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A. Renieri : TWO BEAM TRANSVERSE COHERENT INSTABILITY IN COLLIDING BEAM RINGS. -

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INTRODUCTION. -

This work deals with a new transverse instability which can appear in the electron-positron or electron-electron storage rings. This instability can grow only when two beams are intersecting in the vacuum chamber.

In the dynamics of the instability we have two different mechanism,

1) every particle of one bunch is affected by the wake field due to all the preceding particles of the same bunch,
2) at the crossing, every particle of one bunch gives an information related to its transverse displacement from the unperturbed orbit to the particles of the other bunch.

We assume that:

a) one bunch of electrons and one bunch of positrons are circulating in the ring,
b) the two bunches have the same length,
c) the wake fields decay in a time shorter than the revolution time; we neglect the effect of the wake field of one bunch on the other one,
d) we neglect the synchrotron motion,
e) the crossing angle is zero (head-on collision),
f) we neglect the radiation damping (in the equations of motion),
g) we suppose that the radial motion is not coupled with the vertical one.

The condition b) is very well verified (at least in Adone), because, when the beams are intersecting the currents are well below the threshold for the bunch lengthening effect.

We are interested in the wake fields which satisfy the condition c), because these fields can give an instability with the center of mass of the bunch at rest.

Under these conditions, in particular c) and d), the first mechanism cannot generate any instability without the second one, and viceversa. The second mechanism alone, for example, can generate simply a real shift of the oscillation betatron frequency, because there is no delay in the signal transmission; on the other hand if the coupling between the beams is missing, the leading particle cannot see the trailing one, and there is no regenerative mechanism.

I. - THE EQUATIONS OF MOTION. -

Let us consider two bunches of N^+ positrons and of N^- electrons, whose longitudinal density is $\rho^{\pm} = N^{\pm}/L$. We will label the particles with,

σ = longitudinal coordinate $(0 \le \sigma \le L)$
ξ = vertical betatron oscillation amplitude (so that the frequency spread due to octupolar terms can be taken into account).

We will assume that in the vacuum chamber there are M elements generating the wake fields. We will represent all the forces (wake field and bunch-bunch interaction) with δ-functions.

The equations of motion are:

$$\ddot{x}^{\pm}(\xi,\sigma,t) + \bar{\nu}^{\pm 2}(\xi)\, x^{\pm}(\xi,\sigma,t) = K_0^{\pm} \sum_{n=0}^{\infty} \int_0^L \frac{d\lambda}{L} \int_0^{\infty} f(\mu)\, d\mu \;\times$$
$$\times \left\{ \delta\left(t - \frac{2\pi}{\omega_0} n + \frac{\sigma+\lambda}{2}\right) + \delta\left(t - \frac{2\pi}{\omega_0} n - \frac{\pi}{\omega_0} + \frac{\sigma+\lambda}{2}\right) \right\} \times \tag{1}$$
$$\times \left\{ x^{\pm}(\xi,\sigma,t) - x^{\mp}(\mu,\lambda,t) \right\} + \sum_{i=1}^{M} \int_{\sigma}^{L} \frac{d\lambda}{L} \int_0^{\infty} f(\mu)\, d\mu\; \rho_i^{\pm}(\lambda-\sigma) \;\times$$
$$\times\, \delta\left(t - \frac{2\pi}{\omega_0} n - \eta_i^{\pm} + \sigma\right) x^{\pm}(\mu,\lambda,t-(\lambda-\sigma)) ,$$

with the notations:

a) the signs $\pm$ refer to positrons and electrons respectively;

b) $\bar{\nu}^{\pm 2}(\xi) = \nu^2(1 + K^{\pm}\xi^2)$ where ν is the oscillation betatron frequency and $K^{\pm}$ is the octupolar term

(2) $$K^{\pm} = kN^{\mp} + k_{\mu}$$

where $kN^{\mp}$ is the octupolar term due to the bunch-bunch interaction, while k_{μ} is the octupolar term due to the non linearity of the guide and focusing fileds;

c) $K_0^{\pm}$ = bunch-bunch strenght factor

(3) $$K_0^{\pm} = k_0 N^{\mp} ;$$

d) $f(\xi)$ = normalized distribution function of the betatron oscillation amplitudes

(4) $$\int_0^{\infty} f(\xi)\, d\xi = 1$$

we take for $f(\xi)$ the unperturbed distribution function;

e) ω_0 = angular revolution frequency;

f) $\rho_i^{\pm}(\lambda - \sigma)$ = wake field form factors

(5) $$\rho_i^{\pm}(\lambda-\sigma) = N^{\pm}\bar{\rho}_i(\lambda-\sigma) ;$$

g) $\eta_i^{\pm}$ = longitudinal time coordinates of the wake field generating elements ($\eta_i^{-} = \frac{2\pi}{\omega_0} - \eta_i^{+}$);

h) L = time length of the bunches .

(See Appendix E for the expression of $K^{\pm}$, $K_o^{\pm}$).

We make also the assumption

$$x^{\pm}(\zeta, \sigma, t < 0) = 0 , \qquad (6)$$

for this reason the sums start from zero and not from $-\infty$. Condition (6) will enable us to use Laplace transform.

The differences between this work and the treatment due to Pellegrini and Sessler[1] are essentially related to the time decay of the wake field and to the non-linearities of the bunch-bunch interaction. In the work[1] the wake field decay is assumed to be longer than the revolution time, so that it is reasonable to consider the bunches as rigid; in the same work is neglected the octupolar term that arises from the bunch-bunch interaction.

II. - THE METHOD OF RESOLUTION. -

We shall derive the collective motion frequencies from eq. (1). The method, which is developed in Appendix A, essentially consists of three steps,

1) we derive the Laplace transform of both members of eq. (1);
2) we find the poles, in the "p" plane, of the transforms of the center of mass coordinates of the bunch longitudinal elements;
3) we obtain the couple of dispersion relations

$$\frac{1}{2\nu^2 K^+ a^2}\int_0^{\infty}\frac{f(\zeta)\,d\zeta}{\frac{\zeta^2}{2a^2}-y^+} = G^+ , \qquad \frac{1}{2\nu^2 K^- a^2}\int_0^{\infty}\frac{f(\zeta)\,d\zeta}{\frac{\zeta^2}{2a^2}-y^-} = G^- , \qquad (7)$$

where G^+ and G^- satisfy the equation (see Appendix B, eq. (B. 7):

$$T(G^+, G^-) = 0 , \qquad (8)$$

with the position

$$a = \text{transverse beam dimension}$$

$$y^{\pm} = \frac{1}{2\nu^2 K^{\pm} a^2}\left(2\nu(\omega - i\delta) + \frac{K_o^{\pm}\omega_o}{2\pi}\right) , \qquad (9)$$

where ω and δ are the real and the imaginary shift of the collective oscillation frequency

$$p\Big|_{pole} = i(\nu + n\omega_o + \omega) + \delta \qquad (n = \text{integer}). \qquad (10)$$

III. - BUNCHES WITH $N^+ = N^-$. -

If $N^+ = N^- = N$, eq. (7) and (8) become :

$$\frac{1}{2\nu^2 K a^2}\int_0^{\infty}\frac{f(\zeta)\,d\zeta}{\frac{\zeta^2}{2a^2}-y} = G \qquad (11)$$

(12) $$T(G,G) = \overline{T}(G) = 0,$$

where we put $K^+ = K^- = K$, $y^+ = y^- = y$.

Let us assume

(13) $$\bar{\varsigma}_i(\lambda - \sigma) = k_i e^{-\frac{\lambda - \sigma}{\tau}} .$$

Under this condition eq. (12) becomes (see Appendix B):

(14) $$1 = \frac{e^{G\overline{H} - \mu L} - 1}{G\overline{H} - \mu L} \; \frac{G^2 H\overline{H}}{\mu LGH \pm (G\overline{H} - \mu L)}$$

where we have put

(15) $$\overline{H} = N\overline{h} = \sum_{i=1}^{M} \frac{k_i \omega_o}{4\pi} \; ; \qquad H = Nh = \frac{K_o \omega_o}{2\pi} \; ; \qquad \mu = i\nu + \frac{1}{\tau} .$$

From eq. (14) we derive

(16) $$G(H, \overline{H}, \mu L) = \frac{1}{N\overline{h}} \; S(\mu L, \frac{h}{\overline{h}}) ,$$

where S is the solution of equation

(17) $$1 = \frac{e^{S - \mu L} - 1}{S - \mu L} \; \frac{S^2 (\frac{h}{\overline{h}})}{\mu L S(\frac{h}{\overline{h}}) \pm (S - \mu L)} .$$

Eq. (11) becomes

(18) $$\int_0^\infty \frac{f(\xi)\, d\xi}{\frac{\xi^2}{2a^2} - y} = \frac{2\nu^2 K a^2}{N\overline{h}} \; S(\mu L, \frac{h}{\overline{h}}) .$$

If we substitute eq. (2) in eq. (18), we obtain

(19) $$\int_0^\infty \frac{f(\xi)\, d\xi}{\frac{\xi^2}{2a^2} - y} = \frac{2\nu^2 a^2}{\overline{h}} \; S(\mu L, \frac{h}{\overline{h}})(K + \frac{K_\mu}{N}) .$$

Eq. (19) is a very unusual dispersion relation. Indeed we usually have:

(20) $$\int_0^\infty \frac{f(\xi)\, d\xi}{\frac{\xi^2}{2a^2} - y} = \frac{1}{N(u + iv)} .$$

Eq. (20) gives us the threshold automatically, that is the value of N for which y is real with $v \neq 0$ ($\delta \to 0^-$). Eq. (19) determinates y indipendently from N (practically), because, with the currents which usually are circulating in the ring, we have (see Appendix E):

(21) $$\frac{K_\mu}{N} \approx 10^{-3} K .$$

The threshold will be found with the following method,

1) we derive y from eq. (19) (with a fixed $f(\xi)$ as function of $\frac{2\nu^2 K a^2}{\bar{h}} S(\mu L, \frac{\bar{h}}{h})$;

2) from eq. (9) we derive

(22) $$\delta = -(\nu^2 K a^2 \, \mathrm{II}_m \, y) N .$$

The motion will be stable if $\delta < 0$, that is if

(23) $$K \, \mathrm{II}_m \, y > 0 .$$

If eq. (23) is not satisfied, the threshold will be fixed by the radiation damping, that is from the equation

(24) $$\delta = \frac{1}{\tau_R}$$

where τ_R is the radiation damping time constant.

Thus we obtain

(25) $$N_{th} = \frac{1}{\tau_R \nu |K| a^2 |\mathrm{II}_m y|} .$$

We must solve eq. (17) in order to derive N_{th}. Generally we have (see Appendix E and F):

(26) $$\left| \frac{\bar{h}}{h} \right| << 1 .$$

It is easy to solve eq. (17) under the conditions (26) and in two extreme cases:

(27) $$(1) \quad \frac{L}{\tau} << 1 , \qquad (2) \quad \frac{L}{\tau} >> 1$$

we obtain (see Appendix C):

(28) $$S_1 = \pm \frac{\bar{h}}{h} e^{\pm \frac{i\nu L}{6} \frac{\bar{h}}{h}} , \qquad S_2 = \pm \frac{\bar{h}}{h} e^{\pm i(\nu\tau) \frac{\tau}{L} \frac{\bar{h}}{h}} .$$

Letting

(29) $$\psi_1 = \frac{\nu L}{6} \frac{\bar{h}}{h} , \qquad \psi_2 = (\nu\tau) \frac{\tau}{L} (\frac{\bar{h}}{h}) , \qquad \lambda = \frac{2\nu^2 K a^2}{h}$$

and neglecting $\frac{K_\mu}{N}$ with respect to K, eq. (19) becomes:

(30) $$\int_0^\infty \frac{f(\xi) d\xi}{\frac{\xi^2}{2a^2} - y} = \pm \lambda e^{\pm i \psi_n} \qquad (n = 1, 2)$$

we have (see Appendix E):

(31) $$\lambda \approx 1 , \qquad |\psi_n| << 1 \qquad (\lambda > 0 \text{ always}) .$$

If we put

(32) $$y = r_o e^{i\theta}, \qquad \xi = \sqrt{2}\,a x, \qquad f(\sqrt{2}\,a x) = \frac{g(x)}{\sqrt{2}\,a},$$

eq. (30) becomes:

(33) $$\pm \lambda e^{\pm i\psi_n} = \int_0^\infty \frac{(x^2 - r_o\cos\theta)\, g(x)\, dx}{x^4 + r_o^2 - 2 r_o x^2 \cos\theta} + i\, r_o \sin\theta \int_0^\infty \frac{g(x)\, dx}{x^4 + r_o^2 - 2 r_o x^2 \cos\theta}$$

Our method of resolution will be typically eurystic. If we put

$$\theta = \pi - \phi_n, \qquad \text{with } |\phi_n| << 1,$$

eq. (33) becomes

(34) $$\pm \lambda e^{\pm i\psi_n} \approx \int_0^\infty \frac{g(x)\, dx}{x^2 + r_o} + i\, r_o \phi_n \int_0^\infty \frac{g(x)\, dx}{(x^2 + r_o)^2};$$

from eq. (31) we derive that the sign + must be chosen in eq. (34). Then we obtain:

(35) $$\int_0^\infty \frac{g(x)\, dx}{x^2 + r_o} = \lambda, \qquad \operatorname{arctg} \phi_n = \frac{\int_0^\infty \frac{g(x)\, dx}{x^2 + r_o}}{r_o \int_0^\infty \frac{g(x)\, dx}{(x^2 + r_o)^2}}\, \psi_n .$$

From the first of eq. (35) we derive r_o as function of λ, while from the second one we derive ϕ_n as function of r_o (that is of λ) and ψ_n. If we put

(36) $$\eta = \frac{1}{r_o} \int_0^\infty \frac{g(x)\, dx}{x^2 + r_o} \left\{ \int_0^\infty \frac{g(x)\, dx}{(x^2 + r_o)^2} \right\}^{-1} .$$

we obtain $(\eta(\lambda)\psi_n << 1 \Rightarrow \operatorname{arctg}\phi_n \approx \phi_n)$:

(37) $$r_o = r_o(\lambda), \qquad \phi_n = \eta(\lambda)\psi_n .$$

The stability condition $K\, \Pi_m y > 0$ becomes

(38) $$K\, \Pi_m y = r_o(\lambda)\, \eta(\lambda)\, \psi_n K > 0 ;$$

from eq. (29), (31) and (38) we derive

(39) $$(\text{stability condition}) \Longrightarrow \bar{h} > 0 ;$$

while if we have

(40) $$\bar{h} < 0 ,$$

the threshold is given by (see eq. (25)):

(41) $$N_{th}^{(n)} = \frac{1}{\tau_R \nu |K| a^2 r_o \left(\frac{2\nu^2 K a^2}{h}\right) \eta \left(\frac{2\nu^2 K a^2}{h}\right)} \frac{1}{\psi_n} .$$

Substituting eq. (29) in eq. (41) and letting

(42) $$A = \frac{|h|}{|\bar{h}| \tau_R \nu |K| a^2 r_o \left(\frac{2\nu^2 K a^2}{h}\right) \eta \left(\frac{2\nu^2 K a^2}{h}\right)} ,$$

we obtain

(43) $$N_{th}^{(1)} = A \frac{6}{\nu} \frac{1}{L} \qquad \left(\frac{L}{\tau} \ll 1\right)$$
$$N_{th}^{(2)} = A \frac{1}{\nu L} \left(\frac{L}{\tau}\right)^2 \qquad \left(\frac{L}{\tau} \gg 1\right) .$$

IV. - BUNCHES WITH $N^+ \neq N^-$. -

The resolution of eq. (7) and (8) with $N^+ \neq N^-$ is very difficult. We limit ourselves to the study of the particular case

(44) $$\frac{L}{\tau} \gg 1 .$$

If we put

(45) $$\Delta = \frac{2\nu(\omega - i\delta)}{2\nu^2 K a^2} ,$$

eq. (7) becomes (λ is defined in eq. (29)) :

(46) $$\int_0^\infty \frac{f(\xi)\, d\xi}{\left(\frac{\xi^2}{2a^2} - \frac{1}{\lambda}\right) - \frac{\Delta}{N^\mp}} = \lambda h N^\mp h^\pm .$$

We still assume that $\frac{K_\mu}{N^\pm} \ll K$ (see eq. (21)).

With the condition (44) and letting

(47) $$x^\pm = G^\pm \bar{h} N^\pm , \qquad q = \frac{h}{\bar{h}} ,$$

eq. (8) becomes (see Appendix B) :

(48) $$1 - \frac{\mu L x^+ x^- q^2}{(x^+ - \mu L)(x^- - \mu L) - (\mu L)^2 x^+ x^- q^2} \left(\frac{x^-}{x^- - \mu L} + \frac{x^+}{x^+ - \mu L} \right) =$$
$$= \frac{x^+ x^- q^2}{(x^+ - \mu L)(x^- - \mu L)} \frac{1}{(x^+ - \mu L)(x^- - \mu L) - (\mu L)^2 x^+ x^- q^2} ,$$

where μ is defined in eq. (15).

In the limit $q \gg 1$ (see eq. (26)), the solutions of eq. (48) are (see Appendix C):

(49) $$x^{\pm} = \frac{\mu}{1 + \mu L q \alpha^{\mp 1}} ,$$

where α is a complex parameter which satisfies the equation

(50) $$N^{-} \mathcal{F}(\lambda h N^{-} G^{+}(\alpha)) = N^{+} \mathcal{F}(\lambda h N^{+} G^{-}(\alpha)) ,$$

where $\mathcal{F}(x)$ is the solution of the equation

(51) $$\int_0^{\infty} \frac{f(\xi)\, d\xi}{\left(\frac{\xi^2}{2a^2} - \frac{1}{\lambda}\right) - \mathcal{F}(x)} = x .$$

If we neglect the terms of the second order in $\frac{N^+ - N^-}{N^+ + N^-}$, we obtain (see Appendix D):

(52) $$\delta = - \frac{\lambda}{2\nu} \bar{h} \left(\frac{\tau}{L}\right) r_o(\lambda)\, \eta(\lambda)\, \nu\tau \frac{N^+ N^-}{\left(\frac{N^+ + N^-}{2}\right)} ,$$

where $r_o(\lambda)$ and $\eta(\lambda)$ are defined by eq. (35) and (36). If $\bar{h} < 0$, the threshold is given by:

(53) $$\left. \frac{N^+ N^-}{\left(\frac{N^+ + N^-}{2}\right)} \right|_{th} = \frac{h}{|\bar{h}|\, \tau_R \nu |K| a^2 r_o\left(\frac{2\nu^2 K a^2}{h}\right) \eta\left(\frac{2\nu^2 K a^2}{h}\right)} \frac{1}{\nu\tau} \frac{L}{\tau} .$$

This result is therefore similar to eq. (43), valid for $N^+ = N^-$, with the correspondence

(54) $$N \longrightarrow \frac{N^+ N^-}{\left(\frac{N^+ + N^-}{2}\right)} .$$

Another case of easy solution is when the Landau damping is neglected: letting

(55) $$f(\xi) = \delta(\xi) ,$$

we obtain :

(56) $$r_o(\lambda) = \frac{1}{\lambda} , \quad \eta(\lambda) = 1 \quad \Longrightarrow \quad \left. \frac{N^+ N^-}{\left(\frac{N^+ + N^-}{2}\right)} \right|_{th} = \frac{2L}{\tau^2 \tau_R \bar{h}} ,$$

eq. (56) is obtained without neglecting the terms of the second, and higher, order in

$$\frac{N^+ - N^-}{N^+ + N^-} .$$

V. - CONCLUSIONS. -

The instability mechanism that we have studied in the previous sections can help us to understand the behaviour of the electron and positron beams, when they are in the vecuum chamber simultaneously.

The experimental results of the two beam operation in Adone are:

a) the maximum luminosity, at zero crossing angle, with radial and vertical motion coupled, goes like γ^6, and the corresponding limit current in each beam goes like γ^4;

b) when the crossing angle is different from zero, the limit current is essentially the same as in the case of head-on collision;

c) it has been observed that, when one of the beams is more intense than the limit value, there is only a partial loss of the other beam, which becomes stable again at lower intensity.

This behaviour cannot be explained by the incoherent beam-beam instability, whose limit is related to the strong beam transverse density, while the experimental results tend to suggest a threshold on the beam current.

Point c) above can be interpreted by means of eq. (53), that gives a threshold depending on the product of the two currents.

Points a) and b) can also be explained by the instability studied here; to account for the γ^4 dependence of the threshold current, one has to assume that, in Adone, the wake field time decay is of the same order of magnitude of the bunch length, that is $L/\tau \simeq 1$; remembering that:

$$\frac{Ka^2}{\bar{h}} \quad \text{does not depend on } \gamma \text{ (see Appendix E),}$$

$$\bar{h} \propto \frac{1}{\gamma} \quad \text{(through the relativistic electron mass),} \tag{57}$$

$$L \propto \gamma^{1.5}, \quad \tau_R \propto \frac{1}{\gamma^3} \quad \text{(see reference (2)).}$$

one obtains, for the energy dependence of the threshold current in our model :

$$N_{th} \propto \begin{cases} \gamma^{4-1.5} & \text{when } L/\tau \ll 1 \\ \gamma^{4+1.5} & \text{when } L/\tau \gg 1 . \end{cases} \tag{58}$$

The computation of the absolute value of the current threshold requires the knowledge of the $\bar{h}$, the wake field parameter; in Appendix F $\bar{h}$ is evaluated from the single beam thresholds, assuming that the instability mechanism is the "head-tail" effect(3); the two beam threshold turns out to be, in Adone, approximately 1 mA at 500 MeV, which is very close to the experimental limit (about 1 mA per bunch).

We want to remark that in the case of $L/\tau \ll 1$ the energy dependence of this effect would be masked by the incoherent beam-beam effect, which, in the coupling condition, gives a γ^3 law for the threshold current. We want to point out also that the calculations are based on the hypothesis that the two beams have the same external focusing fields; in the case of different single beam betatron frequencies, the threshold currents should be higher.

In this paper the synchrotron motion has not been taken into account; as it is not required to obtain the regenerative action, we think that the main features of this type of instability do not strongly depend on it. It is however in the intention of the author to complete the study of the dynamics, introducing the synchrotron motion, in order to find a probe for the diagnostics and, possibly, the cure of the instability.

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APPENDIX A - DEVELOPMENT OF THE METHOD. -

The Laplace transform of eq. (1) is

$$\tilde{x}^{\pm}(\xi,\sigma,p)(p^2+\overline{\nu}^{\pm 2}(\xi)) = K_o^{\pm}\int_0^L \frac{d\lambda}{L}\int_0^{\infty} f(\mu)d\mu \sum_{n=0}^{\infty} e^{-p(\frac{2\pi}{\omega_o}n-\frac{\sigma+\lambda}{2})} \times$$

$$\times\left\{x^{\pm}(\xi,\sigma,\frac{2\pi}{\omega_o}n-\frac{\sigma+\lambda}{2}) - x^{\mp}(\mu,\lambda,\frac{2\pi}{\omega_o}n-\frac{\sigma+\lambda}{2})\right\} + K_o^{\pm}\int_0^L \frac{d\lambda}{L} \times$$

(A. 1)
$$\times\int_0^{\infty} f(\mu)d\mu \sum_{n=0}^{\infty} e^{-p(\frac{2\pi}{\omega_o}n+\frac{\pi}{\omega_o}-\frac{\sigma+\lambda}{2})}\left\{x^{\pm}(\xi,\sigma,\frac{2\pi}{\omega_o}n+\frac{\pi}{\omega_o}-\frac{\sigma+\lambda}{2}) -\right.$$

$$\left.- x^{\mp}(\mu,\lambda,\frac{2\pi}{\omega_o}n+\frac{\pi}{\omega_o}-\frac{\sigma+\lambda}{2})\right\} + \sum_{i=1}^{M}\int_{\sigma}^{L}\frac{d\lambda}{L}\int_0^{\infty} f(\mu)d\mu\, \varrho_i^{\pm}(\lambda-\sigma) \times$$

$$\times \sum_{n=0}^{\infty} e^{-p(\frac{2\pi}{\omega_o}n+\eta_i^{\pm}-\sigma)} x^{\pm}(\mu,\lambda,\frac{2\pi}{\omega_o}n+\eta_i^{\pm}-\lambda) + \alpha^{\pm}(\xi,\sigma)p + \beta^{\pm}(\xi,\sigma).$$

where we have

$$\tilde{x}^{\pm}(\xi,\sigma,p) = \int_0^{\infty} e^{-pt}\, x^{\pm}(\xi,\sigma,t)\, dt\,,$$

and

$$\alpha^{\pm}(\xi,\sigma) = x^{\pm}(\xi,\sigma,0^+)\,, \qquad \beta^{\pm}(\xi,\sigma) = \dot{x}^{\pm}(\xi,\sigma,0^+)\,,$$

where we put

$$\lim_{t\to 0^+} f(t) = f(0^+)\,.$$

It is easy to show that

$$\sum_{n=0}^{\infty} e^{-p\alpha n} f(\alpha n-\lambda) = \frac{1}{2\alpha}\sum_{n=-\infty}^{+\infty}\tilde{f}(p-\frac{2\pi i}{\alpha}n)\, e^{-(p-\frac{2\pi i}{\alpha}n)\lambda}$$

If we put

$$X^{\pm}(\xi,\sigma,\lambda,p) = \sum_{n=-\infty}^{+\infty}\tilde{x}^{\pm}(\xi,\sigma,p-i\omega_o n)\, e^{-(p-i\omega_o n)\lambda}$$

eq. (A. 1) becomes:

$$\tilde{x}^{\pm}(\xi,\sigma,p)(p^2+\overline{\nu}^{\pm 2}(\xi)) = \frac{K_o^{\pm}\omega_o}{4\pi}\int_0^L\frac{d\lambda}{L}\int_0^{\infty} f(\mu)d\mu\, e^{p(\frac{\sigma+\lambda}{2})}\left\{X^{\pm}(\xi,\sigma,\frac{\sigma+\lambda}{2},p) -\right.$$

(A. 2)
$$\left.- X^{\mp}(\mu,\lambda,\frac{\sigma+\lambda}{2},p)\right\} + \frac{K_o^{\pm}\omega_o}{4\pi}\int_0^L\frac{d\lambda}{L}\int_0^{\infty} f(\mu)d\mu\, e^{p(\frac{\sigma+\lambda}{2}-\frac{\pi}{\omega_o})} \times$$

(A.2)
$$\times\left\{X^{\pm}\left(\xi,\sigma,\frac{\sigma+\lambda}{2}-\frac{\pi}{\omega_0},p\right) - X^{\mp}\left(\mu,\lambda,\frac{\sigma+\lambda}{2}-\frac{\pi}{\omega_0},p\right)\right\} + \sum_{i=1}^{M}\frac{\omega_0}{4\pi}\int_{\sigma}^{L}\frac{d\lambda}{L}\int_0^{\infty} f(\mu)d\mu \;\times$$
$$\times \quad X^{\pm}(\mu,\lambda,\lambda-\eta_i^{\pm},p)\, e^{p(\sigma-\eta_i^{\pm})}\, \rho_i^{\pm}(\lambda-\sigma) + \alpha^{\pm}(\xi,\sigma)p + \beta^{\pm}(\xi,\sigma).$$

We have

$$X^{\pm}(\xi,x,y,p-i\omega_0 n) = X^{\pm}(\xi,x,y,p) \qquad (n = \text{integer}).$$

In this way, with some algebra, we obtain from eq. (A.2) :

(A.3)
$$X^{\pm}(\xi,\sigma,y,p) = \frac{K_0^{\pm}\omega_0}{4\pi}\int_0^L\frac{d\lambda}{L}\int_0^{\infty}f(\mu)\,d\mu\;\; A^{\pm}\left(\xi,\frac{\sigma+\lambda}{2}-y,p\right)\left\{X^{\pm}\left(\xi,\sigma,\frac{\sigma+\lambda}{2},p\right) -\right.$$
$$\left. - X^{\mp}\left(\mu,\lambda,\frac{\sigma+\lambda}{2},p\right)\right\} + \frac{K_0^{\pm}\omega_0}{4\pi}\int_0^L\frac{d\lambda}{L}\int_0^{\infty}f(\mu)\,d\mu\;\; A^{\pm}\left(\xi,\frac{\sigma+\lambda}{2}-\frac{\pi}{\omega_0}-y,p\right) \times$$
$$\times\left\{X^{\pm}\left(\xi,\sigma,\frac{\sigma+\lambda}{2}-\frac{\pi}{\omega_0},p\right) - X^{\mp}\left(\mu,\lambda,\frac{\sigma+\lambda}{2}-\frac{\pi}{\omega_0},p\right)\right\} +$$
$$+\sum_{i=1}^{M}\frac{\omega_0}{4\pi}\int_{\sigma}^{L}\frac{d\lambda}{L}\int_0^{\infty}f(\mu)d\mu\;\; A^{\pm}(\xi,\sigma-\eta_i^{\pm}-y,p)\;\; X^{\pm}(\mu,\lambda,\lambda-\eta_i^{\pm},p) \;\times$$
$$\times \quad \rho_i^{\pm}(\lambda-\sigma) + Q^{\pm}(\xi,\sigma,y,p),$$

where

(A.4)
$$A^{\pm}(\xi,x,p) = \sum_{n=-\infty}^{+\infty}\frac{e^{(p-i\omega_0 n)x}}{(p-i\omega_0 n)^2+\bar{\nu}^{\pm 2}(\xi)}$$

(A.5)
$$Q^{\pm}(\xi,\sigma,x,p) = \sum_{n=-\infty}^{+\infty}\frac{\alpha^{\pm}(\xi,\sigma)(p-i\omega_0 n)+\beta^{\pm}(\xi,\sigma)}{(p-i\omega_0 n)^2+\bar{\nu}^{\pm 2}(\xi)}\; e^{-(p-i\omega_0 n)x}$$

We look for a solution of eq. (15) in the neighbourhood of

$$p = i(\nu + n\,\omega_0 + \omega) + \delta \qquad (n = \text{integer})$$

where we have

(A.6)
$$|\omega| \ll \nu \quad , \quad |\delta| \ll \nu$$

From the eq. (A.6) and from the equation

(A.7)
$$\bar{\nu}^{\pm 2}(\xi) = \nu^2(1 + K^{\pm}\xi^2),$$

we obtain :

$$A^{\pm}(\xi,x,p) \approx \frac{-e^{i\nu x}}{2\nu(\omega - i\delta) - \nu^2 K^{\pm}\xi^2}$$

$$Q^{\pm}(\xi,\sigma,x,p) \approx -\frac{\alpha^{\pm}(\xi,\sigma) i\nu + \beta^{\pm}(\xi,\sigma)}{2\nu(\omega - i\delta) - \nu^2 K^{\pm}\xi^2} e^{-i\nu x}$$

If we put

$$a^{\pm}(\xi,p) = \frac{-1}{\nu(2(\omega - i\delta) - \nu K^{\pm}\xi^2)} \ , \quad b^{\pm}(\xi,\sigma,p) = -\frac{\alpha^{\pm}(\xi,\sigma) i\nu + \beta^{\pm}(\xi,\sigma)}{\nu(2(\omega - i\delta) - \nu K^{\pm}\xi^2)}$$

$$Y^{\pm}(\xi,\sigma,y,p) = X^{\pm}(\xi,\sigma,y,p)\, e^{i\nu y} \ ,$$

eq. (15) becomes :

$$\begin{aligned} Y^{\pm}(\xi,\sigma,y,p) = {} & \frac{K_o^{\pm}\omega_o}{4\pi} a^{\pm}(\xi,p) \int_0^L \frac{d\lambda}{L} \int_0^\infty f(\mu) d\mu \left\{ Y^{\pm}\left(\xi,\sigma,\frac{\sigma+\lambda}{2},p\right) - \right. \\ & \left. - Y^{\mp}\left(\mu,\lambda,\frac{\sigma+\lambda}{2},p\right)\right\} + \frac{K_o^{\pm}\omega_o}{4\pi} a^{\pm}(\xi,p) \int_0^L \frac{d\lambda}{L} \int_0^\infty f(\mu) d\mu \left\{ Y^{\pm}\left(\xi,\sigma,\frac{\sigma+\lambda}{2} - \frac{\pi}{\omega_o},p\right) - \right. \\ & \left. - Y^{\mp}\left(\mu,\lambda,\frac{\sigma+\lambda}{2} - \frac{\pi}{\omega_o},p\right)\right\} + \sum_{i=1}^{M} \frac{\omega_o}{4\pi} a^{\pm}(\xi,p) \int_\sigma^L \frac{d\lambda}{L} \int_0^\infty f(\mu) d\mu \ \times \\ & \times \ \rho_i^{\pm}(\lambda - \sigma)\, e^{i\nu(\sigma - \lambda)}\, Y^{\pm}(\mu,\lambda,\lambda - \eta_i^{\pm},p) + b^{\pm}(\xi,\sigma,p). \end{aligned} \tag{A. 8}$$

From eq. (A. 8) we obtain:

$$Y^{\pm}(\xi,\sigma,y,p)\Big|_{\xi,\sigma,p = \text{const}} = \text{const}$$

So we can put

$$Z^{\pm}(\xi,\sigma,p) = Y^{\pm}(\xi,\sigma,y,p) \ .$$

Then eq. (A. 8) becomes :

$$\begin{aligned} Z^{\pm}(\xi,\sigma,p) = {} & \frac{K_o^{\pm}\omega_o}{2\pi} a^{\pm}(\xi,p) \int_0^L \frac{d\lambda}{L} \int_0^\infty f(\mu) d\mu \left\{ Z^{\pm}(\xi,\sigma,p) - Z^{\mp}(\mu,\lambda,p) \right\} \ + \\ & + \sum_{i=1}^{M} \frac{\omega_o}{4\pi} a^{\pm}(\xi,p) \int_\sigma^L \frac{d\lambda}{L} \int_0^\infty f(\mu) d\mu \ \rho_i^{\pm}(\lambda - \sigma)\ e^{i\nu(\sigma - \lambda)} Z^{\pm}(\mu,\lambda,p) + b^{\pm}(\xi,\sigma,p) \end{aligned} \tag{A. 9}$$

If we put

$$\frac{K_o \omega_o}{2\pi} = H ,$$

eq. (A. 9) becomes:

$$Z^{\pm}(\xi, \sigma, p) = \frac{-H^{\mp} a^{\mp}(\xi, p)}{1 - H^{\pm} a^{\pm}(\xi, p)} \int_0^L \frac{d\lambda}{L} \int_0^{\infty} f(\mu) d\mu \; Z^{\mp}(\mu, \lambda, p) + \frac{a^{\pm}(\xi, p)}{1 - H^{\pm} a^{\pm}(\xi, p)} \quad x$$

(A. 10)

$$x \int_{\sigma}^{L} \frac{d\lambda}{L} \int_0^{\infty} f(\mu) d\mu \; e^{i\nu(\sigma - \lambda)} \sum_{i=1}^{M} \frac{\omega_o}{4\pi} \rho_i^{\pm}(\lambda - \sigma) Z^{\pm}(\mu, \lambda, p) + \frac{b^{\pm}(\xi, \sigma, p)}{1 - H^{\pm} a^{\pm}(\xi, p)}$$

Letting

$$Y^{\pm}(\sigma, p) = \int_0^{\infty} f(\mu) d\mu \; Z^{\pm}(\mu, \sigma, p) ,$$

$$G^{\pm}(p) = \int_0^{\infty} \frac{f(\xi) a^{\pm}(\xi, p)}{1 - H^{\pm} a^{\pm}(\xi, p)} d\xi$$

(A. 11)

$$q^{\pm}(\sigma, p) = \int_0^{\infty} \frac{b^{\pm}(\xi, \sigma, p) f(\xi)}{1 - H^{\pm} a^{\pm}(\xi, p)} d\xi ,$$

from eq. (A. 10) we derive

$$Y^{\pm}(\sigma, p) = G^{\pm}(p) \left\{ - H^{\mp} \int_0^L \frac{d\lambda}{L} \; Y^{\mp}(\lambda, p) + \frac{\omega_o}{4\pi} \int_{\sigma}^{L} \frac{d\lambda}{L} \sum_{i=1}^{M} \rho_i^{\pm}(\lambda - \sigma) \; x \right.$$

(A. 12)

$$\left. x \;\; e^{i\nu(\sigma - \lambda)} \, Y^{\pm}(\lambda, p) \right\} + q^{\pm}(\sigma, p) .$$

APPENDIX B - DERIVATION OF EQUATION $T(G^+, G^-) = 0$. -

We shall solve eq. (A. 12) with a wake field of the form (13), that is with

(B. 1)

$$\rho_i^{\pm}(\lambda - \sigma) = N^{\pm} k_i \, e^{-\frac{\lambda - \sigma}{\tau}} .$$

Letting

(B. 2)

$$\bar{H}^{\pm} = N^{\pm} \sum_{i=1}^{M} \frac{k_i \omega_o}{4\pi} ,$$

eq. (A. 12) becomes:

$$Y^{\pm}(\sigma,p) = G^{\pm}(p)\left\{-H^{\pm}\int_0^L \frac{d\lambda}{L}\, Y^{\mp}(\lambda,p) + \bar{H}^{\pm}\int_\sigma^L \frac{d\lambda}{L}\, e^{(i\nu+\frac{1}{\tau})(\sigma-\lambda)} \times Y^{\pm}(\lambda,p)\right\} + q^{\pm}(\sigma,p)\,. \tag{B.3}$$

It is easy to verify that the solutions of eq. (B.3) are given by (μ is defined in eq. (15)):

$$Y^{\pm}(\sigma,p) = Y_0^{\pm}(p)\, e^{-(G^{\pm}\bar{H}^{\pm}-\mu L)\sigma} + \bar{Y}^{\pm}(\sigma,p) + C^{\pm}(p)\;, \tag{B.4}$$

where $\bar{Y}^{\pm}(\sigma,p)$ are particular solutions of the equations

$$\frac{\partial}{\partial\sigma}\bar{Y}^{\pm}(\sigma,p) + \left(\frac{G^{\pm}\bar{H}^{\pm}}{L}-\mu\right)\bar{Y}^{\pm}(\sigma,p) = \frac{\partial}{\partial\sigma}q^{\pm}(\sigma,p) - \mu\, q^{\pm}(\sigma,p)\,,$$

which satisfy the relations

$$\int_0^L \bar{Y}^{\pm}(\sigma,p)\, d\sigma = 0\,,$$

$C^{\pm}$ are given by the expression

$$C^{\pm}(p) = \frac{\mu G^{\pm} H^{\pm} L^{-1}}{\left(\frac{G^+\bar{H}^+}{L}-\mu\right)\left(\frac{G^-\bar{H}^-}{L}-\mu\right) - \mu^2 G^+G^-H^+H^-} \times$$

$$\times \left\{ Y_0^{\mp}\left(1-e^{-(G^{\mp}\bar{H}^{\mp}-\mu L)}\right) + Y_0^{\pm}\,\frac{\mu G^{\mp}H^{\mp}}{\frac{G^{\pm}\bar{H}^{\pm}}{L}-\mu}\left(1-e^{-(G^{\pm}\bar{H}^{\pm}-\mu L)}\right)\right\}$$

and $Y_0^{\pm}$ are the solution of the system

$$\alpha^+ Y_0^+ + \beta^- Y_0^- = \gamma^+\;, \qquad \beta^+ Y_0^+ + \alpha^- Y_0^- = \gamma^-$$

where we have put

$$\alpha^{\pm} = e^{-(G^{\pm}\bar{H}^{\pm}-\mu L)} + D^{\pm}\mu G^{\mp}H^{\mp}\left(1-e^{-(G^{\pm}\bar{H}^{\pm}-\mu L)}\right)$$

$$\beta^{\pm} = D^{\mp}\left(\frac{G^{\mp}\bar{H}^{\mp}}{L}-\mu\right)\left(1-e^{-(G^{\pm}\bar{H}^{\pm}-\mu L)}\right) \tag{B.5}$$

$$D^{\pm} = \frac{(G^{\pm})^2 H^{\pm}\bar{H}^{\pm}/(G^{\pm}\bar{H}^{\pm}-\mu L)}{\left(\frac{G^+\bar{H}^+}{L}-\mu\right)\left(\frac{G^-\bar{H}^-}{L}-\mu\right) - \mu^2 G^+G^-H^+H^-}$$

$$\gamma^{\pm} = q^{\pm}(L,p) - \bar{Y}^{\pm}(L,p)$$

The poles (in p) of x_0 and y_0 gives us the collective motion frequencies. These poles are the solutions of the equation

$$\begin{vmatrix} \alpha^+, & \beta^- \\ \beta^+, & \alpha^- \end{vmatrix} = 0 \tag{B. 6}$$

Eq. (B. 5) and (B. 6) yield

$$T(G^+, G^-) = \left\{ e^{-(x^+ - \mu L)} + \frac{\mu L (x^+)^2 x^- q^2}{(x^- - \mu L)(x^+ - \mu L) - (\mu L)^2 x^+ x^- q^2} \; \frac{1 - e^{-(x^+ - \mu L)}}{x^+ - \mu L} \right\} \times$$

$$\times \left\{ e^{-(x^- - \mu L)} + \frac{\mu L (x^-)^2 x^+ q^2}{(x^- - \mu L)(x^+ - \mu L) - (\mu L)^2 x^+ x^- q^2} \; \frac{1 - e^{-(x^- - \mu L)}}{x^- - \mu L} \right\} + \tag{B. 7}$$

$$- \frac{(x^+)^2 (x^-)^2 \; q^2 (1 - e^{-(x^+ - \mu L)}) (1 - e^{-(x^- - \mu L)})}{(x^- - \mu L)(x^+ - \mu L) - (\mu L)^2 \; x^+ x^- \; q^2} = 0$$

where

$$x^{\pm} = G^{\pm} \, \bar{h} \, N^{\pm} \; , \qquad q = h / \bar{h} \; . \tag{B. 8}$$

APPENDIX C - DERIVATION OF THE SOLUTIONS OF EQ. (B. 7). -

a) Case $N^+ \neq N^-$, $q \gg 1$, $L/\tau \gg 1$.

Eq. (B. 7) becomes

$$\frac{x^+ x^-}{(x^+ - \mu L)(x^- - \mu L)} = \frac{1}{(q \mu L)^2} \; ; \tag{C. 1}$$

the solutions of (C. 1) are given by :

$$x^{\pm} = \frac{\mu}{1 + \mu L \, q \, \alpha^{\mp 1}} \; . \tag{C. 2}$$

b) Case $N^+ = N^- = N$ $(x^+ = x^- = S)$.

Eq. (B. 7) becomes

$$(e^{S - \mu L} - 1) \left(\frac{S}{S - \mu L} \right)^2 - \mu L \frac{S}{S - \mu L} = \pm \frac{1}{q} \tag{C. 3}$$

eq. (17) and (C. 3) are the same equation.

b1) $L/\tau \gg 1$.

The solution of eq. (C. 3) is

(C. 4) $$S = \pm \frac{1}{q} \, e^{\pm i(\nu\tau)\frac{\tau}{L}\frac{1}{q}} ,$$

where we use the condition

(C. 5) $$\nu L \ll \frac{L}{\tau} ,$$

and we put

(C. 6) $$e^{S-\mu L} = e^{(S-i\nu L)} \, e^{-L/\tau} \approx 0 .$$

b2) $L/\tau \ll 1$.

We have the system (we can put $\mu L = i\nu L + \frac{L}{\tau} \approx i\nu L$)

(C. 7) $$\mathrm{Re}\left[(e^{S-i\nu L} - 1)\left(\frac{S}{S-i\nu L}\right)^2 - i\nu L \frac{S}{S-i\nu L}\right] = \pm \frac{1}{q}$$

$$\mathrm{Im}\left[(e^{S-i\nu L} - 1)\left(\frac{S}{S-i\nu L}\right)^2 - i\nu L \frac{S}{S-i\nu L}\right] = 0$$

If we put

(C. 8) $$S = \sigma e^{i\theta} , \qquad \mathcal{J} = \nu L$$

eq. (C. 7) become :

(C. 9) $$\frac{\sigma}{D}\left\{\frac{\sigma}{D}\left[(e^{\sigma\cos\theta}\cos(\sigma\sin\theta - \mathcal{J}) - 1)(\sigma^2 - \mathcal{J}^2\cos(2\theta) - 2\mathcal{J}\sin\theta) - e^{\sigma\cos\theta}\sin(\sigma\sin\theta - \mathcal{J})(2\mathcal{J}\sigma\cos\theta - \mathcal{J}^2\sin(2\theta))\right] + \mathcal{J}^2\cos\theta\right\} = \pm \frac{1}{q}$$

$$\frac{\sigma}{D}\left[\mathcal{J}(e^{\sigma\cos\theta}\cos(\sigma\sin\theta - \mathcal{J}) - 1)(2\sigma\cos\theta - \mathcal{J}\sin(2\theta)) + e^{\sigma\cos\theta} \times \sin(\sigma\sin\theta - \mathcal{J})(\sigma^2 - \mathcal{J}^2\cos(2\theta) - 2\mathcal{J}\sigma\sin\theta)\right] - \mathcal{J}(\sigma - \mathcal{J}\sin\theta) = 0$$

where :

(C. 10) $$D = \sigma^2 + \mathcal{J}^2 - 2\mathcal{J}\sigma\sin\theta$$

We look for a solution of eq. (C. 9) in the limit $q \gg 1$; it can be shown that $\sigma \to 0$ and $\sin\theta \to 0$ for $q \to \infty$, so, from the second of equations (C. 9) we obtain:

(C. 11) $$\sin\theta = \sigma \frac{\mathcal{J} - \sin\mathcal{J}}{\mathcal{J}^2} \approx \frac{\mathcal{J}\sigma}{6} \qquad (\mathcal{J} = \nu L \ll 1).$$

From the first of equations (C. 9) we obtain :

(C. 12) $$\sigma \cos\theta = \pm \frac{1}{q}$$

so we have two sets of solutions :

(C. 13)
$$\sigma = \frac{1}{q}, \qquad \theta = \frac{\pi}{6q}$$
$$\sigma = -\frac{1}{q}, \qquad \theta = -\frac{\pi}{6q}$$

Eq. (C.13) show that actually :

(C. 14)
$$\sigma \longrightarrow 0$$
$$\sin\theta \longrightarrow 0 \qquad \text{for} \quad q \longrightarrow \infty .$$

APPENDIX D - BUNCHES WITH $N^+ \neq N^-$. -

We shall solve eq. (46) with the conditions (49) and (50). If we put

(D. 1) $$\alpha = \beta e^{i\theta}$$

we obtain :

(D. 2) $$\Delta = N^{\mp}\left(-\frac{1}{\lambda} + r_o\left(\lambda\left(\frac{N^-}{N^+}\beta\right)^{\pm 1}\right)e^{\,i\frac{(\nu\tau\pm\theta)}{qL}}\,\beta^{\pm 1}\,\eta\left(\lambda\left(\frac{N^-}{N^+}\beta\right)^{\pm 1}\right)\right) .$$

In order to satisfy eq. (50), we must put (up to the second order terms in $\left(\frac{N^+-N^-}{N^++N^-}\right)$) :

(D. 3)
$$\beta = \frac{N^+}{N^-}\left(1 - \frac{1}{\lambda}\,\frac{\lambda r_o(\lambda)-1}{\eta(\lambda)r_o(\lambda)}\,\frac{N^+-N^-}{N^++N^-}\right)$$
$$\theta = \nu\tau\,\frac{N^--N^+}{N^++N^-}\,\frac{1-\dfrac{(r_o(\lambda)-\lambda)^2}{\lambda r_o(\lambda)}}{1-\left(\dfrac{N^+-N^-}{N^++N^-}\right)^2\dfrac{(r_o(\lambda)-\lambda)^2}{\lambda r_o(\lambda)}}$$

Eq. (D. 2), (D. 3) and

(D. 4) $$r_o(\lambda(1+\varepsilon)) \cong r_o(\lambda)\,(1-\varepsilon\,\eta(\lambda)) ,$$

yield eq. (52).

APPENDIX E - EVALUATION OF $K^{\pm}$, $K_o^{\pm}$. -

The transverse force between two relativistic particles of opposite charge traveling in opposite direction (see fig. E1), is given by

(E. 1) $$F(x,t) \approx 2\,\frac{e^2}{c}\,\frac{\delta(t)}{x}$$ (c = velocity of light ; $\delta(t)$ = Dirac function) .

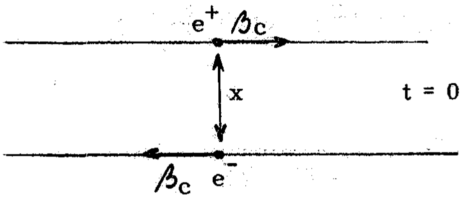

FIG. E1

Integration of $F(x,t)$ over the transverse distribution of one bunch gives the total force exerted by a longitudinal element of one bunch over one particle of the other bunch. The result is indipendent enough from the transverse distribution (we are interested in the order of magnitude only); we make the calculation with the " gaussian thread model distribution", that is we have :

(E. 2) $$F(\delta,t) = 2\left(\frac{e^2}{c\sqrt{2\pi}\,a}\int_{-\infty}^{+\infty}\frac{e^{-\frac{x^2}{2a^2}}}{x-\delta}\,dx\right)\delta(t)\,\frac{N}{L} \; .$$

where :

δ = distance between the particle and the center of mass of the longitudinal element of the bunch,

a = transverse beam dimension, L = time length of the bunches.

We have :

$$\bar{F} = -2\left(\frac{e^2\sqrt{2}}{ac}\,e^{-\frac{\delta^2}{2a^2}}\int_0^{\frac{\delta}{\sqrt{2}\,a}} e^{x^2}\,dx\right)\delta(t)\,\frac{N}{L} \; ,$$

we obtain up to the third order in $\frac{\delta}{\sqrt{2}\,a}$

(E. 3) $$\bar{F}(\delta,t) = -2\sqrt{2}\,\frac{e^2}{ac}\left(\frac{\delta}{\sqrt{2}\,a}\right)\left\{1-\frac{2}{3}\left(\frac{\delta}{\sqrt{2}\,a}\right)^2\right\}\delta(t)\,\frac{N}{L} \; .$$

In order to obtain the order of magnitude of the osctupolar term due to the crossing, we evaluate the quantity

$$\nu^2 K\,\xi^2 = -\frac{1}{T}\int_0^T d\tau\,\frac{\omega_o}{\pi}\int_0^{\frac{2\pi}{\omega_o}} dt\int_0^L d\lambda\,\bar{F}_1\!\left(\xi\,\mathrm{sen}\!\left(2\pi\,\frac{\tau}{T}\right),t\right)\frac{1}{m_o\gamma} = -\,\frac{r_o\,c\,\omega_o}{3\,\pi a^4\gamma}\,N\,\xi^2 \; ,$$

where :

$$\bar{F}_1(x,t) = \frac{2}{3}\,\frac{e^2}{a^4 c}\,\delta(t)\,\frac{N}{L}x^2 \; ; \qquad r_o = \frac{e^2}{m_o c^2} \; ;$$

$T = \frac{1}{\nu} 2\pi$ (ν = betatron frequency) ; ω_o = revolution frequency ;

$\gamma = (1-\beta^2)^{-1/2} = E/m_o c^2$; m_o = electron rest mass ;

ξ = betatron oscillation amplitude.

So we have (see eq. (A. 7)):

(E. 4) $$\bar{\nu}(\xi) = \nu(1 + K\xi^2)^{1/2} \approx \nu(1 + \frac{K}{2}\xi^2) ,$$

with

(E. 5) $$\frac{K}{2} = - \frac{r_o R}{6\pi a^4 \gamma Q^2} N ,$$

where

R = mean ring radius ; $Q = \frac{\nu}{\omega_o}$; $\bar{Q}(\xi) = \frac{\bar{\nu}(\xi)}{\omega_o}$.

from eq. (E. 4) we obtain

(E. 6) $$\frac{\partial \bar{Q}(\xi)}{\partial \xi^2} = - \frac{r_o RN}{6\pi a^4 \gamma Q} .$$

Inserting in eq. (E. 5) the following values

(E. 7) $R = 17$ m ; $N = 2 \times 10^9$; $a = 5 \times 10^{-2}$ cm ; $\gamma = 10^3$; $Q = 3$,

we obtain :

(E. 8) $$\frac{\partial \bar{Q}(\xi)}{\partial \xi^2} \approx -3 \text{ cm}^{-2} .$$

The octupolar term of the ring is (in Adone):

(E. 9) $$\left| \frac{\partial Q_r(\xi)}{\partial \xi^2} \right| \cong 2 \times 10^{-3} \text{ cm}^{-2} ,$$

then we see that it is much smaller than the previous one.

From eq. (E. 3) we obtain:

(E. 10) $$\frac{K_o}{L} = -2 \frac{r_o c}{a^2 \gamma} \frac{N}{L} .$$

So we have (see eq. (15)):

(E. 11) $$H = - \frac{r_o c\, \omega_o N}{\pi a^2 \gamma} = - \frac{r_o c^2}{R \pi a^2} \frac{N}{\gamma} .$$

Inserting in eq. (E. 11) the numerical values (E. 7) we obtain:

(E. 12) $$h = \frac{H}{N} \approx 5 \times 10^3 \ \sec^{-2} \ .$$

From eq. (E. 5) and (E. 10) we obtain:

(E. 13) $$\frac{h}{\nu^2 K a^2} = 3 \ .$$

APPENDIX F - EVALUATION OF $\bar{h}$. -

We shall derive $\bar{h}$ from the experimental thresholds wich are found in Adone with one beam alone, without feedback. We make the assumptions:

a) the head-tail effect[3] is the instability mechanism with one beam in the ring;
b) the distribution function of the vertical betatron oscillation amplitudes is

$$f(\xi) = \frac{\xi}{a} e^{-\frac{\xi^2}{2a^2}} \ .$$

We obtain the dispersion relation

(F. 1) $$\frac{1}{\nu^2 K a^4} \int_0^\infty \frac{\xi \, e^{-\frac{\xi^2}{2a^2}}}{y - \frac{\xi^2}{2a^2}} \, d\xi = \frac{1}{\bar{\varsigma}} e^{-i\theta} \ ,$$

where θ depends on the mode of instability while $\bar{\varsigma}$ depends on H also. We are interested on the minimum value of $\bar{\varsigma}$ compatibly with the observed threshold in Adone, then we assume the mode of instability is the mode (0, 0). So we have:

(F. 2) $$\bar{\varsigma} \approx H \ , \qquad \mathrm{arctg}\,\theta = \frac{2\bar{\nu}_o \Delta}{\pi} \approx 0.6 \ .$$

At the threshold we obtain from eq. (F1):

(F. 3) $$\overline{\mathrm{Ei}}(y) = \pm \pi \,\mathrm{cotg}\,\theta \qquad \left(\begin{array}{ll} + & \text{if } K > 0 \\ - & \text{if } K < 0 \end{array} \right)$$
$$\bar{\varsigma} = - \frac{\nu^2 |K| a^2}{\pi} e^y \sin\theta$$

where

$$\overline{\mathrm{Ei}}(x) = \frac{1}{2} \left\{ \mathrm{Ei}(x + i0) - \mathrm{Ei}(x - i0) \right\} \ ; \qquad \mathrm{Ei}(x) = - P \int_{-x}^{\infty} \frac{e^{-t}}{t} \, dt$$

If we assume

(F. 5) $K \cong -1.3 \times 10^{-3} cm^{-2}$; $a \approx 5 \times 10^{-2}$ cm ; $\nu \approx 6 \times 10^{7} sec^{-1}$,

we obtain :

$$|\bar{H}| \cong |\bar{S}| \geq 0.2 \times 10^{10} \; sec^{-2}$$

then we have

$$|\bar{h}| = \frac{|\bar{H}|}{N_{th}} \gtrsim \frac{1}{I_{th}(mA)} \; sec^{-2} \; ,$$

where (in Adone)

$$N = 2 \times 10^{9} \; I \; (mA) \; .$$

We find experimentally

$$I_{th}(\gamma = 10^{3}) \sim 1 \; mA$$

so the minimum value of $\bar{h}$ is

(F. 6) $$\bar{h}(\gamma = 10^{3}) \approx 1 \; sec^{-2} \; .$$

We have $\bar{h} \propto \frac{1}{\gamma}$ (through the relativistic electron mass), then from eq. (F. 6) we obtain:

(F. 7) $$\bar{h} \cong \frac{10^{3}}{\gamma} \; sec^{-2} \; .$$

The order of magnitude of $N_{th}^{(1)}$ ($\frac{L}{\tau} \simeq 1$) is given by (see eq. (43), (E. 13) and (F. 7)):

(F. 8) $$N_{th}^{(1)} \approx 10^{-3} \frac{\gamma}{\tau_R L} \frac{1}{r_o(\frac{3}{2}) \eta(\frac{3}{2})}$$

we obtain (enough independently on the distribution function $f(\xi)$) :

$$r_o(\tfrac{3}{2}) \approx 1 \; , \qquad \eta(\tfrac{3}{2}) \approx 1 \; ;$$

with $\gamma = 10^{3}$, $\tau_R \approx 3 \times 10^{-1}$ sec , $L \approx 2 \times 10^{-9}$ sec , eq. (F. 8) becomes :

(F. 9) $$N_{th}^{(1)} \approx 2 \times 10^{9} \longrightarrow I_{th} \simeq 1 \; mA$$

and actually in Adone the circulating current per bunch (with crossing) at 500 MeV is about 1 mA.