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On the Reggeized Multiperipheral Model.

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Summary. — The problem of the convergence of the unitarity series, saturated with Reggeized multiperipheral graphs, is formulated in a clear mathematical way. The asymptotic approximation is extensively treated.

1. – Introduction.

In this last year, starting from the C.G.L. multi-Regge equation, strong interest has been devoted to multiperipheral models for the high-energy scattering amplitude (1).

A group-theoretical approach to the C.G.L. equation was proposed by CHEW and DE TAR (2) at zero momentum transfer and its partial diagonalization

(1) G. F. CHEW, M. L. GOLDBERGER and F. E. LOW: *Phys. Rev. Lett.*, **22**, 208 (1969); I. G. HALLIDAY and L. M. SANDERS: *Nuovo Cimento*, **60** A, 177 (1969); L. CANESCHI and A. PIGNOTTI: *Phys. Rev.*, **180**, 1525 (1969); W. R. FRAZER and C. H. MEHTA: University of California, San Diego preprint, UCSD-10Plo-60 (1969); J. S. BALL and G. MARCHESINI: Berkeley, preprint, UCRL-19282 (1969).

(2) G. F. CHEW and C. DE TAR: *Phys. Rev.*, **180**, 1577 (1969).

was obtained by CHEW and FRAZER (3) and more recently by MUELLER and MUZINICH (4). CIAFALONI, DE TAR and MISHELOFF (5) studied the kinematical complications which arise at spacelike momentum transfer and performed the corresponding little-group expansion.

In these developments some problems, whose arrangement is not completely satisfactory, arise. First of all we outline the fact that the substitution of the unitarity graph series by an integral equation, also if it is a useful mathematical tool, is not unambiguous. Actually, it can be shown that the integral equation has several solutions and obviously the physically relevant one is that we can identitify with the unitarity graph series sum. So it seems an essential point to specify what convergence means for the unitarity series. Another difficulty is due to the fact that we have to perform a partial diagonalization with respect to a noncompact group and only well-behaved functions, *i.e.* square summable on the group parameters, can be expanded in a definite way.

The above-mentioned difficulties can be tackled if we consider the problem in the framework of a clearer mathematical formalism: a possible way consists of associating to each unitarity graph a pseudobilinear functional in some dense subspaces of the Hilbert spaces in which the « crossed partial-wave analysis » is performed. The structure of these spaces has been introduced in ref. (6). So in this formalism the problem is to sum an infinite series of pseudobilinear functionals.

Another important point is the fact that the many-particle production amplitudes, parametrized by means of the Chew, Bali and Pignotti variables (7), are not free from kinematical singularities, so the imaginary part itself cannot show the correct analyticity properties. This difficulty arises from the well-known fact that a scattering amplitude constructed by a single Regge pole has not the correct analyticity properties and an infinite family of trajectories must be exchanged. These last properties can be achieved by means of a generalization of the model for the two-body scattering amplitude developed by COSENZA, SCIARRINO and TOLLER (8) in which these difficulties are eliminated (multi-Lorentz model) (9).

A unified treatment of these problems, in the framework of a definite math-

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(3) G. F. CHEW and W. R. FRAZER: Berkeley, preprint, UCRL-18681 (1968); *Phys. Rev.*, to be published.

(4) A. H. MUELLER and I. J. MUZINICH: Brookhaven National Laboratory report BNL-13728 (1969).

(5) M. CIAFALONI, C. DE TAR and M. N. MISHELOFF: Berkeley preprint, UCRL-19286 (1969).

(6) M. TOLLER: *Nuovo Cimento*, **54** A, 295 (1968).

(7) N. F. BALI, G. F. CHEW and A. PIGNOTTI: *Phys. Rev.*, **163**, 1572 (1967).

(8) G. COSENZA, A. SCIARRINO and M. TOLLER: *Nuovo Cimento*, **57** A, 263 (1968); **62** A, 999 (1969).

(9) M. TOLLER: Ref. TH-1026-CERN-Geneva (1969).

ematical structure, will be the object of a future work. In this paper however, we show, as an example, how this mathematical formalism allows us to treat completely the problem, at least when the asymptotic approximation (3) for the projected graphs is considered (spinless particles involved).

2. – The unitarity-graph series convergence.

The unitarity condition for a generical reaction $(1)+(2)\to(3)+(4)$, can be written in the following way:

$$(2.1)\qquad -i([f_3\otimes f_4], (T-T^\dagger)[f_1\otimes f_2]) = ([f_3\otimes f_4], TT^\dagger[f_1\otimes f_2])\,,$$

where $T=-i(S-I)$ is a normal operator $([T, T^\dagger]=0)$, f_i are single-particle states and the matrix elements in (2.1) define two bounded functionals on the topological product $\mathscr{H}^I\times\mathscr{H}^F$: $\mathscr{H}^I=\mathscr{H}^1\otimes\mathscr{H}^2$ and $\mathscr{H}^F=\mathscr{H}^3\otimes\mathscr{H}^4$ are the Hilbert spaces of the initial and final states respectively. Following ref. (6) we can also define two pseudobilinear functionals

$$(2.2)\qquad \Phi(f_4\otimes Jf_2, f_1\otimes Jf_3) = -i([f_3\otimes f_4], (T-T^\dagger)[f_1\otimes f_2])\,,$$

$$(2.3)\qquad \Xi(f_4\otimes Jf_2, f_1\otimes Jf_3) = ([f_3\otimes f_4], TT^\dagger[f_1\otimes f_2])\,,$$

on $\mathscr{H}^B\times\mathscr{H}^A$, where $\mathscr{H}^B$ and $\mathscr{H}^A$ are the Hilbert spaces $\mathscr{H}^4\otimes\overline{\mathscr{H}}^2$ and $\mathscr{H}^1\otimes\overline{\mathscr{H}}^3$ which contain the pseudostates (10) of the reaction $f_4\otimes Jf_2$, $f_1\otimes Jf_3$. J is the antilinear mapping which transforms a Hilbert space $\mathscr{H}$ in its adjoint $\overline{\mathscr{H}}$. We observe that these functionals are defined only in the dense subspaces $\mathscr{L}^B\subset\mathscr{H}^B$, $\mathscr{L}^A\subset\mathscr{H}^A$ which contain elements of factorized type and their finite sums. The unitarity condition becomes

$$(2.4)\qquad \Phi(f^B, f^A) = \Xi(f^B, f^A)\,.$$

As shown in ref. (10) the Hilbert spaces $\mathscr{H}^A$, $\mathscr{H}^B$ can be decomposed as a direct integral of Hilbert spaces $\mathscr{H}^A_Q$, $\mathscr{H}^B_Q$. In the equal-mass case (EE) we have $M_1=M_3=M_A$, $M_2=M_4=M_B$ and the explicit form of the projection of the functionals Φ and Ξ at zero momentum transfer is

$$(2.5)\qquad \Phi_{Q=0}(f^B_{Q=0}, f^A_{Q=0}) = \int \mathrm{d}\mu^+(P_B)\,\mathrm{d}\mu^+(P_A)\bar{f}^B_{Q=0}(P_B)\,\mathrm{Im}\,M(P_A, P_B; P_A, P_B) f^A_{Q=0}(P_A)\,,$$

$$(2.6)\qquad \Xi_{Q=0}(f^B_{Q=0}, f^A_{Q=0}) = \sum_{n=2}^{\infty}\chi_{n+2}(f^B_{Q=0}, f^A_{Q=0})\,,$$

(10) S. Ferrara and G. Mattioli: *Nuovo Cimento*, **65** A, 25 (1970).

where

$$(2.7)\qquad \chi_{n+2}(f^B_{Q=0}, f^A_{Q=0}) = \int d\mu^+(P_B)\, d\mu^+(P_A) \prod_{i=1}^{N} d\mu^+(P_n) \cdot$$

$$\cdot |M(P_0, P_1, \ldots, P_n, P_{n+1}; P_A, P_B)^2|\, \delta^4\Big(P_A + P_B - \sum_{i=0}^{n+1} P_i\Big) \bar{f}^B_{Q=0}(P_B)\, f^A_{Q=0}(P_A)\,.$$

$d\mu^+(p)$ is the invariant measure on the mass hyperboloid and $f^B_{Q=0}(P_B) = f_4(P_B)\bar{f}_2(-P_B)$, $f^A_{Q=0}(P_A) = f_1(P_A)\bar{f}_3(-P_A)$ are the wave functions of the pseudostates at zero momentum transfer. The function $M(P_0, \ldots, P_{n+1}; P_A, P_B)$ is defined by the relation

$$(2.8)\qquad \big([\varphi_0(P_0) \otimes \ldots \otimes \varphi_{n+1}(P_{n+1})],\, T[\varphi_A(P_A) \otimes \varphi_B(P_B)]\big) =$$

$$= (2\pi)^{1-\frac{3}{2}n}\, \delta^4\Big(P_A + P_B - \sum_{i=0}^{n+1} P_i\Big) M(P_0, \ldots, P_{n+1}; P_A, P_B)$$

and $P_0 = P_A$, $P_{n+1} = P_B$ (from now on we will omit the index $Q = 0$).

Following ref. ([2]) we now use the B.C.P. group-theoretical variables and we obtain

$$(2.9)\qquad \operatorname{Im} M(P_A, P_B; P_A P_B) = \mathscr{M}(a_B^{-1} a_A)\,,$$

$$(2.10)\qquad |M(P_0, \ldots, P_{n+1}; P_A, P_B)|^2 = \mathscr{N}(W_1, \ldots, W_{n+1}; g_1, \ldots, g_{n+1})\,,$$

where $W_1 = (P_A + P_0)^2 = Q_1^2$ and

$$W_i = Q_i^2 = (P_{i-1} + Q_{i-1})^2 \qquad (i = 2, \ldots, n+1)\,.$$

The Lorentz transformation a_A (a_B) connects the rest frame of the particle A (B) to the laboratory system; g_i are elements of the little groups H_{W_i} of Q_i and are defined by the relations

$$(2.11)\qquad g_i = g_{i-1}^{-1} a_{i-1}^{-1} a_i \qquad (i = 1, \ldots, n+1)\,,$$

where a_i are Lorentz transformations which connect the systems in which the four-momenta have the form

$$(2.12)\qquad \begin{aligned} Q_i &= Q_i^0\,, \\ Q_{i+1} &= g_i Q_{i+1}^0\,, \end{aligned} \qquad Q_i^0 = \begin{cases} (\sqrt{W_i}, 0, 0, 0)\,, & \text{if } W_i \geqslant 0\,, \\ (0, 0, 0, \sqrt{-W_i})\,, & \text{if } W_i \leqslant 0\,, \end{cases}$$

to the laboratory system and q_i are boosts in the direction of the z-axis. The following relations hold:

$$(2.13)\qquad a_0 = a_A r_A\,, \qquad a_{i+1} = a_i q_i g_{i+1}\,, \qquad a_B = a_{n+1} q_{n+1} r_B\,,$$

and the elements r_A and r_B are simple rotations.

The group elements can be parametrized in the usual way:

$$(2.14)\qquad g_i = u_z(\mu_i)\, a_x(\zeta_i)\, u_z(\nu_i)\,,$$

where $u_z(\mu_i)$, $u_z(\nu_i)$ are rotations about the indicated axis and $a_x(\zeta_i)$ is a rotation about the x-axis if $W_i > 0$ and a boost if $W_i < 0$.

The single multi-Regge contribution to the function (2.10) is

$$(2.15)\qquad \mathcal{N}(W_1, \dots, W_{n+1}; g_1, \dots, g_{n+1}) = g^{2(n+2)} \beta^A(W_1)(\cosh \zeta_1)^{2\alpha(W_1)} \cdot \\ \cdot \beta(W_1, W_2)(\cosh \zeta_2)^{2\alpha(W_2)} \dots (\cosh \zeta_{n+1})^{2\alpha(W_{n+1})} \beta^B(W_{n+1})\,,$$

where the functions β are the square moduli of the residue functions, g is a coupling constant and angular dependence is neglected.

If we express the phase space in group-theoretical variables we finally obtain

$$(2.16)\qquad \chi_{n+2}(f^B, f^A) = g^{2(n+2)} 2^{-2} (2\pi)^6 \int \frac{\delta^3_{M_A^2}(r_A)}{M_A^2} \beta^A(W_1)\, \tilde{\Delta}(W_1, M_A^2, \mu^2) \cdot \\ \cdot \frac{\delta^3_{W_1}(g_1)}{W_1} (\cosh \zeta_1)^{2\alpha(W_1)} \cdot \tilde{\Delta}(W_1, W_2, \mu^2) \dots \frac{\delta^3_{M_B^2}(r_B)}{M_B^2} \cdot \\ \cdot \beta^B(W_{n+1})\, \tilde{\Delta}(W_{n+1}, M_{B_1}^2, \mu^2) \bar{f}^B(P_B) f^A(P_A)\, \mathrm{d}W_1 \dots \mathrm{d}W_{n+1}\, \mathrm{d}a_0 \dots \mathrm{d}a_{n+1}\, \mathrm{d}a_A\, \mathrm{d}a_B\,.$$

The symbols used have the following meaning: $\tilde{\Delta}(a, b, c) = \Delta^{\frac{1}{2}}(a, b, c)\theta(\Delta(r, b, c))$ where $\Delta(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$ is the triangular function and θ is the step function, μ is the mass of the produced particles, which are assumed all identical, and $\delta^3_W(a)$ is a δ-like distribution on the Lorentz group with the property

$$(2.17)\qquad \int_{O_{3,1}} f(a)\, \delta^3_W(a)\, \mathrm{d}a = \int_{H_W} f(g)\, \mathrm{d}g\,.$$

The last expression can be rewritten as

$$(2.18)\qquad \chi_{n+2}(f^B, f^A) = g^{2(n+2)} \int B_n(a_A, a_{n+1}, W_{n+1})\, 2^{-2} (2\pi)^6 \beta^B(W_{n+1}) \cdot \\ \cdot \tilde{\Delta}(W_{n+1}, M_B^2, \mu^2) \frac{\delta^3_{M_B^2}(r_B)}{M_B^2} \bar{f}^B(a_B) f^A(a_A)\, \mathrm{d}W_{n+1}\, \mathrm{d}a_{n+1}\, \mathrm{d}a_B\, \mathrm{d}a_A\,,$$

where

$$B_n(a_A, a_{n+1}, W_{n+1}) = \int B_{n-1}(a_A, a_n, W_n)\beta(W_n, W_{n+1}, \mu^2)\cdot \tag{2.19}$$
$$\cdot \frac{\delta^3_{W_{n+1}}(g_{n+1})}{|W_{n+1}|}(\cos\zeta_{n+1})^{2\alpha(W_{n+1})}\,\mathrm{d}W_n\,\mathrm{d}a_n$$

and we have

$$B_0(a_A, a_1, W_1) = \int \frac{\delta^3_{M_A^2}(r_A)}{M_A^2}\,\beta^A(W_1)\tilde{\Delta}(W_1, M_A^2, \mu^2)\,\frac{\delta^3_{W_1}(g_1)}{|W_1|}(\cosh\zeta_1)^{2\alpha(W_1)}\,\mathrm{d}a_0\,. \tag{2.20}$$

If we define the kernel

$$G(W^1, a'^{-1}a, W) = \beta(W', W)|W^{-1}|\,\delta^3_W(g)(\cosh\zeta)^{2\alpha(W)} \tag{2.21}$$

and the function

$$A_0(r_B, W_{n+1}) = 2^{-2}(2\pi)^6\beta^B(W_{n+1})\,\tilde{\Delta}(W_{n+1}, M_B^2, \mu^2)\,\frac{\delta^3_{M_B^2}(r_B)}{M_B^2} \tag{2.22}$$

we can write the formula (2.18) as

$$\chi_{n+2}(f^B, f^A) = g^{2(n+2)}\int \mathrm{d}a_B\,\mathrm{d}a_A\bar{f}(a_B)f(a_A)\int \mathrm{d}W_1\ldots\mathrm{d}W_{n+1}\cdot \tag{2.23}$$
$$\cdot\,\mathrm{d}a_0\ldots\mathrm{d}a_{n+1}B_0(a_A, a_1, W_1)\ldots G(W_n, a_n^{-1}a_{n+1}, W_{n+1})A_0(r_B, W_{n+1})\,.$$

It can be shown that the functionals $\chi_n(f^B, f^A)$ are bounded provided $\mathrm{Re}\,\alpha(0) < -\frac{1}{2}$; in order to do this it is convenient to use a symmetrized kernel

$$\tilde{G}(W', a'^{-1}, a, W) = \left|\frac{W}{W'}\right|^{\frac{1}{2}} G(W', a'^{-1}a, W)\,, \tag{2.24}$$

so we obtain

$$\chi_{n+2}(f^B, f^A) = g^{2(n+2)}\int \mathrm{d}a_B\,\mathrm{d}a_A\bar{f}(a_B)f(a_A)\int \mathrm{d}W_1\ldots\mathrm{d}W_{n+1}\cdot \tag{2.25}$$
$$\cdot\,\mathrm{d}a_0\ldots\mathrm{d}a_{n+1}\tilde{B}_0(a_A, a_1, W_1)\ldots\tilde{G}(W_n, a_n^{-1}, a_{n+1}, W_{n+1})\tilde{A}_0(r_B, W_{n+1})\,,$$

where

$$\begin{cases} \tilde{B}_0(a_A, a_1, W_1) = |W_1|^{\frac{1}{2}}B_0(a_B, a_1, W_1)\,,\\ \tilde{A}_0(r_A, W_{n+1}) \;\;= |W_{n+1}|A_0(r_B, W_{n+1})\,. \end{cases} \tag{2.26}$$

Then we can write

$$|\chi_{n+2}(f^B, f^A)| < |g|^{2(n+2)}\|B_0\|\,\|\tilde{G}\|^n\,\|\tilde{A}_0\|\,\|f^B\|\,\|f^A\|\,. \tag{2.27}$$

The norms of the integral operators $\widetilde{G}$, $\widetilde{B}_0$, $\widetilde{A}_0$ satisfy the following inequalities ([11]):

$$(2.28)\qquad \begin{cases} \|\widetilde{G}\| \leqslant \left[\int \mathrm{d}W\,\mathrm{d}W'\left(\int \mathrm{d}a|\widetilde{G}(W, a, W')|\right)^2\right]^{\frac{1}{2}}, \\ \|\widetilde{B}_0\| \leqslant \left[\int \mathrm{d}W\left(\int \mathrm{d}a|\widetilde{B}_0(a, W)|\right)^2\right]^{\frac{1}{2}}, \\ \|\widetilde{A}_0\| \leqslant \left[\int \mathrm{d}W\left(\int \mathrm{d}r_B|\widetilde{A}_0(r_B, W)|\right)^2\right]^{\frac{1}{2}}, \end{cases}$$

and the integrals converge if $\operatorname{Re}\alpha(0) < -\frac{1}{2}$. Then, for these values of $\alpha(0)$, the functionals are bounded: they are the matrix elements of bounded operators ([12]), so we can study the convergence problem in the operator norm (uniform topology). Consider $\operatorname{Re}\alpha(0) < -\frac{1}{2}$, performing the $O_{3,1}$ expansion of the bounded functionals ([13]) $\chi_{n+2}(f^B, f^A)$, the unitarity equation at zero momentum transfer reads

$$(2.29)\qquad \Phi(f^B, f^A) = \sum_{n=0}^{\infty} g^{2(n+2)} \sum_M \int_{-i\infty}^{i\infty} \mathrm{d}\lambda (M^2 - \lambda^2)\, \chi_{n+2}^{M\lambda}(f^{BM\lambda}, f^{AM\lambda}),$$

where the projected functionals are bounded and generated by Hilbert-Schmidt operators ([12])

$$(2.30)\qquad \chi_{n+2}^{M\lambda}(f^{BM\lambda}, f^{AM\lambda}) = (f^{BM\lambda}, \widetilde{A}^{M\lambda}\, \widetilde{G}^{M\lambda^n}\, \widetilde{B}_0^{M\lambda} f^{AM\lambda}).$$

If we choose $f^B(a_A)$, $f^A(a_A)$ in the set of continuous functions with compact support, which is a dense set in L^2, these functionals are analytic in the strip $|\operatorname{Re}\lambda| < -2\alpha(0)$, and are uniformly bounded with respect to λ in every closed strip inside $|\operatorname{Re}\lambda| < -2\alpha(0)$. For $g^2 < \|\widetilde{G}\|^{-1}$ the projected series converges in the operator norm, uniformly with respect to g and λ, M so exchanging the integration with the sum in (28) we have

$$(2.31)\qquad \Phi(f^B, f^A) = \sum_M g^4 \int_{-i\infty}^{i\infty} \mathrm{d}\lambda (M^2 - \lambda^2) \sum_{n=0}^{\infty} (f^{BM\lambda}, \widetilde{A}_0^{M\lambda}\, \widetilde{G}^{M\lambda^n} B_0^{M\lambda} f^{AM\lambda}) =$$

$$= \sum_M g^4 \int_{-i\infty}^{i\infty} \mathrm{d}\lambda (M^2 - \lambda^2)(f^{BM\lambda}, \widetilde{A}_0^{M\lambda}[\widetilde{B}_0^{M\lambda} + \widetilde{R}^{M\lambda}\, \widetilde{B}_0^{M\lambda}] f^{AM\lambda})$$

([11]) This result can be obtained by using some results of R. L. LIPSAN: *Bull. Am. Math. Soc.*, **7**, 652 (1967).

([12]) M. A. NAIMARK: *Normed Rings*, Chap. I (Groningen, 1964), p. 90.

([13]) See ref. ([6]).

and the resolvent operator $\tilde{R}^{M\lambda}$ satisfies the equation

$$\tilde{R}^{M\lambda} = \tilde{G}^{M\lambda} + \tilde{R}^{M\lambda}\,\tilde{G}^{M\lambda}\,; \tag{2.32}$$

its kernel satisfies the corresponding equation

$$\tilde{R}^{M\lambda}_{J'm'Jm}(W', W) = g^2\,\tilde{G}^{M\lambda}_{J'm'Jm}(W', W) + + \sum_{J''m''} g^2 \int \tilde{R}^{M\lambda}_{J'm'J''m''}(W', W)\,\tilde{G}^{M\lambda}_{J''m''Jm}(W'', W)\,\mathrm{d}W''\,. \tag{2.33}$$

When g increases in such a way that $|g|^2 > \|\tilde{G}\|^{-1}$ the functional (2.31) becomes unbounded ([10]) and the projected sum exhibits poles which cross the integration path in the inversion formula in (2.31). However the quantity $\Phi(f^B, f^A)$ which has a well-defined physical meaning (see (2.2) and (2.3)) in correspondence to pseudostates, is finite so we may evaluate it performing an analytic continuation in g. To do this we rewrite (2.31) as

$$\Phi(f^B, f^A) = \sum_M g^2 \int_{-i\infty}^{i\infty} \mathrm{d}\lambda (M^2 - \lambda^2)\,\Psi^{M\lambda}(\tilde{A}_0^{M\lambda}\tilde{B}_0^{M\lambda} + \tilde{A}_0^{M\lambda}\tilde{R}^{M\lambda}\tilde{B}_0^{M\lambda})\,, \tag{2.34}$$

where the functions $\Psi^{M\lambda}$ are so defined

$$\Psi^{M\lambda} = (f^{B^{M\lambda}}, f^{A^{M\lambda}}) \tag{2.35}$$

and are analytic ([6]) in the strip $|\mathrm{Re}\,\lambda| \leqslant 2$ if $f^B \in \mathscr{L}^B$, $f^A \in \mathscr{L}^A$. The analytic continuation in g is obtained by modification of the integration path in the right-hand side of (2.34), using the analyticity properties of the resolvent $\tilde{R}^{M\lambda}$ and of the functions $\Psi^{M\lambda}$.

3. – The asymptotic approximation.

In order to investigate the unitarity series to higher values of $\mathrm{Re}\,\alpha(0)$ $(\mathrm{Re}\,\alpha(0) \geqslant -\frac{1}{2})$ we assume the asymptotic approximation for the projected kernel. Consider $\mathrm{Re}\,\alpha(0) < 0$, the projection of the kernel $\tilde{G}$ is given by ([14])

$$\tilde{G}^{M\lambda}_{J'm'Jm}(W', W) = \int_{O_{3,1}} \tilde{G}(W', a, W)\,\mathscr{D}^{M\lambda}_{J'm'Jm}(a)\,\mathrm{d}a\,. \tag{3.1}$$

([14]) The explicit expression of the matrix elements of the irreducible representations of $O_{3,1}$ can be found in A. SCIARRINO and M. TOLLER: *Journ. Math. Phys.*, **8**, 1252 (1967).

For $W > 0$ the integration is over the compact group O_3 and the result defines an entire function of λ in all the λ-plane, so it is consistent with our approximation to completely disregard this contribution. For $W < 0$ the integration is over the noncompact group $O_{2,1}$ and the integral defines an analytic function of λ in the strip $|\mathrm{Re}\,\lambda| < -2\alpha(0)$ $(\alpha'(0) \geqslant 0)$ (15):

$$(3.2)\qquad \tilde{G}^{M\lambda}_{J'm'Jm}(W', W) = \beta(W', W)\frac{\Delta^{\frac{1}{2}}(W', W, \mu^2)}{|WW'|^{\frac{1}{2}}}\,\delta_{0m}\,\delta_{0m'}(\cosh\zeta)^{2\alpha(W)}\mathscr{D}_{J'0J0}(qg)\,\mathrm{d}g\,.$$

It is easy to find that

$$q a_x(\zeta) \underset{\zeta\to\infty}{=} a_z(\chi)\,a_x(\zeta) \sim u_p(\theta)\,a_z(\eta)\,u_p\left(-\frac{\pi}{2}\right),$$

where $\cosh\eta = \cosh\chi\cosh\zeta$ and $\mathrm{tg}(\theta/2) = e^{-\chi}$, $\cosh\hbar\chi = \mu^2 - W - W'/4|WW'|^{\frac{1}{2}}$.

Using the asymptotic approximation for the representation matrices (14)

$$(3.3)\qquad \mathscr{D}^{M\lambda}_{J'0J0}(q a_x(\zeta)) \simeq \frac{1}{2}\,\mathscr{R}^{J'}_{0M}(\theta)\,a^{M\lambda}_{J'}\,b^{M\lambda}_{J}\,\mathscr{R}^{J}_{M0}\left(-\frac{\pi}{2}\right)(\cosh\chi\cosh\zeta)^{-\lambda-1}$$

we obtain (we put $\mathscr{D}^{M\lambda}_{J'0J0}(a) = \mathscr{D}^{M\lambda}_{J'J}(a)$)

$$(3.4)\qquad G^{M\lambda}_{J'J}(W', W) = \frac{1}{2}(\cosh\chi)^{-\lambda-1}\mathscr{R}^{J'}_{0M}(\theta)\,a^{M\lambda}_{J'}\,b^{M\lambda}_{J}\,\mathscr{R}^{J}_{M0}\left(-\frac{\pi}{2}\right)\frac{\beta(W', W)}{|WW'|^{\frac{1}{2}}}\,\frac{\Delta^{\frac{1}{2}}(W, W', \mu^2)}{\lambda - 2\alpha(W)}\,.$$

This is just the pole approximation considered by CHEW and FRAZER: we observe that this kernel is justified only for λ near $2\alpha(0)$. In fact this approximation describes exactly the part of the kernel which is singular near this point but an important nonsingular correction may be present. In particular it does not possess the symmetry property for $\lambda \to -\lambda$. We will discuss later, in the analytic continuation in $\alpha(0)$ and g, the possible effect of neglecting mirror and satellite cuts.

In this approximation the analyticity properties of the projected resolvent derive from the Fredholm theory of compact operators.

The Hilbert-Schmidt condition for the kernel is

$$(3.5)\qquad \sum_{J'J}\int \mathrm{d}W\,\mathrm{d}W'\,|\tilde{G}^{M\lambda}_{J'J}(W', W)|^2 < \infty\,.$$

(15) For $W<0$ $\theta(\Delta(W, W', \mu^2)) = 1$, so $\tilde{\Delta}(W, W', \mu^2) = \Delta^{\frac{1}{2}}(W, W', \mu^2)$.

It can be shown that

$$\mathscr{R}^{J'}_{0M}(\theta)\, a^{M\lambda}_{J'} b^{M\lambda}_{J} \mathscr{R}^{J}_{M0}\left(-\frac{\pi}{2}\right) \sim J^{2\lambda}, \tag{3.6}$$

so we use the symmetrized kernel

$$G^{M\lambda}_{J'J}(W', W) = J'^{-\lambda}\, \tilde{G}^{M\lambda}_{J'J}(W', W)\, J^{\lambda}. \tag{3.7}$$

The sum over J', J converges only if $\operatorname{Re}\lambda < -\frac{1}{2}$. In order to work with $\operatorname{Re}\lambda \geqslant -\frac{1}{2}$ we observe that the kernel (3.7) is factorized with respect to J', J so also the resolvent must have the same form

$$R^{M\lambda}_{J'J}(W', W) = \mathscr{R}^{J'}_{0M}(\theta)\, a^{M\lambda}_{J'} R^{M\lambda}(W', W)\, b^{M\lambda}_{J} \mathscr{R}^{J}_{M0}\left(-\frac{\pi}{2}\right) \tag{3.8}$$

and $R^{M\lambda}(W', W)$ satisfies the integral equation

$$R^{M\lambda}(W', W) = G^{M\lambda}(W', W) - \int \mathrm{d}W''\, R^{M\lambda}(W', W'')\, G^{M\lambda}(W'', W)\, S^{M\lambda}(W'', W), \tag{3.9}$$

where

$$G^{M\lambda}(W', W) = \frac{1}{2}\, \frac{\beta(W', W)}{\lambda - 2\alpha(W)}\, \frac{1}{|WW'|^{\frac{1}{2}}}\, \Delta^{\frac{1}{2}}(W', W, \mu^2) \tag{3.10}$$

and

$$S^{M\lambda}(W', W) = \sum_{J} b^{M\lambda}_{J} \mathscr{R}^{J}_{M0}\left(-\frac{\pi}{2}\right) \mathscr{R}^{J}_{M0}(\theta)\, a^{M\lambda}_{J} = (\operatorname{tgh}\chi)^{-\lambda-1} d^{\lambda}_{0M}(\operatorname{ctgh}\chi). \tag{3.11}$$

The function d^{λ}_{0M} is a representation matrix of the group $O_{2,1}$ [14]. The result (3.11) is not trivial; in fact the (infinite) sum in the right-hand side can be performed only by means of an extension of the theorems given in ref. [16] on the traces of the irreducible representation operators of $SL_{2,C}$.

The kernel of the eq. (3.9) is given by

$$K^{M\lambda}(W', W) = \frac{\beta(W', W)}{\lambda - 2\alpha(W)}\, |WW'|^{\lambda/2} [\Delta(W', W, \mu^2)]^{-\lambda/2} d^{\lambda}_{0M}(\operatorname{ctgh}\chi) \tag{3.12}$$

and the residues $\beta(W', W)$ are singular at $W', W = 0$ and behave like $|W'|^{-\alpha(W')}|W|^{-\alpha(W)}$.

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[16] M. H. NAIMARK: *Linear Representations of the Lorentz Group*, Chap. III (London, 1964), p. 137.

Then the kernel (3.12) is of H. S. type ([17]) for $\operatorname{Re}\lambda > 2\operatorname{Re}\alpha(0) - |M| - 1$ with a cut starting from $2\alpha(0)$ and going to the left (A.F.S. cut). We observe that the equation for the resolvent is of Fredholm type near the cut where the asymptotic approximation is very good. However the analytic properties of the resolvent of the eq. (3.9) in all the strip $|\operatorname{Re}\lambda| < 2$ can be investigated (for any value of $\alpha(0)$ which does not violate the Froissart bound) following a technique developed by TIKTOPOULOS ([18]). It is possible to separate the kernel in two parts:

$$K^{M\lambda}(W', W) = C^{M\lambda}(W', W) + H^{M\lambda}(W', W) , \tag{3.13}$$

where $H^{M\lambda}(W, W')$ is a H.S. kernel for $\operatorname{Re}\alpha(0) < 1$ and $|\operatorname{Re}\lambda| < 2$ and $C^{M\lambda}(W', W)$ is of finite rank:

$$C^{M\lambda}(W', W) = \sum_{i=1}^{4} \psi_i^{M\lambda}(W', \alpha(0)\, \bar{\varphi}_i^{M\lambda}(W, \alpha(0)) . \tag{3.14}$$

This decomposition can be carried out using an expansion near W, $W' = 0$ for the terms of the kernel which are not factorized.

The resolvent $R_C^{M\lambda}(W', W)$ of $C^{M\lambda}(W', W)$ has the form

$$R_C^{M\lambda}(W', W) = \frac{1}{d(M, \lambda, \alpha(0))} \sum_{i=1}^{4} f_i(M, \lambda, \alpha(0))\, \psi_i^{M\lambda}(W', \alpha(0))\, \bar{\varphi}_i^{M\lambda}(W, \alpha(0)) , \tag{3.15}$$

where $d(M, \lambda, \alpha(0))$, the Fredholm determinant of the factorized kernel $C^{M\lambda}$, and the functions $f_i(M, \lambda, \alpha(0))$ are constructed by means of the scalar products of the « vectors » $\psi_i^{M\lambda}(W, \alpha(0))$ and $\bar{\varphi}_i^{M\lambda}(W, \alpha(0))$. These last functions are meromorphic with poles at $\lambda = 2\alpha(0) - |M| - n$ which are cancelled by the same poles which are present in the function $d(M, \lambda, \alpha(0))$ so the function $R_C^{M\lambda}(W', W)$ is meromorphic with the poles given by the zeros of $d(M, \lambda, \alpha(0))$. The resolvent operator of $K^{M\lambda}$ is given by

$$R^{M\lambda} = (I + R_C^{M\lambda})[I - H^{M\lambda}(I + R_C^{M\lambda})]^{-1} - I . \tag{3.16}$$

Then we have that $R^{M\lambda}$ is analytic in λ, α, g until

$$\operatorname{Re}\alpha(0) < -\frac{1}{2}, \qquad |\operatorname{Re}\lambda| < 2 , \qquad |g|^2 < \frac{1}{\|G\|} ,$$

([17]) The $|M|$-dependence arises from the fact that the function $d^{\lambda}_{0M}(W, W')$ behaves like $(WW')^{|M|/2}$ for $W, W' \sim 0$.

([18]) G. TIKTOPOULOS: *Phys. Rev.*, **133** B, 413 (1964).

apart from a cut starting from $2\alpha(0)$. When $\alpha(0)$ and g increase the resolvent becomes meromorphic in the strip $|\mathrm{Re}\lambda|<2, \mathrm{Re}\alpha(0)<1$. The analytic continuation of the pseudolinear functional Φ is obtained by modification of the integration path in the inversion formula (2.34). However we observe that in the analytic continuation the contribution of the mirror cut at $\lambda=-2\alpha(0)$ and of the satellite cuts at $\lambda=\pm 2\alpha(0)-n$ cannot be neglected and the asymptotic approximation is no longer valid. In particular a satellite mirror cut can lie near the Lorentz cut at $\lambda=2\alpha(0)$ and could in principle give a big contribution. Only a future deeper analysis can answer this question. In this connection we note that in the Bethe-Salpeter [19] ladder model the Regge and the mirror poles exhibit a sort of orthogonality which forbid their interference.

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([19]) L. SERTORIO and M. TOLLER: *Nuovo Cimento*, **33**, 413 (1964).

RIASSUNTO

Si dà una chiara formulazione matematica del problema della convergenza della serie dell'unitarietà, saturata con grafici multiperiferici reggeizzati. Una dettagliata trattazione viene data per l'approssimazione asintotica.

О реджеизованной мультипериферической модели.

Резюме (*). — Четким математическим образом формулируется проблема сходимости унитарных рядов, насыщенных с помощью реджеизованных мультипериферических графиков. Широко рассматривается асимптотическое приближение.

(*) *Переведено редакцией.*