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S. Ferrara and A. F. Grillo: SOME REMARKS ON INFINITE VENEZIANO REPRESENTATIONS. -

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ABSTRACT. -

We investigate a class of integral representations for dual-resonance models which preserve all physical properties of the original Veneziano amplitude.

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In this note we point out some properties of a class of integral representations describing dual-resonance amplitudes, mainly in connection with the generalization of the Veneziano amplitude to a series of satellite terms. Recently this problem has been extensively investigated by many authors(1). In particulars some explicit examples of such representations and their convergence problems have been studied by Mandelstam, Matsuda, Khuri, Sivers and Yellin(1).

A very instructive example was that proposed by Matsuda for scalar particles :

$$(1) \qquad M(s,t;\lambda) = -\beta \frac{\Gamma(-\alpha_s)\Gamma(-\alpha_t)}{\Gamma(-\alpha_s-\alpha_t)}\ {}_2F_1\left(-\alpha_s, -\alpha_t; \frac{1-\alpha_s-\alpha_t}{2}; \frac{1-\lambda}{2}\right)$$

$$(\alpha_s = as + b)$$

This amplitude has all the good properties of a single Veneziano term for $0 < \lambda \leq 1$ and in fact it reduces to Veneziano for $\lambda = 1$. It is possible to rewrite this function by means of the following integral representation:

$$(2) \qquad M(s,t;\lambda) = \beta \int_0^1 x^{-\alpha_s - 1} (1-x)^{-\alpha_t - 1} \left[1+2x(1-x)(\lambda-1)\right]^{\frac{\alpha_s+\alpha_t}{2}} dx$$

We observe that for $\lambda = 1/2$ we obtain the u-channel amplitude proposed by Mandelstam in the framework of a SU(6) quark model.

On the other hand such models, with a finite number of terms, have been used in phenomenological applications by many authors: in particular Rubinstein, Squires and Chaichian(2), starting from a five-point function, derived a formula very similar to that proposed by Altarelli and Rubinstein(3) for the $p\bar{n} \to 3\pi$ scattering at treshold.

A more general expression which includes all these models as particular cases can be written in the following form(4):

$$(3) \qquad A(s,t;\phi) = \int_0^1 x^{-\alpha_s - 1} (1-x)^{-\alpha_t - 1} \phi(x)^{-\alpha_s - 1} \phi(1-x)^{-\alpha_t - 1} dx$$

where in order to preserve the pole structure and real analyticity of the amplitude, $\phi(x)$ must be real analytic and different from zero in $[0, 1]$. We want to point out that the asymptotic behaviour of the representation (3) in general violates the Regge limit. In fact we have:

$$(4) \qquad A(s,t;\phi) \underset{s\to\infty}{\simeq} \phi(0)^{-\alpha_t - 1} \phi(1)^{-\alpha_s - 1} (-\alpha_s)^{\alpha_t} \cdot \int_0^\infty y^{-\alpha_t - 1} e^{-y\left(1+\frac{\phi'(1)}{\phi(1)}\right)} dy$$

and the Regge behaviour requires that $\phi(1) = 1$ and that the integral converges i. e. $\phi'(1) > -1$. Note that this formula is valid for $\mathrm{Re}\,\alpha_s < 0$, but it can be easily analytically continued in the whole complex plane except for the real axis. Then we obtain:

$$(5) \qquad A(s,t;\phi) \underset{s\to\infty}{\simeq} \phi(0)^{-\alpha_t - 1} (1+\phi'(1))^{\alpha_t} (-\alpha_s)^{\alpha_t} \Gamma(-\alpha_t)$$

An interesting consequence of these constraints is that the residue at the pole $\alpha(s) = n$ is a polynomial of order n in t. This can be seen if we explicitely construct the residue functions.

Expanding in a power series the function $\eta(1-x)^{-\alpha_k-1} = \left[(1-x)\phi(1-x)\right]^{-\alpha_t-1}$ near x=0, we obtain:

(6)
$$A(s,t;\phi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \chi^{(n)}(\alpha_t) \int_0^1 x^{-\alpha_s-1+n} \phi(x)^{-\alpha_s-1} dx =$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \chi^{(n)}(\alpha_t) \frac{1}{n-\alpha_s}\left[1 - \int_0^1 x^{n-\alpha_s} \frac{d}{dx}\phi(x)^{-\alpha_s-1} dx\right]$$
$$\left[\chi^{(n)}(\alpha_t) = \frac{d^n}{dx^n}\eta(x)^{-\alpha_t-1}\Big|_{x=0}\right]$$

The residue at the n-th pole is then:

(7)
$$R_n(\alpha_t) = \frac{(-1)^n}{n!}\chi^{(n)}(\alpha_t)\phi(0)^{-n-1} -$$
$$- \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}\chi^{(k)}(\alpha_t)\left[1 - \frac{d^{n-k}}{dx^{n-k}}\phi(x)^{-n-1}\Big|_0^1\right]$$

The following compact expression of $\chi^{(n)}(\alpha_t)$

(8)
$$\chi^{(n)}(\alpha_t) = \sum_{k=0}^{n} \sum_{i_1 \ldots i_J} (-1)^{n+k} \frac{n!}{k!} \frac{\Gamma(\alpha_t+1+m)\Gamma(\alpha_t+1+k)}{\left[\Gamma(\alpha_t+1)\right]^2} \cdot$$
$$\cdot \prod_{j=1}^{r} \frac{1}{i_j!}\left(\frac{\phi^{(J)}(1)}{J!}\right)^{i_J} \phi(1)^{-\alpha_t-1}$$

($\sum_{i_1 \ldots i_J}$ is extended to the solutions of the partition equation $\sum_{J=1}^{r} J i_J = n - k$ and $m = \sum_{J=1}^{r} i_J$) shows that $Rn(\alpha_t)$ is a polynomial of order n in t only if $\phi(1) = 1$. It is easy to check that at least on the leading trajectory the residues are all positive. Moreover it can be shown that, in order to have an exponential decrease for $s, t \rightarrow \infty$ at fixed u (in any ray except the real axis) the following further conditions must be verified:

(9)
$$\phi(0) = 1, \qquad \phi'(0) \geq 0$$

Note that for the amplitude (1) our conditions are equivalent to the constraint $0 < \lambda \leq 1$.

Moreover the representation (3) can be written as a sum of satellite terms; in fact if we expand the function $\Phi(x)$ near $x = 0$ and $\Phi(1-x)$ near $x = 1$ we can write :

$$(10) \quad A(s,t;\Phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} C_{n-m}(\alpha_s)\, C_m(\alpha_t)\, B(n-m-\alpha_s,\ m-\alpha_t)$$

where $B(x,y)$ is the Euler Beta-function and $C_n(\alpha_s)$ are polynomials of order n in s. We observe that, in this case, the representation (10) can be reordered as a series of Veneziano satellite terms (see Khuri, ref. (1)).

For what concerns the Lorentz content of the amplitude (3), it is in principle possible to constraint the residues (7) in such a way that the poles fit into a single irreducible 0 (3, 1)-representation[5] (Toller pole). Nevertheless these conditions are necessary but not sufficient in order that the series (10) converges[6]. In this connection Domokos and Domokos[7] recently suggested that duality is not compatible with Lorentz irreducibility in a single channel. A more refined projection must be performed in order to have a simple group theoretical interpretation of dual amplitudes.

As a final point we note that the integral representation (3) allows a natural extension to the N-point dual amplitude[8], at least in the form proposed by Chan. In fact, following the notations of ref. (8) we can write :

$$(11) \quad A_N(1,2,\ldots,N;\Phi) = \int_0^1 \prod_p dx_p \prod_p \left(x_p \Phi(x_p)\right)^{-\alpha_p - 1} \cdot \prod_{p' \neq (1,J)} \delta\left(-1 + x_{p'} + \prod_{\bar{p}'} x_{\bar{p}'}\right)$$

where the conditions on the x_p-variables are the usual ones.

However the factorization of the N-point function is in principle very complcate[9]. Nevertheless there is some feeling that the N-point amplitude factorizes if the function $\Phi(x)$ satisfies some particular conditions[10].

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(2) - H. R. Rubinstein, E. J. Squires and M. Chaichian, Phys. Letters 30, 189 (1969).

(3) - G. Altarelli and H. R. Rubinstein, to be published in Phys. Rev.

(4) - We point out, however, that eq. (3) can be generalized again by writing

$$A(s,t;\phi,\psi) = \int_0^1 x^{-\alpha_s-1}(1-x)^{-\alpha_t-1}\phi(x)^{-\alpha_s-1}\phi(1-x)^{-\alpha_t-1}\psi(x)\,dx$$

where $\psi(x) = \psi(1-x)$ is real analytic and without zeros in $[0, 1]$.

(5) - M. Paciello, L. Sertorio and B. Taglienti, Nuovo Cimento 62A, 713 (1969).

(6) - P. Di Vecchia, F. Drago and S. Ferrara, Phys. Letters 29B, 114 (1969).

(7) - G. Domokos and S. Kovesi-Domokos, NYO-4076-3, Argonne (1969).

(8) - For a review of the subject and a complete list of references, see H. M. Chan, CERN Preprint TH-1057.

(9) - S. Fubini and G. Veneziano, MIT Preprint; K. Bardakci and S. Mandelstam, Berkeley Preprint; D. Gross, CERN Preprint TH-1048.

(10) - We thank Dr. G. Benfatto and Dr. G. Rossi for having emphasized this point.