{\mathbf{p}}) = K_{l\tau m}^{\rho\rho'}D_{l\tau m}^{\rho}(\hat{\mathbf{p}}). \tag{B.11}$$

If we consider the transformation properties of (B.1) under the operator $L_1 \pm iL_2$ it turns out that $K^{\rho\rho'}_{l\tau m} = K^{\rho\rho'}_{l\tau m+1}$ for every m, then finally the convolution integral may be written:

$$(KD)^{\rho\rho'}_{l\tau m}(\hat{\mathbf{p}}) = K^{\rho\rho'}_{l\tau} D^{\rho}_{l\tau m}(\hat{\mathbf{p}}), \tag{B.12}$$

and this is the final result.

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[5] See Ref. [3].

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