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A.G. Ruggiero[(x)] and V.G. Vaccaro[(ϕ)] : THE WAKE FIELD OF AN OSCILLATING PARTICLE IN THE PRESENCE OF CONDUCTIVE PLATES WITH RESISTIVE TERMINATIONS AT BOTH ENDS. -

PURPOSE -

The transverse coherent instabilities for bunched beam observed in ADONE[(1)], the Italian 1.5 GeV electron-positron storage ring, seem to be due to the presence of the clearing field electrodes placed inside the vacuum chamber. The unstable frequency is very close to the frequency of the betatron oscillations.

Since it has been shown experimentally that there is no interaction between bunches, one expects that the instability is due to a coupling of the two fundamental oscillation modes of a particle: the phase synchrotron oscillation and the transverse betatron oscillation. The coupling between these two modes of oscillation could be determined by the wake fields produced by all the particles in the bunch.

The instability observed in ADONE can be explained by the following "head-tail effect" model[(2)]. The wake field produced by a particle, generally, decreases with increasing distance from the particle, and depends on the electric properties of the surrounding media (in our case, the vacuum chamber wall with the conductive plates insertion). The field has a discontinuous behaviour. In fact it acts only on the particles following the source. Thus, each particle can be considered at the same time as object and subject of the total field inside the bunch. The field is produced by the transverse betatron oscillation, but the amount of field seen by a particle depends on its position within the bunch. This position changes according to the phase oscillation of the particle. The field seen by a particle is zero for a particle at the "head" of the bunch, and reaches a maximum for a particle at the "tail" of the bunch.

On the other hand, one particle produces identical fields when it is at the "head" and when it is at the "tail" of the bunch; but in the former case the field is seen by all par

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ticles of the bunch while in the latter case it has no effect.

Purpose of this paper is to find the expression for the wake field of a particle in the presence of conductive plates. Our results have already been used in a coupled betatron-synchrotron dynamics investigation to explain the instability in ADONE[2].

We will use cylindrical geometry, and we will consider only the case of conductive plates terminated at both ends by resistive terminations.

METHOD -

Our method is based on the Fourier expansion of the expression for the electric dipole to be associated to the oscillating particle. The Fourier transformation produces a sort of coasting beam, of radius a = 0, executing dipole mode transverse oscillations.

In a previous paper[3] we have written down the fields produced by the transformed electric dipole in presence of conductive plates terminated at both ends. The anti-transformation operation applied to these fields will give us the wake field produced by the single oscillating particle.

Performing the anti-transformation we will distinguish two cases that correspond to two different physical situations and that require different mathematical approaches. The two cases are

i) Plates terminated at least at one end by the characteristic impedance.
ii) Plates not terminated by the characteristic impedance at either end.

Since the synchrotron oscillation frequency is in general much smaller than the betatron oscillation frequency, we shall in the following assume the relative phase of a particle with respect to the synchronous one to be a constant.

1. - INTRODUCTION -

We consider a charged particle traveling inside a circular accelerator vacuum chamber. Assuming the radius R of the closed orbit to be much larger than the radius b of the vacuum chamber we can treat the particle as traveling along a straight cylindrical pipe of radius b. Let z be the axis of the pipe, and y one of the two transverse directions.

The motion equations of the particle can be written as

$$z = vt + \sigma \tag{1.1a}$$

$$y = \xi e^{i\nu t}, \tag{1.1b}$$

where v is the velocity of the bunch that contains the particle in consideration, and σ is the linear phase with respect to the "synchronous particle". The phase σ oscillates in time with frequency $\Omega/2\pi$.

In the r. h. side of eq.(1.1b), ξ is the transverse oscillation amplitude, to be taken in the following analysis as a constant, and $\nu/2\pi$ is very close to the betatron oscillation frequency. Generally, one has

$$\Omega << \nu ,$$

so that, since $\dot{y} >> \dot{\sigma}$, σ will be taken in the following simply as a constant.

We associate to our particle the following electric dipole per unit length

$$p = e\,\xi\, e^{i\nu t}\,\delta(z-vt-\sigma).$$

A double Fourier expansion of p(z, t) gives

$$p = \iint \tilde{p}(k,\omega)\, e^{-i(kz-\omega t)}\,dk\,d\omega$$

with

$$\tilde{p}(k,\omega) = \frac{1}{(2\pi)^2}\iint p(z,t)e^{i(kz-\omega t)}dzdt$$

(1. 2)

$$= \frac{e\zeta e^{ik\sigma}}{2\pi}\,\delta(\nu+kv-\omega)\,.$$

We consider the element of transformed electric dipole

$$p(k,\omega)dkd\omega = \tilde{p}(k,\omega)e^{-i(kz-\omega t)}dkd\omega\,.$$

It produces an element of force per unit charge $f(k,\omega)dkd\omega$ in the vicinity of the closed orbit ($r = 0$). This force is obtained by solving the Maxwell equations and satisfying the usual boundary conditions taking the electric properties of the surrounding media into account.

In the case where the amplitude displacement ζ is small compared to the vacuum chamber radius b, we can write

(1. 3) $$f(k,\omega)dkd\omega = g(z,k,\omega)p(k,\omega)dkd\omega\,.$$

The form factor $g(z,k,\omega)$ depends only on the electric properties of the surrounding media and not on the particle motion. In the case where there is a periodic electric structure along the vacuum chamber wall, the function $g(z,k,\omega)$ is periodic in z and has the same periodicity of the structure.

The total force produced by the particle is obtained by anti-transforming $f(k,\omega)$,

(1. 4) $$F(z,t) = \iint g(z,k,\omega)\tilde{p}(k,\omega)e^{-i(kz-\omega t)}dkd\omega.$$

Substituting eq. (1. 2) in eq. (1. 4), we have

(1. 5) $$F(z,t) = \frac{e\zeta e^{i\omega t}}{2\pi}\int g(z,k,\nu+kv)\,e^{-ik(z-vt-\sigma)}dk.$$

In our analysis we find it more convenient to replace the function $g(z,k,\omega)$ with its average value, $g(k,\omega)$, over one vacuum chamber wall-equipment structure periodicity. That is equivalent to taking the mean value of the force $F(z,t)$ over one period of the particle revolution motion.

2. - THE g-FORM FACTOR FOR THE CASE OF CONDUCTIVE TERMINATED PLATES -

The force components $\tilde{f}_x(k,\omega)$, $\tilde{f}_y(k,\omega)$ and $\tilde{f}_z(k,\omega)$ produced by the transformed dipole $\tilde{p}(k,\omega)$ in the presence of conductive terminated plates, and in the vicinity of the vacuum chamber axis ($r = 0$), are obtained from eqs. (5. 4) and (5. 5) of ref. (3). We have

(2. 1a) $$\tilde{f}_x(k,\omega) = -\mathcal{A}\left[\sum_h(V_h-\beta_p A_h)e^{-i\frac{h}{R}z}\right]\left[\sum_p\frac{\partial U_p}{\partial x}\right]$$

(2. 1b) $$\tilde{f}_y(k,\omega) = -\mathcal{A}\left[\sum_h(V_h-\beta_p A_h)e^{-i\frac{h}{R}z}\right]\left[\sum_p\frac{\partial U_p}{\partial y}\right]$$

(2. 2) $$\tilde{f}_z(k,\omega) = i\mathcal{A}\left\{\sum_h\left[(\frac{h}{R}+k)V_h-\frac{\omega}{c}A_h\right]e^{-i\frac{h}{R}z}\right\}\left[\sum_p U_p\right]$$

with

(2.3) $$U_p = \left(\frac{r}{b}\right)^p (\Phi_p \cos p\varphi + \Theta_p \sin p\varphi),$$

where Φ_p and Θ_p are two coefficients described in section 4 of ref. (3), and depending on the angular position of the plate with respect to the dipole oscillation axis (x=0).

V_h and A_h are two potentials given by eqs. (4.5) of ref. (3), both divided by the quantity $\mathcal{A}$ that, in our case, takes the form

(2.4) $$\mathcal{A} = \frac{\tilde{p}(k,\omega)}{\pi b^2} \frac{b_1}{C} \frac{\beta_p - \beta_w}{1-\beta_w^2},$$

where

$$b_1 = b \int \cos\varphi \, d\varphi,$$

and the integral extends over the transverse size of the plate.

Besides it is

- c, light velocity
- $\beta_p = v/c$
- $\beta_w = \omega/kc$
- C, capacitance per unit length of each plate.

Isolating the transformed dipole $\tilde{p}(k,\omega)$ at the r.h. side of eqs.(2.1) and (2.2) we get the form factor $g(z,k,\omega)$ taking different expressions for the three space directions.

To replace $g(z,k,\omega)$ with its average value $g(k,\omega)$ over one periodicity of the electric boundary structure (that we assume to correspond to one revolution of the particle) is equivalent to taking into consideration only the term with h = 0 at the r.h. side of eqs. (2.1) and (2.2)

Thus we have

$$\tilde{f}_x(k,\omega) = -\mathcal{A}(V_o - \beta_p A_o) \sum_p \frac{\partial U_p}{\partial x}$$

$$\tilde{f}_y(k,\omega) = -\mathcal{A}(V_o - \beta_p A_o) \sum_p \frac{\partial U_p}{\partial y}$$

$$\tilde{f}_z(k,\omega) = ik\mathcal{A}(v_o - \beta_w A_o) \sum_p U_p.$$

We observe that the generic component $\tilde{f}_i(k,\omega)$ depends on the transverse coordinates r and φ (or x and y) through the potential functions U_p(p = 0, 1, 2, ...). We also see that the variables k and ω do not enter the expression for U_p, therefore when we perform the anti-transformation of $\tilde{f}_i(k,\omega)$ the dependence on the transverse coordinates can be simply taken as a constant factor.

Performing some simple transformations we obtain, using the same notation as in ref. (3),

(2.4a) $$\tilde{f}_x(k,\omega) = \mathcal{B}_x P^{trans}(k,\omega)\, \tilde{p}(k,\omega)$$

(2.4b) $$\tilde{f}_y(k,\omega) = \mathcal{B}_y P^{trans}(k,\omega)\, \tilde{p}(k,\omega)$$

(2.5) $$\tilde{f}_z(k,\omega) = i\,\mathcal{B}_z P^{long}(k,\omega)\, \tilde{p}(k,\omega),$$

where

$$\beta_x = -\left(\frac{Ml}{2\pi R}\right)\left(\frac{b_1/C}{\pi b^2}\right)\left(\sum_p \frac{\partial U_p}{\partial x}\right)$$

$$\beta_y = -\left(\frac{Ml}{2\pi R}\right)\left(\frac{b_1/C}{\pi b^2}\right)\left(\sum_p \frac{\partial U_p}{\partial y}\right)$$

and

$$\beta_z = \left(\frac{Ml}{2\pi R}\right)\left(\frac{2b_1/lC}{\pi b^2}\right)\left(\sum_p U_p\right).$$

P^{trans} is given by eq. (6. 6) of ref. (3), and P^{long} by eq. (6. 8) of the same reference but replacing β with β_p.

In the following we will limit ourselves to investigate only the two transverse components of the wake field produced by the oscillating particle. Since the r. h. side of both eqs. (2. 4a) and (2. 4b) are similar, apart from the factors β_x and β_y that will never enter our subsequent analysis, we write for one of the two transverse components

(2. 6) $$\tilde{f}(k,\omega) = \beta P^{trans}(k,\omega)\tilde{p}(k,\omega).$$

In what follows the indices x and y for the two components will be dropped.

From eq. (1. 3) one easily obtains

(2. 7) $$g(k,\omega) = \beta P^{trans}(k,\omega),$$

and from eq. (1. 5)

(2. 8) $$F(z,t) = \frac{1}{2}\mathcal{A}\int P(k, \nu + kv)\, e^{-ik(z-vt-\sigma)}dk$$

where

(2. 9) $$\mathcal{A} = \mathcal{A}_o e^{i\omega t},$$

$$\mathcal{A}_o = \frac{e\gamma}{\pi l}\beta ,$$

and $P(\omega,k) = P^{trans}(\omega,k)$.

Introducing

$$\phi = \frac{\omega l}{2c} \quad \text{and} \quad \theta = \frac{kl}{2}$$

We write

(2. 10) $$F(\chi,t) = \mathcal{A}\int P(\theta, \eta + \beta_p\theta)\, e^{i\theta\chi}\, d\theta$$

with

$$\eta = \frac{\nu l}{2c},$$

and $\chi = 2(z'-z)/l$ the reduced distance between the point of coordinate z where we want to know the fields and the field source having coordinate $z' = vt + \sigma$.

3. - ANTI-TRANSFORMATION TO OBTAIN THE EXPRESSION FOR THE WAKE FIELD -

The integral at the r.h. side of eq. (2.10) can be solved taking θ as a complex quantity and using the residues method.

Making use of eqs. (6.6) and (3.5) of ref. (3) we can decompose the integral at the r.h. side of eq. (2.10) into nine integrals.

We have

$$2i\,\frac{F(\chi,t)}{a} = 2i\,F_o(\chi,t) =$$

(3.1)

$$\begin{aligned}
&= (1-\beta_p)(1-r_1)\int \frac{\beta_p-\beta_w}{1-\beta_w^2}\,\frac{\tilde{r}_2-\beta_w}{1-\beta_w}\,\frac{e^{-2i\phi}}{\theta\Delta}\,e^{i\theta\chi}\,d\theta\;+\\
&- (1-\beta_p)(1-r_1)\int \frac{\beta_p-\beta_w}{1-\beta_w^2}\,\frac{\tilde{r}_2-\beta_w}{1-\beta_w}\,\frac{e^{-2i\theta}}{\theta\Delta}\,e^{i\theta\chi}\,d\theta\;+\\
&+ (1-\beta_p)(1+r_2)\int \frac{\beta_p-\beta_w}{1-\beta_w^2}\,\frac{\tilde{r}_1+\beta_w}{1-\beta_w}\,\frac{e^{2i\theta}}{\theta\Delta}\,e^{i\theta\chi}\,d\theta\;+\\
&- (1-\beta_p)(1+r_2)\int \frac{\beta_p-\beta_w}{1-\beta_w^2}\,\frac{\tilde{r}_1+\beta_w}{1-\beta_w}\,\frac{e^{2i\theta}}{\theta\Delta}\,e^{i\theta\chi}\,d\theta\;+\\
&- (1+\beta_p)(1-r_2)\int \frac{\beta_p-\beta_w}{1-\beta_w^2}\,\frac{\tilde{r}_1+\beta_w}{1+\beta_w}\,\frac{e^{2i\theta}}{\theta\Delta}\,e^{i\theta\chi}\,d\theta\;+\\
&+ (1+\beta_p)(1-r_2)\int \frac{\beta_p-\beta_w}{1-\beta_w^2}\,\frac{\tilde{r}_1+\beta_w}{1+\beta_w}\,\frac{e^{-2i\phi}}{\theta\Delta}\,e^{i\theta\chi}\,d\theta\;+\\
&- (1+\beta_p)(1+r_1)\int \frac{\beta_p-\beta_w}{1-\beta_w^2}\,\frac{\tilde{r}_2-\beta_w}{1+\beta_w}\,\frac{e^{2i\phi}}{\theta\Delta}\,e^{i\theta\chi}\,d\theta\;+\\
&+ (1+\beta_p)(1+r_1)\int \frac{\beta_p-\beta_w}{1-\beta_w^2}\,\frac{\tilde{r}_2-\beta_w}{1+\beta_w}\,\frac{e^{-2i\theta}}{\theta\Delta}\,e^{i\theta\chi}\,d\theta\;+\\
&- 2i\int \frac{(\beta_p-\beta_w)^2}{1-\beta_w^2}\,e^{i\theta\chi}\,d\theta.
\end{aligned}$$

We have used the following notation, derived from that used in ref. (3),

$\beta_w = \omega / kc$;

r_1, r_2 are the two termination impedances at the ends of each conductive plate. These impedances are normalized to the characteristic impedance (see Ref. (3));

$$\tilde{r}_i = r_i \beta_w \frac{1-\beta_w \beta_p}{\beta_w - \beta_p} \; ; \quad i = 1, 2,$$

and

$$\Delta = -2 \left[(r_1+r_2)\cos 2\phi + i\,(1 + r_1 r_2)\sin 2\phi \right] \tag{3.2}$$

Since a has the dimension of a force per unit charge, $F_o(\chi, t)$ is an a-dimensional quantity.

We have derived the expression (3.1) for $F_o(\chi, t)$ assuming that the termination impedances r_1 and r_2 are constant, i. e. we are here referring only to the case of resistive terminations.

The integration path of the integrals at the r. h. side of eq. (3.1) must be chosen in the complex θ-plane in such way as to contain the real axis and to be short-circuited at infinity on the upper or on the lower half-plane.

The integrands will then vanish identically at infinity. We see that we must choose different integration paths for the nine integrals at the r. h. side of eq. (3.1). At this end we observe that the quantity Δ, eq. (3.2), can also be written as follows

$$\Delta = -\, r_s \left[(1+W)\, e^{2i\phi} + (1-W)\, e^{-2i\phi} \right] \tag{3.3}$$

with

$$r_s = r_1 + r_2$$
$$r_p = r_1 r_2 / (r_1 + r_2)$$
$$W = r_p + 1/r_s.$$

We distinguish two cases.

i) $W = 1$. This case corresponds to a plate matched to its characteristic impedance at least at one of the two ends ($r_i = 1$). The quantity Δ, taken as a function of θ, has no zero; then the only singularities in the integrands at the r. h. side of eq. (3.1) are given by the quantities $(1-\beta_w)$ and $(1+\beta_w)$ in the denominator.

ii) $W \neq 1$. The plate is not terminated by its characteristic impedance; hence, the quantity Δ, as we will show in the following, has an infinite number of zeros. In this case, we have to take into account two kinds of singularities in the integrands of eq. (3.1): the zeros of the function Δ, and the zeros of the quantities $(1-\beta_w)$ and $(1+\beta_w)$.

The singularities mentioned in the two above cases are all poles of the first or, at most of the second order. Poles of the second order are given by quantities such as $(1 \pm \beta_w)^2$, while poles given by the function Δ are of the first order.

Each of the above mentioned poles corresponds to a physical resonance of the structure formed by the conductive plate with its terminations and the vacuum chamber wall. The poles given by quantities such as $(1+\beta_w)$ or $(1-\beta_w)$ produce the resonances for which

$$k = \pm\, \omega / c,$$

i. e. the wave length $2\pi/k$ of the transformed electric dipole is equal to the wavelength $2\pi c/\omega$ of the signal induced on the plate. That occurs for those Fourier components of the electric dipole, associated to the oscillating particle, that travel at the velocity of light ($|\beta_w| = 1$), although the velocity of the particle is less than the velocity of light.

The poles given by the zeros of the quantity Δ are to be related to the electrical resonances of the equivalent circuit formed by the termination impedances and the impedance of the transmission line structure formed by the plate and the vacuum chamber wall.

Furthermore, we observe that the quantity θ which appears at the denominator of each of the integrands at the r. h. side of eq. (3. 1) is not a pole; in fact it can been shown that the function $P(\theta, \eta + \beta_p\theta)$ takes on a finite value for $\theta = 0$.

4. - CASE OF CONDUCTIVE PLATES TERMINATED BY THE CHARACTERISTIC IMPEDANCE. -

For matched conductive plates, we have

$$W = 1.$$

The function Δ has no zeros; in fact from (3. 3), it is

(4. 1) $$\Delta = - 2\, r_s\, e^{2i\phi}.$$

One has $W = 1$ when:

i) $r_1 = r_2 = 1$, i. e. the plate is matched at both ends, and when
ii) $r_1 = 1$, $r_2 \neq 1$, or
iii) $r_1 \neq 1$, $r_2 = 1$, i. e. the plate is matched at only one end.

As mentioned in the preceding section, only poles given by the quantities such as $(1 - \beta_w)$ or $(1 + \beta_w)$ contribute to the integrals of eq. (3. 1).

The scaled force $F_o(\chi, t)$ assumes different values in the following intervals of :

interval n. 1 ,			χ	$<$	$-2(1 - \beta_p)$
interval n. 2 ,	$-2(1 - \beta_p)$	$<$	χ	$<$	0
interval n. 3 ,	0	$<$	χ	$<$	$4\beta_p$
interval n. 4 ,	$4\beta_p$	$<$	χ	$<$	$2(1 + \beta_p)$
interval n. 5 ,	$2(1 + \beta_p$	$<$	χ		.

One can easily verify that

$$P(\theta, \phi) = \frac{1 - \beta_p}{2} \qquad \text{for} \quad \beta_w = 1$$

and

$$P(\theta, \phi) = \frac{1 + \beta_p}{2} \qquad \text{for} \quad \beta_w = 1,$$

thus it is easily seen that $F_o(\chi, t)$ is zero in the intervals n. 1 and 5 of χ .

Since the intervals n. 2 and 4 have lengths equal to $2(1-\beta_p)$, generally a very small quantity since β_p is very close to one, we do not calculate $F_o(\chi, t)$ in these intervals. The length of the third interval practically corresponds to twice of plate length (since $\beta_p \sim 1$).

It can be shown[x] that $P(\theta, \phi)$ is an invariant under exchange of the order of the termination impedances. Thus we can simply consider the case (ii) with $r_1 = 1$; besides, we

(x) - See eq. (8. 2) of ref.(3) which gives a more exotic expression for $P(\theta, \phi)$.

will have

$$r_s = 1 + r_2,$$

allowing r_2 to take also values other than 1.

Using the fact that $F_o(\chi, t)$, i. e. the sum of all integrals at the r. h. side of eq. (3. 1), is zero for very large $|\chi|$, and that the first two integrals are zero because they are multiplied by $(1-r_1)$, we have

$$iF_o = \frac{1-\beta_p^2}{2} \frac{1-r_2}{1-{}^2_w} e^{-4i\eta} \oint \frac{\beta_w}{1-\beta_w^2} \frac{e^{i\theta(\chi - 4\beta_p)}}{\theta} d\theta +$$

$$+ (1+\beta_p) \frac{e^{-2i}}{1+r_2} \oint \frac{\beta_w}{\theta} \frac{(r_2+\beta_p)-\beta_w(1+r_2\beta_p)}{(1+\beta_w)^2(1-\beta_w)} e^{i\theta\left[\chi - 2(1+\beta_p)\right]} d\theta.$$

The circuitation of the integrals at the r. h. side must be taken along the semi-circle at infinity in the lower θ half-plane.

Using

$$\beta_w = \frac{\phi}{\theta} = \frac{\eta + \beta_p \theta}{\theta}$$

and the residues method, we obtain

$$F_o(\chi, t) = -\pi e^{-i\eta(\chi/1+\beta_p)} \left[\beta_p + i\eta \frac{\chi - 2(1+\beta_p)}{1+\beta_p}\right] +$$

(4. 2)

$$- \pi \frac{1-r_2}{1+r_2} \frac{1-\beta_p}{4} e^{-i\eta(\chi/1+\beta_p)} \left(1 - e^{-4i(\eta/1+\beta_p)}\right),$$

(for $0 < \chi < 4\beta_p$).

The expression (4. 2) for the scaled force $F_o(\chi, t)$ contains two terms. The first term does not depend on r_2, while the second does depend on r_2 through the factor

$$\frac{1-r_2}{1+r_2} .$$

Assuming $r_2 \geq 0$ we see that $(1-r_2)/(1+r_2)$ varies from -1 ($r_2 = \infty$) to + 1(r_2=0). Besides the second term vanishes for $r_2 = 1$, i. e. when the second of the plate is also terminated by its characteristic impedance; thus for $r_1 = r_2 = 1$ we simply have

$$(4.3) \qquad F_o(\chi, t) = -\pi e^{-i\eta(\chi/1+\beta_p)} \left[\beta_p + i\eta \frac{\chi - 2(1+\beta_p)}{1+\beta_p}\right] .$$

We observe that, in general, η is a small quantity, and the first term in the squared brackets at the r. h. side of eq. (4. 3) gives therefore the foremost contribution to $F_o(\chi, t)$ as long as $\beta_p \gg 2\eta$ is verified. Furthermore, even in the case $r_2 \neq 1$, the second term at the r. h. side of eq. (4. 2) is, in first approximation, an imaginary quantity(because η is small) governed by the factor $\eta(1-\beta_p)$. Thus in all cases where W = 1 and the imaginary part can be neglected with respect to the real one, we can write in good approximation.

$$F_o(\chi, t) \sim -\pi\beta_p, \qquad \text{for} \qquad 0 < \chi < 4\beta_p$$

$$(4.4) \qquad = 0 \,, \qquad \text{elsewhere.}$$

5. - CASE OF CONDUCTIVE PLATES NOT TERMINATED BY THE CHARACTERISTIC IMPEDANCE -

If one has $W \neq 1$, the expression (3. 3) for Δ must be used. In this case Δ has an infinite number of zeros so that, to apply the residues method to the integration of the r. h. side of eq. (3. 1), one must take into account not only the singularities given by quantities such as $(1 + \beta_w)$ and $(1-\beta_w)$ but also the poles given by the zeros of the function Δ.

i) Contribution of the $(1-\beta_w^2)$ poles.

We see that all integrals at the r. h. side of the equation (3. 1) except the last one, are multiplied by the following factor

$$(5.1) \qquad \frac{e^{i\theta(\chi-\alpha)}}{e^{2i\eta}(1+W)e^{2i\beta_p\theta}+(1-W)e^{-2i\beta_p\theta}e^{-2i\eta}}$$

where α takes one of the following values

$$-2, \quad -2\beta_p, \quad 2\beta_p \quad \text{and} \quad 2.$$

For $\chi < 0$ it is convenient to take as integration path of the integral at the r. h. side of eq. (2. 10) the boundary of the lower θ half-plane. In fact, when the imaginary part of θ is very large, the quantity (5. 1) goes simply as

$$(5.2) \qquad \sim e^{i\theta(\chi-\alpha-2\beta_p)}$$

On the other hand, for $\chi > 0$ it is convenient to take the integration path in the upper θ half-plane. In this case, the quantity (5. 1) approaches the expression

$$(5.3) \qquad \sim e^{i\theta(\chi-\alpha+2\beta_p)}$$

as the imaginary part of θ becomes very large.

It is soon verified that for $|\chi| > 2(1-\beta_p)$ the nine integrals at the r. h. side of eq. (3. 1) must all keep the same sign, and that the corresponding quantities to be integrated are all reduced to zero for $|\theta| \to \infty$. Thus, since the sum of the integrals for $|\chi| \to \infty$ is zero, we have that the contribution of the $(1-\beta_w)$ and $(1+\beta_w)$ poles only exists in the interval $|\chi| < 2(1-\beta_p)$ and is identically zero for $|\chi| > 2(1-\beta_p)$.

Since $\beta_p \sim 1$, we shall leave out the calculation of the scaled force $F_o(\chi, t)$ in the interval $|\chi| < 2(1-\beta_p)$, and we shall assume that the substantial contribution to $F_o(\chi, t)$ is simply given by the poles due to the zeros of the function Δ.

ii) Contribution of the poles due to the zeros of Δ.

We must again distinguish the two cases: $|\chi| < 2(1-\beta_p)$ and $|\chi| > 2(1-\beta_p)$. In the first case the residues method requests the use of different integration paths for the nine integrals at the r. h. side of eq. (3. 1) according to the behaviour of the quantity (5. 1). We decide to leave out of the subsequent analysis also the contribution of the poles due to the zeros of Δ to the force $F_o(\chi, t)$ in the range $|\chi| < 2(1-\beta_p)$, and we limit ourselves to giving the expression for $F_o(\chi, t)$ for $|\chi| > 2(1-\beta_p)$.

In this range of χ, the integration path of the nine integrals of eq. (3.1) must follow the same semi-circle at infinity in the θ-complex plane.

For $\chi < 0$ one must take $\mathrm{Im}(\theta) \to -\infty$, and
for $\chi > 0$ one must take $\mathrm{Im}(\theta) \to +\infty$.

We re-write Δ as

(5.4) $$\Delta = \alpha \sin 2 (\phi - i\, \phi^*)$$

with

(5.5) $$\alpha^2 = 4(r_2^2 - 1)(1 - r_1^2)$$

and ϕ^* given by

(5.6) $$t_{gh}\, 2\phi^* = 1/W$$

From the last equation we obtain

(5.7a) $$\phi^* = \frac{1}{4} \log \frac{1 + W}{1 - W}$$

or

(5.7b) $$\phi^* = \frac{1}{4} \log \frac{(1+r_1)(1+r_2)}{(r_1+1)(1-r_2)}$$

Writing

$$(-1)^p D = \frac{(1+r_1)(1+r_2)}{(r_1-1)(1-r_2)}$$

with

(5.8) $$D = \left| \frac{(1+r_1)(1+r_2)}{(r_1-1)(1-r_2)} \right|$$

and

$$(-1)^p = \mathrm{sign} \left\{ \frac{(1+r_1)(1+r_2)}{(r_1-1)(1-r_2)} \right\}, \quad \text{(p is 0 or 1)},$$

we see that ϕ^* takes on an infinite number of values the s-th of which is

(5.9) $$\phi_s^* = \frac{1}{4} \log D - i\, \frac{\pi}{2} \left(s + \frac{p}{2}\right)$$

It is easy to verify that since r_1 and r_2 are two real positive numbers, the real part of ϕ_s^*

$$\mathcal{R}e(\phi_s^*) = \frac{1}{4} \log D$$

is also a positive quantity. Then, since the poles in the θ - plane are placed at the points of complex coordinate

(5.10a) $$\theta_s = \frac{1}{\beta_p} \left[\frac{\pi}{2} \left(s + \frac{p}{2}\right) - \eta \right] + i\, \frac{\log D}{4 \beta_p}$$

(s = ---, -3, -2, -1, 0, 1, 2, 3, ---),
and are thus all in the upper θ half-plane, it follows that the scaled force $F_o(\chi, t)$ is identically zero for $\chi < -2(1-\beta_p)$, and we have therefore only to consider the case $\chi > 2(1-\beta_p)$.

Defining

(5.10b) $$\bar{\theta} = \frac{\log D}{4\beta_p}$$

and by the residues method we obtain, for $\chi > 2(1-\beta_p)$,

(5.11) $$F_o(\chi, t) = \pi i\, e^{-i(\eta/\beta_p)\chi} \cdot e^{i(\pi p/4\beta_p)\chi} \cdot e^{-\bar{\theta}\chi} \cdot \mathcal{F}(\chi),$$

where

(5.12) $$\mathcal{F}(\chi) = \sum_s f(\theta_s) e^{i(\pi/2)s(\chi/\beta_p)},$$

with

(5.13) $$f(\theta_s) = P(\theta_s, \eta + \beta_p \theta_s) \cdot \frac{\Delta}{\alpha \beta_p},$$

a periodic function in χ with period $4\beta_p$. This function will be investigated in more detail in the following. We here want to stress that, as one can see from eq. (5.11), $F_o(\chi, t)$ has a general oscillatory behaviour with amplitude decreasing with χ in the manner shown by the factor

$$e^{-\bar{\theta}\chi}$$

appearing in the r.h. side of eq. (5.11).

We could define $\bar{\theta}$ as the "inverse scaled damping distance" at which the field is reduced by a factor 1/e due to the dissipation at the resistive termination impedances of the current induced on the plate.

6. - APPLICATIONS -

We now wish to apply the result of the preceding section to some particular cases.

i) Floating plate.

$$r_1 = r_2 = \infty$$

We easily get from eq. (5.10b)

$$\bar{\theta} = 0.$$

It can also be shown that $g(\theta_s)/\alpha$ is limited for any s.

We have no damping of the wake field produced by a particle in presence of a floating plate.

ii) Shorted plate.

$$r_1 = r_2 = 0$$

we still have $\bar{\theta} = 0$, and $g(\theta_s)/\alpha$ limited for any s.

Also in this case there is no damping of the wake field.

iii) Plate open at one end.

Taking one of the two termination impedances to be very large, and letting r be the other impedance, one has

$$\bar{\theta} = \frac{1}{4\beta_p} \log \left| \frac{1+r}{1-r} \right| .$$

iv) Plate with equal termination impedances.

If $r_1 = r_2 = r$, then

$$\bar{\theta} = \frac{1}{2\beta_p} \log \left| \frac{1+r}{1-r} \right|$$

v) Plate with small termination impedances.

We assume

$$r_1 \ll 1 \qquad \text{and} \qquad r_2 \ll 1$$

then it is

$$\bar{\theta} = \frac{r_1 + r_2}{\beta_p}$$

vi) Plate terminated by its characteristic impedance.

One can try to apply the expression (5. 11) for $F_o(\chi, t)$ to the case where r_1, or r_2, or both, are equal to one (W = 1). We see from eqs. (5. 8) and (5. 10a) that as W approaches 1, the imaginary part of θ_s, $\bar{\theta}$, increases until it becomes infinite at W = 1. Thus it seems that for this case the wake field is damped as soon as it is produced. Actually (eq. (5. 5)) is zero, so that, one must calculate the limit of $g(\theta_s)/\alpha$ for $W \rightarrow 1$ and any s.

Careful computation gave again the result of eq. (4. 2), obtained in section 4.

vii) Plate with small detuning.

We take $r_1 = 1 + x_1$, $r_2 = 1 + x_2$ with x_1 and x_2 very small quantities with respect to 1.

We obtain

$$\bar{\theta} = \frac{1}{4\beta_p} \log \frac{2}{|x_1 \; x_2|}$$

For a detuning of 1%, i. e. $x_1 = x_2 = 10^{-2}$ we have that the "damping distance" of the wake field corresponds to about 20% of the plate length.

Decreasing the deturning the "damping distance" is further reduced.

7. - INSPECTION OF THE FUNCTION $\mathcal{F}(\chi)$ -

The function $\mathcal{F}(\chi)$ has been defined in section 5 by eqs. (5. 12) and (5. 13). The coefficient $f(\theta_s)$ can be written in the following form

$$f(\theta_s) = \frac{\beta_w}{2i\theta_s \alpha \beta_p} \; \frac{\beta_p - \beta_w}{1 - \beta_w^2} \left\{ \frac{1-\beta_p}{1-\beta_w} \left[(r_1 - 1)(e^{-2i\beta_w\theta_s} - e^{-2i\theta_s}) + \right.\right.$$

$$\left. + (1 + r_2(e^{2i\theta_s} - e^{2i\beta_w\theta_s}) \right] - \frac{1+\beta_p}{1+\beta_w} \left[(1-r_2)(e^{2i\theta_s} - e^{-2i\beta_w\theta_s}) + \right.$$

$$\left.\left. - (1 + r_1(e^{2i\beta_w\theta_s} - e^{-2i\theta_s}) \right] \right\} +$$

$$
- \frac{\beta_w}{2i\theta_s \alpha \beta_p} \frac{1-\beta_w\beta_p}{1-\beta_w^2} \left\{ \frac{1-\beta_p}{1-\beta_w} \left[r_2(1-r_1)(e^{-2i\beta_w\theta_s} - e^{-2i\theta_s}) + \right. \right.
$$

(7. 1)

$$
\left. + r_1(1+r_2)(e^{2i\theta_s} - e^{2i\beta_w\theta_s}) \right] - \frac{1+\beta_p}{1+\beta_w} \left[r_1(1-r_2)(e^{2i\theta_s} - e^{-2i\beta_w\theta_s}) + \right.
$$

$$
\left. \left. + r_2(1+r_1)(e^{2i\beta_w\theta_s} - e^{-2i\theta_s}) \right] \right\} .
$$

θ_s is given by eq. (5. 10a), and β_w by

$$
\beta_w = \frac{\eta + \beta_p \theta_s}{\theta_s} \tag{7. 2}
$$

We try now to give an analytic expression for the function $\mathcal{F}(\chi)$. For this purpose we make the following assumptions and remarks.

a) We assume $\eta = \nu l/2c$ to be a very small quantity with respect to one. This is verified, to our knowledge, for all existing accelerating and storage rings. Then, whenever possible, we will drop all the small quantities in η or in powers of η .

b) As $|s|$ increases and η and $\bar{\theta}$ can be neglected θ_s approaches the value $\pi/2(s/\beta_p)$. In this connection we observe that $\bar{\theta}$ takes the highest values for matched or nearly matched plates, but $\bar{\theta}$ is less than 4 also for detuning of the order of 0, 1% of the characteristic impedance.

c) We assume that β_p is close enough to one that it is possible to make use the following approximated relations

$$
\frac{1}{\beta_p} = 1 + x \qquad \text{with} \qquad x \ll 1
$$

and

$$
1 - \beta_p = \frac{1-\beta_p^2}{2} = \frac{1}{2\gamma_p^2} .
$$

d) From eqs. (7. 1) and (7. 2) and using

$$
\beta_p \theta_s \sim \frac{\pi}{2} s \qquad \text{for} \qquad |s| \gg 1
$$

we obtain that $f(\theta_s)$ decreases as θ_s^{-1} so that the convergence of the series at the r. h. side of eq. (5. 12) can be insured.

e) We remark that the $P(\theta, \phi)$ - factor in the integral at the r. h. side of eq. (2. 10) has been derived in ref. (3) for θ small enough to satisfy the relation

$$
2\theta \frac{b}{e} \sqrt{1-\beta_w^2} \ll 1. \tag{7. 3}
$$

For higher values of θ one should multiply the $P(\theta, \phi)$ - function by an extra form factor that is approximately one whenever eq. (7. 3) holds, and decreases exponentially as θ increases further. This extra form factor can be espressed as a combination of modified first order Bessel functions (see, for instance, ref. (4)).

Therefore the exact expression for $f(\theta_s)$ should not decrease as θ_s^{-1} as one can derive from eq. (5.12), but exponentially.

From the above assumptions and remarks we can derive a useful expansion for $\exp(2i\theta)$, that is

$$e^{2i\theta} = e^{2i\phi}\left[1 + i\,\frac{\phi - 2\eta\gamma_p^2}{\beta_p\gamma_p^2} - \left(\frac{\phi - 2\eta\gamma_p^2}{\beta_p\gamma_p^2}\right)^2 + 0(\phi)\right]. \tag{7.4}$$

$0(\phi)$ is a small function when ϕ is small with respect to γ_p^2, and we will omit it in our calculation also when ϕ is very large. In fact in this case we must remember that, as mentioned in points (d) and (e), its contribution to the series defining the function $\mathcal{F}(\chi)$ of $\bar{\phi}$ can be neglected.

Inserting eq. (7.4) for $e^{2i\theta}$ in eq. (7.1), we can write $f(\theta_s)$ as the sum, of three terms corresponding to the three terms at the r.h. side of eq. (7.4). That is

$$f(\theta_s) = f_o(\theta_s) + f_1(\theta_s) + f_2(\theta_s). \tag{7.5}$$

Introducing

$$\phi_s = \eta + \beta_p\theta_s$$

and

$$2\phi_o = \frac{\pi}{2}p + 2i\beta_p\bar{\theta},$$

it results

$$f_o(\theta_s) = (-1)^s\beta_p\,\frac{r_1 - r_2}{\alpha}\,\sin 2\phi_o\,\frac{\phi_s}{\left(\phi_s - \frac{\eta}{1+\beta_p}\right)^2}. \tag{7.6}$$

This quantity is identically zero for either floating or shorted plates, and it gives the most substantial contribution to $f(\theta_s)$ for the intermediate cases. The contribution for floating and shorted plates cases must therefore be looked for in the terms $f_1(\theta_s)$ and $f_2(\theta_s)$.

We have

$$\begin{aligned} f_1(\theta_s) = {} & \frac{\beta_w}{2i\theta_s\alpha\beta_p}\,\frac{\beta_p-\beta_w}{1-\beta_w^2}\left\{\frac{1-\beta_p}{1-\beta_w}\left[(r_1-1)e^{-2i\phi_s} + (1+r_2)e^{2i\phi_s}\right] + \right.\\ & \left. - \frac{1+\beta_p}{1+\beta_w}\left[(1-r_2)e^{2i\phi_s} - (1+r_1)e^{-2i\phi_s}\right]\right\} i\,\frac{\phi_s - 2\eta\gamma_p^2}{\beta_p\gamma_p^2} + \\ & - \frac{\beta_w}{2i\theta_s\alpha\beta_p}\,\frac{1-\beta_w\beta_p}{1-\beta_w^2}\left\{\frac{1-\beta_p}{1-\beta_w}\left[r_2(1-r_1)e^{-2i\phi_s} + r_1(1+r_2)e^{2i\phi_s}\right] + \right.\\ & \left. - \frac{1+\beta_p}{1+\beta_w}\left[r_1(1-r_2)e^{2i\phi_s} + r_2(1+r_1)e^{-2i\phi_s}\right]\right\} i\,\frac{\phi_s - 2\eta\gamma_p^2}{\beta_p\gamma_p^2} = \end{aligned} \tag{7.7}$$

$$= f_{1_{sh}}(\theta_s) + f_{1_{fl}}(\theta_s).$$

It is easily verified that the first term at the r. h. side of eq. (7. 7) contributes substantially to the shorted plate case and can be neglected for the floating plate one. This term has been indicated by $f_{1_{sh}}(\theta_s)$. The second term, indicated by $f_{1_{fl}}(\theta_s)$, contributes to the floating plate case and it can be omitted in the shorted plate one. Thus taking exactly $r_1 = r_2 = 0$ in the former term and $r_1 = r_2 = \infty$ in the latter one, and recalling that, since in both cases $p = 1$, it follows

$$\sin 2\phi_s = (-1)^s,$$

we obtain

(7. 8) $$f_{1_{sh}}(\theta_s) = (-1)^s \frac{\eta \beta_p^2}{2} \frac{0_s}{(\phi_s - \frac{\eta}{1+\beta_p})^2}$$

and

(7. 9) $$f_{1_{fl}}(\theta_s) = (-1)^s \frac{\eta}{2} \frac{\phi_s}{(\phi_s - \frac{\eta}{1+\beta_p})^2}$$

We observe that $f_{1_{sh}}(\theta_s)$ and $f_{1_{fl}}(\theta_s)$ are equal apart from the factor β_p^2 appearing only at the r. h. side of eq. (7. 7). That is nothing but a consequence of the result obtained and discussed in ref. (3) for the wake field produced by a coasting beam perturbation in the presence of conductive plates.

We also observe that the above quantities are small, being multiplied by the small quantity η. When we move away from the floating or shorted plate cases to some intermediate ones, we still obtain for $f_1(\theta_s)$ a quantity multiplied by η or γ_p^{-1} and therefore very much smaller than $f_o(\theta_s)$ given by eq. (7. 6).

Exact computation of the term $f_2(\theta_s)$, i. e. the third one at the r. h. side of eq. (7. 5), would yield small quantities in η^2 or γ_p^{-2} for any couple of values r_1, r_2. We therefore decide to neglect it and consider $f_{1_{sh}}$ and $f_{1_{fl}}$ as given by eqs. (7. 8) and (7. 9), as a good approximation for $f(\theta_s)$ in the cases of shorted and floating plates respectively. Besides, $f_o(\theta_s)$, given by eq. (7. 6) will represent $f(\theta_s)$ well enough for all intermediate cases, except the matched plate one.

For all cases we can write

$$f(\theta_s) = K(r_1, r_2) \frac{2}{\pi} (-1)^s \frac{s + \sigma_1}{(s - \sigma_o)}$$

where, from eqs. (7. 6), (7. 8) and (7. 9), $K(r_1, r_2)$ is, respectively,

$$K(r_1, r_2) = \beta_p \frac{r_1 + r_2}{\alpha} \sin 2\phi_o$$

$$K(r_1, r_2) = \eta \beta_p^2/2,$$

$$K(r_1, r_2) = \eta/2,$$

and

(7.10a) $$\sigma_1 = \frac{p}{2} + \frac{2}{\pi} i \beta_p \bar{\theta}$$

(7.10b) $$\sigma_o = \frac{2\eta/\pi}{1+\beta_p} - \sigma_1 .$$

Introducing

(7.10c) $$y = \frac{2\pi}{T}\left(\chi + \frac{T}{2}\right),$$

with $T = 4\beta_p$ the period of the function $\mathcal{F}(\chi)$, we have

$$\mathcal{F}(\chi) = \sum_s f(\theta_s) e^{i(2\pi/T)s\chi}$$

$$= K(r_1, r_2) \cdot G(y).$$

The function

$$G(y) = \frac{2}{\pi} \sum_s{}' \frac{s-\sigma_1}{(s-\sigma_o)^2} e^{isy}$$

is periodic in y with period 2π. In the n-th period ranging in the interval

$$n < y < 2\pi n \qquad n = 0, 1, 2, \ldots.$$

it is

(7.11) $$G(y) = \left[a_1(y-2\pi n) + a_2\right] e^{i\sigma_o(y-2\pi n)} .$$

The coefficients a_1 and a_2 are given by

(7.12a) $$a_1 = \frac{\sigma_1 + 4\sigma_o}{e^{2\pi i\sigma_o} - 1}$$

and

(7.12b) $$a_2 = \frac{4}{i(e^{2\pi i\sigma_o}-1)} - 2\pi \frac{\sigma_1 + 4\sigma_o}{(e^{2\pi i\sigma_o}-1)^2} e^{2\pi i\sigma_o}$$

Thus, denoting by y_n and $\mathcal{F}_n$ the functions $y(\chi)$ and $\mathcal{F}(\chi)$ in the n-th period, we finally have

(7.13) $$\mathcal{F}_n = K(r_1, r_2) \left[a_1(y_n - 2\pi n) + a_2\right] e^{i\sigma_o(y_n - 2\pi n)} .$$

8. - CONCLUSIONS -

In this paper we have derived the expressions for the wake field of an oscillating particle in the presence of conductive plates terminated at both ends by resistive terminations. We have distinguished two cases: plates matched by their characteristic impedance and plates terminated otherwise; and we have seen that this distinction corresponds to distinguishing between the two possible ways of resonating of the conductive plate structure when excited by a wave current.

Our results are in perfect agreement with our knowledge of the theory of transmission lines of limited length and terminated at both ends. The current travelling along the transmission line axis, is partially absorbed by the termination impedance and partially reflected back every time it reaches an end. The amount of current absorbed depends on the termination impedance, and the remaining current travels back until it reaches the second termination where another fraction is again absorbed. After several reflections at the terminations the current is reduced exponentially. We have seen that in our problem the damping of the signal induced on the plate, and, hence, of the particle wake field, is measured by the parameter $\bar{\theta}$. We have also shown that the wake field has an oscillating behaviour with period equal to $2/\beta_p$ times the length of the plate. The factor $/\beta_p$ appears because of the difference between the particle velocity v and the velocity c of the signal induced on the plate. The periodicity corresponds to two complete reflections of the signal at the transmission line ends.

Our results can be summarized as follows.

a) The wake field produced in the presence of matched plates is "short ranged" in the sense that it acts only on the subsequent particles up to a distance of $2/\beta_p$ times the plate length. This is easily explained by the total absorption of the plate surface current after two complete reflections.

b) Floating or shorted plates. In these cases the current flowing on the plate surface is completely reflected at each end. The wake field produced by an oscillating particle (always backwarded with respect to the particle motion) is thus "long ranged" in the sense that its action extends up to infinity.

c) All other cases of terminated plates behave in a way intermediate between cases a) and b).

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