A.G. Ruggiero and V.G. Vaccaro: THE ELECTRO-MAGNETIC FIELD OF AN INTENSE COASTING BEAM PERTURBATION IN THE PRESENCE OF CONDUCTIVE PLATES TERMINATED AT BOTH ENDS. -

A. G. Ruggiero[x] and V. G. Vaccaro[o]: THE ELECTRO-MAGNETIC FIELD OF AN INTENSE COASTING BEAM PERTURBATION IN THE PRESENCE OF CONDUCTIVE PLATES TERMINATED AT BOTH ENDS. -

1. - INTRODUCTION. -

Coherent instabilities of intense coasting beams have been observed in several particle accelerators or storage rings in the last years. In some cases (MURA, CESAR,.....) these instabilities seemed to be indebted to the presence of conductive plates, such as clearing electrodes or deflecting electrodes, in the accelerator vacuum chamber.

To understand this beam-equipment interaction a theoretical investigation has been started by L. J. Laslett[1]. Using a rectangular geometry and taking the transverse average value for the charges and currents induced on the plates, L. J. Laslett studied the effect of a conductive plate, practically covering all the accelerator circumference, with a single termination placed at the centre of the plate itself.

He found a coupling resonance between the termination impedance and the distributed impedances of the two plate branches separated by the feed-point.

More recently A. G. Ruggiero, P. Strolin and V. G. Vaccaro[2,3], continuing the investigation, used a circular geometry and a more rigorous expression for the induced charges and currents (always averaged along the transverse size of the plate). They investigated the effect of the plates also at very high frequencies.

They assumed again to have a single termination, but the position of the feed-point had been taken as a variable.

The above mentioned authors encountered in their investigation two different types of resonances. The resonances of the first type happen when the termination absorbs current from the plate surface, and at those frequencies for which the total impedance of our plate - termination structure as seen by the feed - point appears as infinitely large. One has resonances of the second type when the termination does not absorb current, and the con-

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ductive plate structure appears as an unloaded transmission line. In the last case the position of the connecting point is very important. In particular one has shown that a plate terminated at one of the two ends can never resonate as an unloaded transmission line.

The following analysis is devoted to work out the fields produced by a coherent oscillating beam in the presence of conductive plates terminated at both ends. The termination impedances are used as variables.

We will take a beam perturbation with z, t–dependence of the kind exp $\left[-i(kz - \omega t)\right]$, travelling along a continuous beam with circular cross section and placed at the centre of a cylindrical straight pipe. The perturbation is taken to be any mode of transverse throbbing configuration. We will investigate also the effect of a longitudinally perturbed beam.

2. - THE METHOD. -

The method used in the following is very alike that used in reference (1) and (3). We have a coasting stationary particle beam of circular cross section with radius a, travelling along the axis of an infinitely long straight circular pipe of inner radius b. Conductive plates, terminated at both the ends, are placed parallel to the vacuum chamber wall with a given longitudinal distribution having periodicity equal to one revolution around the circular accelerator circumference. There are M plates per period, and all identical each to the others.

To the beam we can associate an unperturbed charge and current distribution

$$\rho_o = \frac{e\lambda_o}{\pi a^2} \, \mathcal{H}(a - r), \qquad\qquad \vec{J}_o \equiv (0, \, 0, \, v\rho_o),$$

where

$$\mathcal{H}(x) = 1 \quad \text{for} \quad x > 0$$
$$= 0 \quad \text{for} \quad x < 0,$$

v is the beam velocity, and

$$e\lambda_o = \frac{N e}{2\pi R} \quad \text{is the charge per unit length,}$$

N, number of the particles in the beam

R, radius of the beam closed orbit

e, particle charge.

In the perturbed beam the radius varies as $a + \varsigma \cos m\psi$, where 2m is the multipole number, and

$$\varsigma = \varsigma_o \, e^{-i(kz - \omega t)}.$$

Thus the total charge and current distribution can be expressed in the following way

(2.1) $$\rho = \rho_o + \rho_1, \qquad\qquad \vec{J} = \vec{J}_o + \vec{J}_1.$$

A self-consistent solution satisfying the charge conservation law, in the limit $k \ll b \Big/ \sqrt{1 - \dfrac{v^2}{c^2}}$, is

$$(2.2a) \qquad \mathcal{S}_1 = \frac{e \lambda_o}{\pi a^2} \sum \left[\delta(r-a) - \frac{2}{a} \delta_{m,0} \mathcal{H}(a-r) \right] \cos m \varphi \ ,$$

where $\delta(x)$ is the Dirac function, and $\delta_{m,0}$ is always zero except when $m = 0$.

Similarly we have

$$(2.2b) \qquad \vec{J}_1 \equiv \begin{pmatrix} i(\omega -kv)(\frac{r}{a})^{m-1} \left[1+ \delta_{m,0}(\frac{r^2}{a^2} -1)\right] \mathcal{H}(a-r) \cos m \varphi \\ -i(\omega -kv)(\frac{r}{a})^{m-1} \mathcal{H}(a-r) \sin m \varphi \\ v \left[\delta(r-a) - \frac{2}{a} \delta_{m,0} \mathcal{H}(a-r)\right] \cos m \varphi \end{pmatrix} \frac{e \lambda_o}{\pi a^2} \gamma \ .$$

For a longitudinally perturbed beam we can make still use of eq. (2.1), but now

$$(2.3a) \qquad \mathcal{S}_1 = \frac{e \lambda}{\pi a^2} \mathcal{H}(a-r) \ ,$$

with $e \lambda = e \lambda_1 e^{-i(kz- \omega t)}$ the perturbed charge per unit length, and

$$(2.3b) \qquad \vec{J}_1 \equiv (0, 0, \frac{\omega}{k} \mathcal{S}_1) \ .$$

The perturbed charge and current densities $\mathcal{S}_1$ and $\vec{J}_1$ induce on the conductive plate surface a surface charge density σ and a current per unit length I. We compute σ and I neglecting, in zero-th approximation, the presence of the plates themselves; i.e. we assume that the charges and currents induced on the plate surface are equal to the charges and currents induced on the vacuum chamber wall, these also to be taken without resistivity.

The distribution σ and I can be expressed in the following form

$$(2.4a) \qquad \sigma = \sigma_o \, g \cos m \varphi \, e^{-i(kz- \omega t)} \ ,$$

$$(2.4b) \qquad I = \beta c \sigma \ ,$$

where σ_o has the dimension of a charge per unit of surface, and g and β are two a-dimensional parameters. The three factors σ_o, β and g, have been calculated in the approximation $k \ll b/\sqrt{1-(v^2/c^2)}$, and they can be found in the Appendix. In the case of a longitudinally perturbed beam, the $\cos m \varphi$ factor must be dropped.

In the Appendix we have made use of the following notation, which will be used also in the subsequent analysis:

$$c = \text{light velocity}, \qquad \beta_p = v/c, \qquad \beta_w = \omega/kc \ ,$$

$$\gamma_p = 1/\sqrt{1- \beta_p^2} \ , \qquad \gamma_w = 1/\sqrt{1- \beta_w^2}, \qquad q = k/\gamma_w \ ,$$

4.

With I we are referring to the longitudinal current induced on the plate surface. Really, we should consider also the transverse induced current I_φ .

The charges σ and the currents I and I_φ travel along the plate surface in both the two longitudinal and transverse directions.

It is very complicate to take into account the distribution of σ, I and I_φ, since one need use some bi-dimensional transmission line equations, and the problem would be equivalent to that of an oscillating membrane with some boundary conditions.

To make easier our problem, we decide to integrate the value of I and σ over the transverse size of the plate, and consider this with the surrounding media as some one-dimensional transmission line structure.

We define

$$(2.5) \qquad b_m = b \int \cos m \, \varphi \; d\varphi \; ,$$

where the integral extends over the transverse size of the plate.

The total charge per unit length induced on the plate surface, assumed to be a sector of cylinder at $r = b$, is, hence,

$$(2.6a) \qquad \sigma_t = \sigma_o \, g \, b_m \, e^{-i(kz - \omega t)} \; .$$

Similarly we define the total induced current

$$(2.6b) \qquad I_t = \beta \, c \, \sigma_t \; .$$

With this assumption the transverse current I_φ will not appear in our analysis, and the potentials V_1 and $\vec{A}_1$ at the plate surface are governed by the usual transmission line equations, apart from a forcing term depending only on I_t and σ_t.

Our goal is to investigate the self-forces produced on the beam perturbation through the potential distributions V_1 and $\vec{A}_1$.

3. - THE TRANSMISSION LINE EQUATIONS AND THEIR SOLUTIONS. -

With reference to the Fig. 1 we use the following notation:

$Z_o = 1/cC$, characteristic impedance of the plate-wall structure,
C, capacitance per unit length,
Z_1, Z_2, termination impedances,
$r_1 = Z_1/Z_o$ and $r_2 = Z_2/Z_o$,
z_o , coordinate of the plate centre,
1, plate length,
$2 \, \varphi_o$, transverse angular plate aperture,
$\overline{\varphi}$, angular position of the plate axis with respect to the transverse throbbing oscillation.

We will deal with the transmission line equations in the form as used by J. L. Laslett[1], and subsequently adopted in ref. (3).

These equations are

$$(3.1a) \qquad \frac{1}{c}\frac{\partial V_1}{\partial t} + \frac{\partial A_1}{\partial z} = -Z_o\left(\frac{\partial \sigma_t}{\partial t} + \frac{\partial I_t}{\partial z}\right)$$

and

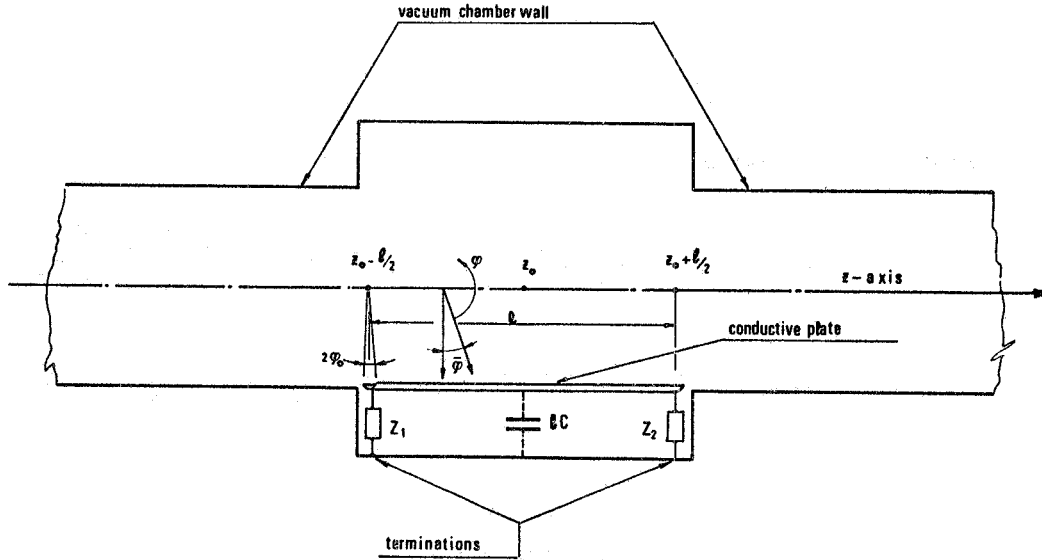$$(3.1b) \qquad \frac{\partial V_1}{\partial z} + \frac{1}{c}\frac{\partial A_1}{\partial t} = 0.$$

FIG. 1

A_1 is the longitudinal component of the vector potential $\vec{A}_1$; the other two components, for the assumptions mentioned in Section 2, are taken to be zero.

Eq. (3.1a) comes from the charge and current conservation law, applied to the gene ric surface plate element of infinitesimal length and extending along all the transverse size of the plate. Eq. (3.1b) is equivalent to impose that the longitudinal electric field E_z on the plate surface is zero, assuming the plate perfectly conductive.

Taking into account eqs. (2.6), eq. (3.1a) can be transformed as

$$(3.1c) \qquad \frac{1}{c}\frac{\partial V_1}{\partial t} + \frac{\partial A_1}{\partial z} = i Z_o kc\,(\beta - \beta_w)\,\sigma_o\,g\,b_m\,e^{-i(kz-\omega t)},$$

Solutions of the above equations are

$$(3.2a) \qquad V_1 = \mathcal{A}\left\{\left[\mathcal{E}_1\,e^{-i\frac{\omega}{c}(z-z_o)} + \mathcal{E}_2\,e^{i\frac{\omega}{c}(z-z_o)}\right]e^{i\omega t} + \beta_w\,e^{-i(kz-\omega t)}\right\}$$

and

$$(3.2b) \qquad A_1 = \mathcal{A}\left\{\left[\mathcal{E}_1\,e^{-i\frac{\omega}{c}(z-z_o)} - \mathcal{E}_2\,e^{i\frac{\omega}{c}(z-z_o)}\right]e^{i\omega t} + e^{-i(kz-\omega t)}\right\},$$

with

$$(3.3) \qquad \mathcal{A} = \sigma_o \frac{b_m}{C} \, g \, \frac{\beta_w - \beta}{1 - \beta_w^2} \, .$$

The coefficients $\mathcal{E}_1$, $\mathcal{E}_2$ can be determined with the usual conditions at the feed--points,

$$(3.4a) \qquad V_1 = r_2 (A_1 + Z_o I_t), \qquad \text{at} \qquad z = z_o + 1/2 \, ,$$

$$(3.4b) \qquad V_1 = -r_1 (A_1 + Z_o I_t), \qquad \text{at} \qquad z = z_o - 1/2 \, .$$

Using these conditions, we obtain

$$(3.5a) \qquad \mathcal{E}_1 = \left[(1-r_1)(\tilde{r}_2 - \beta_w) \, e^{-i(\theta+\phi)} + (1+r_2)(\tilde{r}_1 + \beta_w) \, e^{i(\theta+\phi)} \right] \frac{e^{-ikz_o}}{\Delta} \, ,$$

$$(3.5b) \qquad \mathcal{E}_2 = -\left[(1-r_2)(\tilde{r}_1 + \beta_w) \, e^{i(\theta-\phi)} + (1+r_1)(\tilde{r}_2 - \beta_w) \, e^{-i(\theta-\phi)} \right] \frac{e^{-ikz_o}}{\Delta} \, ,$$

with

$$(3.6) \qquad \tilde{r}_i = r_i \beta_w \frac{1 - \beta_w \beta}{\beta_w - \beta} \, , \qquad i = 1, 2 \, ,$$

$$\theta = kl/2 \qquad \text{and} \qquad \phi = \omega l/2c = \beta_w \theta \, ,$$

and

$$(3.7) \qquad \Delta = -2 \left[(r_1 + r_2) \cos 2\phi + i (1 + r_1 r_2) \sin 2\phi \right] \, .$$

4. - THE POTENTIAL DISTRIBUTION AROUND THE ACCELERATOR CIRCUMFERENCE. -

We denote with V and $\vec{A}$ respectively the scalar and the vector potential in the region of space inside the vacuum chamber. We denote with A the z-component of $\vec{A}$, and we take the transverse components of $\vec{A}$ equal to zero.

Using cylindrical coordinates r, φ, z, we can write the potentials at $r = b$,

$$(4.1a) \qquad V(b, \varphi, z, t) = \overline{V}_1(z) \, \Omega(z) \, \phi(\varphi) \, e^{-i(kz - \omega t)} \, ,$$

$$(4.1b) \qquad A(b, \varphi, z, t) = \overline{A}_1(z) \, \Omega(z) \, \phi(\varphi) \, e^{-i(kz - \omega t)} \, .$$

$\Omega(z)$ and $\phi(\varphi)$ are two functions respectively of z and φ; they are everywhere zero except on the plate surface where they are 1.

Besides we have also used

(4.2a)
$$V_1 = \overline{V}_1(z) \, e^{-i(kz - \omega t)}$$

(4.2b)
$$A_1 = \overline{A}_1(z) \, e^{-i(kz - \omega t)}$$

where V_1 and A_1 are given by (3.2a and b).

Our problem is to look for that V, A potential distribution inside the vacuum chamber matching with the distribution (4.1) at $r = b$.

The potentials V and A satisfy the following wave equation in the region $r < b$

(4.3)
$$\nabla^2 f - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = 0 \, ,$$

where V and A must be used in the place of f.

Before solving the differential equation (4.3) we decompose the r.h. side of equations (4.1) in a double Fourier series. For this purpose we observe that $\overline{V}_1(z) \, \Omega(z)$ and $\overline{A}_1(z) \, \Omega(z)$ are periodic functions in z with period equal to the accelerator circumference, $2\pi R$, and that $\Phi(\psi)$ is periodic in ψ with period 2π. In the more general case where the plate axis forms an angle $\overline{\psi}$ with the throbbing beam oscillation direction (taken as origin of the ψ-coordinate), we have at $r = b$,

(4.4a)
$$V = e^{-i(kz - \omega t)} \sum_{p, h} V_h (\phi_p \cos p\psi + \theta_p \sin p\psi) \, e^{-i \frac{h}{R} z}$$

and

(4.4b)
$$A = e^{-i(kz - \omega t)} \sum_{p, h} A_h (\phi_p \cos p\psi + \theta_p \sin p\psi) \, e^{-i \frac{h}{R} z} \, .$$

It is

$$\phi_p = \frac{1}{\pi} \int_0^{2\pi} \Phi(\psi) \cos p\psi \, d\psi \, , \qquad \theta_p = \frac{1}{\pi} \int_0^{2\pi} \Phi(\psi) \sin p\psi \, d\psi$$

for $p = 1, 2, 3, \ldots\ldots$, and

$$\phi_0 = \frac{1}{2\pi} \int_0^{2\pi} \Phi(\psi) \, d\psi \, , \qquad \theta_0 = 0 \, .$$

Besides, it is

$$V_h = \frac{1}{2\pi R} \int_0^{2\pi R} \overline{V}_1(z) \, \Omega(z) \, e^{i \frac{h}{R} z} \, dz, \qquad A_h = \frac{1}{2\pi R} \int_0^{2\pi R} \overline{A}_1(z) \, \Omega(z) \, e^{i \frac{h}{R} z} \, dz \, .$$

Denoting with z_{os} the coordinate of the centre of the s-th of the M identical plates around a period of $2\pi R$, we have

(4.5a)
$$V_h = \frac{Ml}{2\pi R} \propto_h \mathcal{A} \left\{ \mathcal{E}_1' \frac{\sin(\theta + \frac{hl}{2R} - \emptyset)}{\theta + \frac{hl}{2R} - \emptyset} + \mathcal{E}_2' \frac{\sin(\theta + \frac{hl}{2R} + \emptyset)}{\theta + \frac{hl}{2R} + \emptyset} + \beta_w \frac{\sin\frac{hl}{2R}}{\frac{hl}{2R}} \right\}$$

and

(4.5b)
$$A_h = \frac{Ml}{2\pi R} \propto_h \mathcal{A} \left\{ \mathcal{E}_1' \frac{\sin(\theta + \frac{hl}{2R} - \emptyset)}{\theta + \frac{hl}{2R} - \emptyset} - \mathcal{E}_2' \frac{\sin(\theta + \frac{hl}{2R} + \emptyset)}{\theta + \frac{hl}{2R} + \emptyset} + \frac{\sin\frac{hl}{2R}}{\frac{hl}{2R}} \right\} ,$$

where

$$\mathcal{E}_i' = \mathcal{E}_i e^{ikz_{os}} \qquad i = 1, 2 ,$$

and

(4.6)
$$\propto_h = \frac{1}{M} \sum_s e^{i\frac{h}{R}z_{os}} .$$

Then, the solution of eq. (4.3) taking the value (4.4) at $r = b$ is given by

(4.7a)
$$V = e^{-i(kz - \omega t)} \sum_{p,h} V_h (\emptyset_p \cos p\varphi + \theta_p \sin p\varphi) \frac{I_p(q_h r)}{I_p(q_h b)} e^{-i\frac{h}{R}z} ,$$

(4.7b)
$$A = e^{-i(kz - \omega t)} \sum_{p,h} A_h (\emptyset_p \cos p\varphi + \theta_p \sin p\varphi) \frac{I_p(q_h r)}{I_p(q_h b)} e^{-i\frac{h}{R}z} .$$

$I_p(x)$ is the modified Bessel function of the first kind and p-th order, and

$$q_h = (k + \frac{h}{R}) \sqrt{1 - \frac{\omega^2}{c^2(k + \frac{h}{R})^2}} .$$

5. - THE FIELDS PRODUCED BY THE PERTURBATION. -

The electric ($\vec{E}$) and magnetic ($\vec{H}$) components of the field can be calculated from

$$\vec{E} = -\text{grad } V - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} ; \qquad \vec{H} = \text{rot } \vec{A} .$$

Since the first two components of $\vec{A}$ are zero, we have

$$E_r = -\frac{\partial V}{\partial r} , \qquad E_\varphi = -\frac{1}{r} \frac{\partial V}{\partial \varphi} , \qquad E_z = -\frac{\partial V}{\partial z} - \frac{1}{c} \frac{\partial A}{\partial t}$$

$$H_r = \frac{1}{r} \frac{\partial A}{\partial \varphi} , \qquad H_\varphi = -\frac{\partial A}{\partial r} , \qquad H_z = 0 .$$

Using the expression (4.7) for V and A and denoting with a dash the derivative of I_p with respect to the argument, we have

(5.1a)

$$E_r = -e^{-i(kz-\omega t)} \sum_{p,h} V_h (\emptyset_p \cos p\psi + \theta_p \sin p\psi) q_h \frac{I'_p(q_h r)}{I_p(q_h b)} e^{-i\frac{h}{R}z}$$

$$E_\psi = e^{-i(kz-\omega t)} \sum_{p,h} V_h (\emptyset_p \sin p\psi - \theta_p \cos p\psi) \frac{p}{r} \frac{I_p(q_h r)}{I_p(q_h b)} e^{-i\frac{h}{R}z}$$

$$E_z = i e^{-i(kz-\omega t)} \sum_{p,h} \left[(\frac{h}{R}+k) V_h - \frac{\omega}{c} A_h \right] (\emptyset_p \cos p\psi + \theta_p \sin p\psi) \frac{I_p(q_h r)}{I_p(q_h b)} e^{-i\frac{h}{R}z} ,$$

and

(5.1b)

$$H_r = -e^{-i(kz-\omega t)} \sum_{p,h} A_h (\emptyset_p \sin p\psi - \theta_p \cos p\psi) \frac{p}{r} \frac{I_p(q_h r)}{I_p(q_h b)} e^{-i\frac{h}{R}z}$$

$$H_\psi = -e^{-i(kz-\omega t)} \sum_{p,h} A_h (\emptyset_p \cos p\psi + \theta_p \sin p\psi) q_h \frac{I'_p(q_h r)}{I_p(q_h b)} e^{-i\frac{h}{R}z}$$

$$H_z = 0 .$$

With the expressions (5.1a, b) for the components of $\vec{E}$ and $\vec{H}$, we construct the force per unit charge acting on the beam

$$\vec{F} = \vec{E} + \frac{\vec{V}}{c} \times \vec{H} .$$

The three comptonents of $\vec{F}$ are

(5.2a)

$$F_r = -e^{-i(kz-\omega t)} \sum_{p,h} (V_h - \beta_p A_h) (\emptyset_p \cos p\psi + \theta_p \sin p\psi) q_h \frac{I'_p(q_h r)}{I_p(q_h b)} e^{-i\frac{h}{R}z}$$

(5.2b)

$$F_\psi = e^{-i(kz-\omega t)} \sum_{p,h} (V_h - \beta_p A_h) (\emptyset_p \sin p\psi - \theta_p \cos p\psi) \frac{p}{r} \frac{I_p(q_h r)}{I_p(q_h b)} e^{-i\frac{h}{R}z}$$

(5.2c)

$$F_z = E_z .$$

The two transverse components in a Cartesian frame are given by

(5.3a)

$$F_x = -(F_r \sin\psi + F_\psi \cos\psi) ,$$

$$F_y = F_r \cos\psi - F_\psi \sin\psi ,$$

having denoted with y the axis directed along the throbbing beam oscillation direction.

Generally one is interested in taking only that term of the force-components, F_x, F_y and F_z, which oscillates in the same way as the perturbation; i. e., the term of the series at the r. h. side of each eq. (5. 2) having h = 0. In fact the other terms having the mode number n' = n + h different from the fundamental mode of the perturbation, n=kR, are orthogonal to it and hence do not interact with it. Nevertheless these other force modes having n' $\neq$ n could act on the other perturbated modes eventually present in the beam, so that it could be possible to have a coupling between the different modes of the unstable beam.

This coupling can exist only when one has (as in our case) some discontinuities of the boundary conditions. When this is not the case, as for instance in the uniform resistive wall effect, the coupling cannot be present.

If we limit ourselves to the modes $k \ll b/\gamma$ we have

$$q_h \frac{I'_p(q_h r)}{I_p(q_h b)} \approx \frac{p}{r} \frac{I_p(q_h r)}{I_p(q_h b)} \approx \frac{p}{b} (\frac{r}{b})^{p-1}$$

and

(5. 4a)
$$F_x = - \left[\sum_h (V_h - \beta_p A_h) e^{-i\frac{h}{R} z} \right] \left[\sum_p \frac{\partial U_p}{\partial x} \right] e^{-i(kz - \omega t)} ,$$

(5. 4b)
$$F_y = - \left[\sum_h (V_h - \beta_p A_h) e^{-i\frac{h}{R} z} \right] \left[\sum_p \frac{\partial U_p}{\partial y} \right] e^{-i(kz - \omega t)} ,$$

(5. 5)
$$F_z = i \left\{ \sum_h \left[(\frac{h}{R} + k) V_h - \frac{\omega}{c} A_h \right] e^{-i\frac{h}{R} z} \right\} \left[\sum_p U_p \right] e^{-i(kz - \omega t)} ,$$

U_p is the following potential function

(5. 6)
$$U_p \left[r(x, y), \varphi (x, y) \right] = (\frac{r}{b})^p (\phi_p \cos p\varphi + \theta_p \sin p\varphi) ,$$

with

$$p = 0, 1, 2, \ldots \ldots$$

and

$$r = \sqrt{x^2 + y^2} , \qquad \varphi = - \text{arctg} \frac{x}{y} .$$

We see from eqs. (5.4) and (5.5) that the force components F_x, F_y and F_z are expressed as a combination of all the transverse throbbing configuration modes. The self-force produced by a transversely perturbed beam is obtained taking the term in the series at the r.h. side of each eq. (5.4) having p = m. The other terms can act only on the other p $\neq$ m multipole modes of the unstable beam. Then we see that we can have in a beam two kinds of coupling, the first regards all the transverse throbbing modes and the second all the longitudinal propagation mo des.

Also the force F_z produced by a longitudinally perturbed beam is expressed as a combination of all the transverse p-modes; but in this case all the modes having p $\neq$ 0 are insensitive to any unstable beam configuration.

Besides, we observe that a coupling exists also between perturbed beam oscillations along different directions. Really a longitudinal perturbation produces also transverse field components, and viceversa; a vertical (y) beam oscillation can produce also a radial (x) field component, and viceversa.

We wish to well remark that all these coupling have their origin in the discontinuous nature of the electric properties of the conductive plate - vacuum chamber wall structure.

We observe, at last, that the self-field of the 2m-pole of a throbbing perturbed beam goes as $(a/b)^{m-1}$. So that the dipole mode produces the highest field, the other modes producing field decreasing with m. The relative intensity of the modes is determined by the ratio a/b of the beam size to the vacuum chamber size. In an electron or positron machine a/b is very small so that, apart from the dipole mode, all the other modes produce very weak fields. Instead in a proton machine the ratio a/b can even be close to unity (expecially at the injection or in a full stacked beam in proton storage ring case) and all the modes can produce self-fields of compara ble amount.

6. - THE EFFECT OF THE PLATES DISTRIBUTION AROUND THE ACCELERATOR CIRCUM-FERENCE. -

Inspection of eqs. (4.5) shows that the forces produced by the perturbed beam are pro portional to $Ml/2\pi R$, i.e. the fraction of the accelerator circumference occupied by the conduc tive plates.

The distribution of the plates around the accelerator circumference enters the expres sion for the forces through the factor α_h

$$(6.1) \qquad \alpha_h = \frac{1}{M} \sum_s e^{i\frac{h}{R}z_{os}} \ .$$

We investigate two particular simple cases:

6.1.- Uniform plates distribution.-

We assume that the plates are equally spaced. Thus, denoting with z_{oo} the coordina te of the centre of a reference plate, we have

$$(6.2) \qquad \alpha_h = \frac{e^{i\frac{h}{R}z_{oo}}}{M} \sum_{s=1}^{M} e^{is\frac{2\pi h}{M}} \ ,$$

and by some simple trigonometric relations

$$(6.3) \qquad \alpha_h = e^{i\frac{h}{R}z_{oo}} \frac{\cos\frac{M+1}{M}\pi h + i\sin\frac{M+1}{M}\pi h}{M\sin\frac{\pi h}{M}} \sin\pi h \ ,$$

This quantity is always zero except when h is zero or a multiple of M, in that case

$$(6.4) \qquad \alpha_h = e^{i\frac{h}{R}z_{oo}} (-1)^{\frac{M+1}{M}h} \ .$$

Thus the first significant term in the series at the r.h. side of eqs. (5.4), apart from

that with $h = 0$, is that corresponding to $|h| = M$.

6.2.- Two series of uniformely distributed plates. -

We assume that in each series the distance between two consecutive plates is constant and that the plates of one series superpose those of the other series by a shift of an angle Λ. We have

$$\alpha_h = \frac{e^{i\frac{h}{R}z_{oo}}}{M} \left[\sum_{s=1}^{M/2} e^{is\frac{4\pi h}{M}} + \sum_{s=1}^{M/2} e^{i(s\frac{4\pi h}{M} + \Lambda h)} \right] =$$

(6.5)

$$= e^{i\frac{h}{R}z_{oo}}(1+e^{i\Lambda h}) \frac{\cos\frac{M/2+1}{M/2}\pi h + i\sin\frac{M/2+1}{M/2}\pi h}{M\sin\frac{2\pi h}{M}} \sin\pi h .$$

Also this quantity is generally zero except when h is zero or a multiple of $M/2$, in that case

(6.6)
$$\alpha_h = e^{i\frac{h}{R}z_{oo}} \frac{1+e^{i\Lambda h}}{2} (-1)^{\frac{M+2}{M}h} .$$

From eqs. (6.4) and (6.6) we have

$$\alpha_o = 1$$

in both two cases.

For $h \neq 0$, the α_h-factor depends on the absolute position of the plates with respect to the phase of the perturbed travelling wave, $e^{-i(kz - \omega t)}$.

From eq. (6.4) we have for example for $h = M$

$$\alpha_M = e^{i\frac{M}{R}z_{oo}} (-1)^{M+1} ,$$

and we see that moving the coordinate z_{oo}, i.e. rotating uniformely the plate distribution, we can change the ratio of the imaginary part to the real part of the term with $h = M$ at the r.h. side of eqs. (5.4) and (5.5).

7. - THE SELF-FORCES ACTING ON THE BEAM PERTURBATION. -

The Self-Force components are obtained taking that term with $p = m$ and $h = 0$ at the r.h. side of eqs. (5.4), and taking that term with $p = h = 0$ at the r.h. side of eq. (5.5).

We have

(7.1a)
$$F_x = -(V_o - \beta_p A_o) \frac{\partial U_m}{\partial x} e^{-i(kz - \omega t)}$$

(7.1b)
$$F_y = -(V_o - \beta_p A_o) \frac{\partial U_m}{\partial y} e^{-i(kz-\omega t)} ,$$

and

(7.2)
$$F_z = ik(V_o - \beta_w A_o) \phi_o e^{-i(kz-\omega t)}$$

F_x and F_y are the self-force components produced by a transverse perturbed throbbing beam. F_z is the self-force produced by a longitudinal beam perturbation.

We make inspection of the eqs. (7.1) and (7.2) separately.

7.1.- Throbbing beam.-

Multipole modes ($m \neq 0$). From eqs. (4.5) and (3.3), and eqs. (A.7) and (A.8) that can be found in the Appendix, we have

(7.3)
$$\vec{F} = -\frac{e \lambda_o \gamma}{\pi a^2} (\frac{Ml}{2\pi R}) (\frac{a}{b})^{m+1} (\frac{b_m}{C} grad U_m) P_{trans}$$

where with $\vec{F}$ we have denoted the vector with components F_x and F_y.

We see that the force is proportional to the electric dipole per unit volume

(7.4)
$$\frac{e \lambda_o \gamma}{\pi a^2} ,$$

and to the percent of the accelerator circumference occupied by the conductive plates.

The ratio a/b enter with its $(m+1)$-th power.

The effect of the angular plate position with respect to the throbbing beam oscillation axis is given by the factor (7.5) that represent also the dependence of the force on the transverse displacement

(7.5)
$$\frac{b_m}{C} grad U_m ,$$

where b_m is given by eq. (2.5) and U_m by eq. (5.6).

The dependence of the forces upon the space and time frequencies, respectively k and ω, of the travelling perturbation is described by the factor

(7.6)
$$P_{trans} = \frac{\beta_p - \beta_w}{1 - \beta_w^2} \left\{ \varepsilon'_1 \frac{1 - \beta_p}{1 - \beta_w} \frac{\sin(\theta - \phi)}{\theta} + \varepsilon'_2 \frac{1 + \beta_p}{1 + \beta_w} \frac{\sin(\theta + \phi)}{\theta} + (\beta_w - \beta_p) \right\}$$

where ε'_i are given by eq. (3.5) by replacing β with β_p.

7.2.- Throbbing beam monopole mode (m = 0).-

It is soon seen from eq.(5.2.b) that the azimuthal component F_ϕ of the force for h = 0 and p = 0 is identically zero. The force has only a radial component.

Introducing the potential

$$U_o = \frac{r^2}{2} \phi_o$$

we have, in a cartesian frame,

$$F_x = -(V_o - \beta_p A_o) q^2 \frac{\partial U_o}{\partial x} e^{-i(kz - \omega t)}, \qquad F_y = -(V_o - \beta_p A_o) q^2 \frac{\partial U_o}{\partial y} e^{-i(kz - \omega t)}.$$

From eqs. (4.5) and (3.3), and eqs. (A.8a) and (A.14) in the Appendix, we have

(7.7)
$$\vec{F} = - \frac{e \lambda_o \gamma}{\pi a^2} (\frac{Ml}{2\pi R}) (\frac{a}{b}) (\frac{b_o}{C}) \, g \, rad U_o) k^2 (1 - \beta_p \beta_w) P_{long} \,,$$

where again $\vec{F}$ is the vector with components F_x, F_y. b_o is given by eq. (2.5) with m = 0; it results to be equal to the transverse size $2b \, \psi_o$ of the plate.

P_{long} is the function which describes the dependence of the force $\vec{F}$ on k and ω, it is

(7.8)
$$P_{long} = \frac{\beta - \beta_w}{1 - \beta_w^2} \left\{ \varepsilon_1' \sin(\theta - \phi) + \varepsilon_2' \sin(\theta + \phi) \right\},$$

where β takes now the value β_w. We observe that the coefficients ε_1' and ε_2' contain terms linear in the quantity

$$\frac{1 - \beta_w \beta_p}{\beta_w - \beta} \,,$$

as one can see from eq. (3.5); thus the r.h. side of eq. (7.8) is not identically zero.

We see that the general dependence of $\vec{F}$ on the ratio a/b as $(a/b)^{m+1}$ is respected also for the monopole case, but the force for this case results very small compared to that produced by a multipole throbbing beam, since the presence of the factor $k^2(1 - \beta_p \beta_w)$ at the r.h. side of eq. (7.7), and for the limitation

$$k \ll b \gamma$$

under which our formulas are valid.

7.3.- Longitudinally perturbed beam.-

From eqs. (7.2), (4.5) and (3.3), and eqs. (A.19) and (A.20) in the Appendix, we obtain

(7.9)
$$F_z = i e \lambda (\frac{Ml}{2\pi R}) (\frac{2 \psi_o \phi_o}{\pi 1 C}) P_{long},$$

where P_{long} is still given by eq. (7.8). $2 \psi_o$ is the transverse angular extension of the plate.

We see that in this case the self-force is governed by the perturbed charge per unit length $e \lambda$.

8. - INSPECTION OF THE P-FACTORS. -

Inserting eq. (3.5) with $\beta = \beta_w$ in the expression (7.8) for P_{long}, and with $\beta = \beta_p$ in the expression (7.6) for P_{trans}, we obtain

$$(8.1) \qquad P_{long} = \frac{\beta_w}{2} \frac{2ir_p(\cos 2\phi - \cos 2\theta) - \sin 2\phi}{2\cos 2\phi + 2i\,W\sin 2\phi}$$

and

$$
\begin{aligned}
(8.2) \qquad P_{trans} = \beta_w\ & \frac{-\sin 2\phi\left[(1-\beta_w\beta_p)^2+(\beta_w-\beta_p)^2\right]}{\theta(1-\beta_w^2)^2\,(2\cos 2\phi+2i\,W\sin 2\phi)} + \\[2mm]
& +\beta_w\frac{2ir_p(1-\beta_w\beta_p)^2(\cos 2\phi-\cos 2\theta)}{\theta(1-\beta_w^2)^2(2\cos 2\phi+2i\,W\sin 2\phi)} + \\[2mm]
& +\beta_w\frac{2(\beta_w-\beta_p)(1-\beta_w\beta_p)\sin 2\theta}{\theta(1-\beta w^2)^2(2\cos 2\phi+2i\,W\sin 2\phi)} - \frac{(\beta_w-\beta_p)^2}{1-\beta_w^2} + \\[2mm]
& +\beta_w\frac{2i\,r_s^{-1}(\beta_w-\beta_p)^2(\cos 2\phi-\cos 2\theta)}{\theta(1-\beta_w^2)^2\,(2\cos 2\phi+2i\,W\sin 2\phi)} ,
\end{aligned}
$$

where $r_p = r_1 r_2 / (r_1+r_2)$ is the impedance of the circuit formed by the impedances r_1 and r_2 in parallel, $r_s = r_1+r_2$ is the impedance of the circuit formed by r_1 and r_2 in series, and

$$(8.3) \qquad W = \frac{1+r_1 r_2}{r_1+r_2} = \frac{1}{r_s} + r_p .$$

Eqs. (8.1) and (8.2) must be compared with the companion equations (2.22 a, b) of ref.(3). Tfe former equations reduce to the last ones when one takes $\delta = \pm 1$ and $r_p = W = r$, i.e. when one of the two ends of the plate is open.

At the r.h. side of eq. (8.2) we have the last term in r_s^{-1} that has no equivalent in the companion eq. (2.22 b) of ref. (3). This term vanishes when r_s^{-1} is zero, i.e. for a floating or a single terminated plate.

For small ϕ and θ we get

$$(8.4a) \qquad P_{long} = \frac{\theta}{2} \frac{2i\,r_p\phi(1-\beta_w^2) - \beta_w^2}{1+2i\,W\phi}$$

and

$$(8.4b) \qquad P_{trans} = \frac{2i\,r_p\phi(1-\beta_p^2) - \beta_p^2}{1+2i\,W\phi}$$

These equations are very similar to eqs. (2.23) of ref. (3). The difference is that for a plate shorted at both the ends, the r.h. side of both eqs. (8.4) is exactly zero; this is contrary to the case of a plate shorted at one end and open on the other one.

It is soon verified, from the general expression (8.1) valid for any frequency k or ω, that P_{long} is identically zero for a plate shorted at both ends. However that is not verified for P_{trans} at large values of k or ω.

We consider some particular cases.

a) - Floating plate. -

$$r_p = \infty, \qquad W = \infty, \qquad r_s = \infty$$

(8.5a)
$$P_{trans} = \frac{\beta_w}{\theta} \left(\frac{1-\beta_w\beta_p}{1-\beta_w^2}\right)^2 \frac{\cos 2\phi - \cos 2\theta}{\sin 2\phi} - \frac{(\beta_w-\beta_p)^2}{1-\beta_w^2} .$$

b) - Plate shorted at one end and open on the other one. -

$$r_p = 0, \qquad W = 0, \qquad r_s = \infty$$

(8.5b)
$$P_{trans} = \frac{\beta_w}{\theta} \frac{2(\beta_w-\beta_p)(1-\beta_w\beta_p)\sin 2\theta - \sin 2\phi\left[(1-\beta_w\beta_p)^2+(\beta_w-\beta_p)^2\right]}{2(1-\beta_w^2)^2\cos 2\phi} - \frac{(\beta_w-\beta_p)^2}{1-\beta_w^2} .$$

c) - Plate shorted at both the ends. -

$$r_p = 0, \qquad W = \infty, \qquad r_s = 0$$

(8.5c)
$$P_{trans} = \frac{\beta_w}{\theta} \left(\frac{\beta_w-\beta_p}{1\pm\beta_w^2}\right)^2 \frac{\cos 2\phi - \cos 2\theta}{\sin 2\phi} - \frac{(\beta_w-\beta_p)^2}{1-\beta_w^2} .$$

Eqs. (8.5a) and (8.5b) have already been compared in ref. (3), and it had been shown that it is more advantageous, to reduce the effect of the plates on the beam, to use plates with very high termination impedances rather plates shorted only at one end.

If we refer to plates with two terminations we see from eqs. (8.5a) and (8.5b) that floating plates or shorted plates give rise practically to the same expression for P_{trans}, the only difference being the factor in front of the first term at the r.h. side of the eqs. (8.5a) and (8.5b).

The two functions

(8.6)
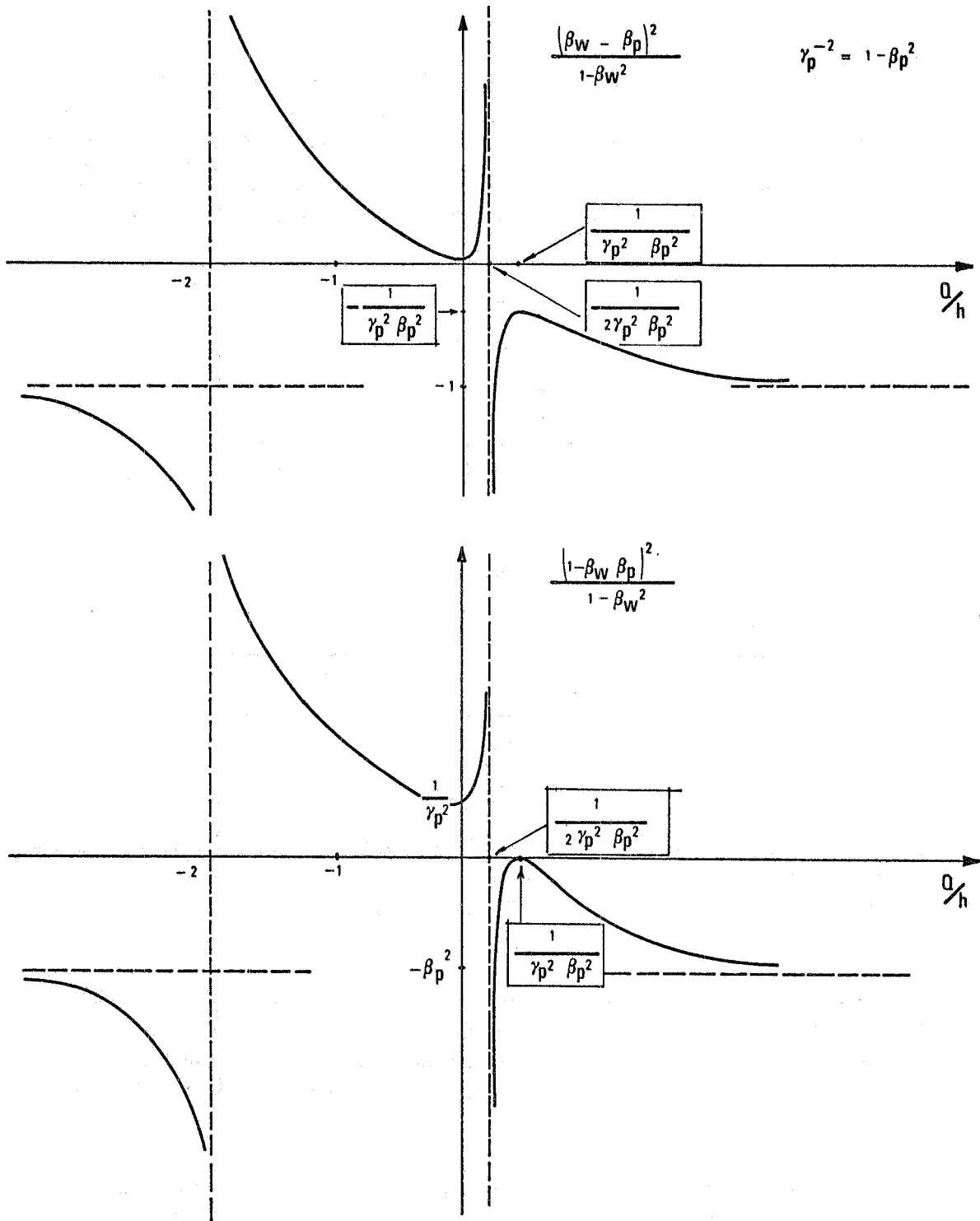$$\frac{(1-\beta_w\beta_p)^2}{1-\beta_w^2} \qquad \text{and} \qquad \frac{(\beta_w-\beta_p)^2}{1-\beta_w^2}$$

FIG. 2

have been plotted in Fig. 2 against $Q/n^{(x)}$, assuming

$$\beta_w \sim \beta_p^{(1+\frac{Q}{n})}.$$

If γ_p is a very large quantity the two functions result identical apart from terms in γ_p^{-2}.

The three pictures in Fig. 3 can help us to understand better the effect of the plate termination impedances. The picture (a) shows a floating plate; the standing wave of the potential distribution exhibits two bellies. The picture (c) shows a plate shorted at both the ends; the standing wave exhibits in this case two zeros at the ends. In both these cases the potential average along the plate length for a fixed time is zero, although the effect on the beam is not zero but given by the difference between the velocity $c\beta_w$ of the perturbation and the velocity c of the induced electric signal on the plate. This difference enters just the r.h. side of eqs. (8.5a) and (8.5b) through the term

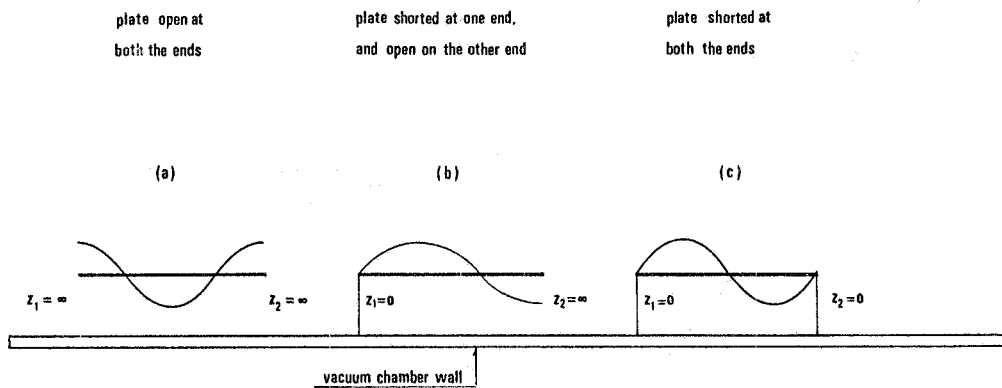$$\frac{\cos 2\phi - \cos 2\theta}{\sin 2\phi}$$

plate open at
both the ends

plate shorted at one end,
and open on the other end

plate shorted at
both the ends

(a)

(b)

(c)

FIG. 3

For small ϕ and θ the above quantity is practically $\phi(1-\beta_w^2)$. At high frequencies the conductive vacuum chamber wall structure resonates as a transmission line shorted or open at both ends, and the resonance frequencies are given by the relation

$$2\phi = \pi \nu ,$$

with ν an integer. This resonance has been described in ref. (3), and it has been shown that the resonances can be damped by dissipation on the plate finite resistivity.

The picture (b) shows a plate shorted at one end and open on the other one. The potential distribution has now a zero at the former end and a maximum at the latter one.

(x) - Q is the wavelength number of the betatron oscillations to be taken with the sign + for a fast beam perturbation, and with the sign - for a slow beam perturbation. Besides it is

$$n = kR.$$

The potential average along the plate length is not zero, and, also taking into account the difference between the velocity of the perturbation and that of the induced signal, we see that this case is the worst with regard to the effect on the beam.

9. - CONCLUSIONS. -

In this paper we have investigated the effect of conductive plates in circular particle accelerators or storage rings excited by some travelling wave beam perturbations. We have de voted this paper only to the case of plates with two terminations placed at the ends of the plates. We avoided to consider the more general case with two terminations but placed anywhere, for two orders of reasons:

a) The mathematical approach would have been very complicated by the increase of the problem variables.

b) In ref. (3) one has shown that in the case of a single terminated plate the best situa tion is to place the termination at one of the two plate ends with the effect to eliminate those re- sonances for which the plate appears as an unloaded transmission line. It is not difficult to verify that also in the case of two terminations the above mentioned resonances disappear when the two terminations are at the plate ends, otherwise we would have always some frequencies for which the plate structure resonates as an unloaded transmission line. Nevertheless we must say that this kind of resonance can happen also for plates terminated at both ends when the plates are either shorted or floating. Our intention writing this paper was to complete the analysis of the conductive plates effect made in refs. (1) and (3) for an intense coasting beam. We summa rize the following aspects of our results.

a) We have used a circular geometry with the beam centered in a straight pipe. The beam perturbation has been taken to be a travelling wave of the kind

$$e^{-i(kz - \omega t)}.$$

We have made use of the following approximation

$$k \ll \gamma/b .$$

b) We have extended our analysis to throbbing beam oscillations as well as to longitu dinal beam oscillations. It has been shown that, for the discontinuous character of the boundary of the problem geometry, several kinds of coupling can be present in a perturbed beam:

- coupling between longitudinal mode oscillations having different wave number k,
- coupling between throbbing beam modes having different m-mode number,
- coupling between the oscillations with different axis.

c) The distribution of the plates around the accelerator circumference can modify the amount of the force of the above mentioned coupling.

d) We have devoted large part of our analysis to the investigation of the self-for- ces, i. e. of those force terms which oscillate in both the longitudinal and transverse directions as the perturbation.

It has been shown that the self-forces have no dependance on the plate longitudinal distribution.

We have seen that the monopole throbbing mode (m=0) gives rise to a very weak self- -force governed by the term

$$k^2 (1 - \beta_p \beta_w) .$$

All the other multipole throbbing modes (m ≠ 0) give rise to forces practically of the same

amount except for the factor $(a/b)^{m+1}$. The self-force of longitudinal perturbed beam results be a small quantity in k.

e) The behaviour of the plate against the frequencies k and ω is measured by the P_{trans} and P_{long} factors.

It has been verified that the effect of the plates excited by a longitudinal perturbation or by a monopole throbbing beam is reduced to zero shorting the plates at both ends.

We have seen also that a shorted or an open plate (at both ends) gives rise also to the smallest effect on a multipole throbbing beam, although this effect can never be reduced to zero.

f) The coupling between the perturbed beam and the transverse geometry of the conductive plates is described by the following three factors

$$b_m \, \text{grad} \, U_m , \qquad \text{for multipole throbbing beam},$$

$$b_o \, \text{grad} \, U_o , \qquad \text{for monopole throbbing beam},$$

$$2 \, \phi_o \, \frac{\psi_o}{\pi} , \qquad \text{for longitudinal perturbation}.$$

We have included also the case of plates with axis forming an angle $\overline{\psi} \neq 0$ with the throbbing beam oscillation axis.

APPENDIX 1. - Derivation of the $\mathcal{G}_o$ and g-factor for any m-mode transverse beam motion: $m \neq 0.$ -

We have seen in Section 2 that the sources of the field are the following charge and current densities (in the limit $k \ll b\gamma$),

(A.1a)
$$\varrho_1 = \frac{e \lambda_o \gamma}{\pi a^2} \delta(r-a) \cos m\varphi$$

and

(A.1b)
$$\vec{J}_1 = \frac{e \lambda_o \gamma}{\pi a^2} \begin{pmatrix} i(\omega - kv)(\frac{r}{a})^{m-1} \mathcal{H}(a-r)\cos m\varphi \\ -i(\omega - kv)(\frac{r}{a})^{m-1} \mathcal{H}(a-r)\sin m\varphi \\ v \delta(r-a)\cos m\varphi \end{pmatrix} .$$

In the case of perfect conductive wall, the z-components of the e.m. fields have the form:

(A.2a)
$$E_z = \gamma \left\{ C_1 r^m \mathcal{H}(a-r) + \left[C_1 r^m + C_1'(r^m - \frac{a^{2m}}{r^m}) \right] \mathcal{H}(r-a) \right\} \cos m\varphi$$

and

(A.2b)
$$H_z = \gamma \left\{ D_1 r^m \mathcal{H}(a-r) + \left[D_1 r^m + D_1'(r^m - \frac{a^{2m}}{r^m}) \right] \mathcal{H}(r-a) \right\} \sin m\varphi .$$

All the other field components can be derived from E_z, H_z, ϱ_1 and $\vec{J}_1$.

The four constants C_1, C_1', D_1 and D_1' are determined by the usual conditions at $r = a$ and $r = b$.

From the two Maxwell equations

$$\text{rot}_\varphi \vec{E} = -i \frac{\omega}{c} H_\varphi , \qquad\qquad \text{rot}_r \vec{H} = \frac{4\pi}{c} J_r + i \frac{\omega}{c} E_r ,$$

and inserting eq. (A.2a) and (A.2b), we obtain, for $r > a$,

(A.3a)
$$E_r = \frac{im\gamma}{k(1 - \beta_w^2)} \left\{ \left[(C_1 + C_1') + \beta_w (D_1 + D_1') \right] r^{m-1} + (C_1' - \beta_w D_1') \frac{a^{2m}}{r^{m+1}} \right\} \cos m\varphi .$$

and

(A.3b)
$$H_\varphi = \frac{im\gamma}{k(1 - \beta_w^2)} \left\{ \left[\beta_w (C_1 + C_1') + (D_1 + D_1') \right] r^{m-1} + (\beta_w C_1' - D_1') \frac{a^{2m}}{r^{m+1}} \right\} \cos m\varphi .$$

Imposing $E_z = 0$ at $r = b$, we obtain

(A.4a)
$$(C_1 + C_1') b^{m-1} - C_1' \frac{a^{2m}}{b^{m+1}} = 0 .$$

Imposing that E_z and H_z are discontinuous crossing $r = a$, and that the amount of the discontinuity is determined by the perturbed surface charge of the beam, we have

(A.4b)
$$C_1' = - \frac{2 i e \lambda_o}{m \, a^{m+1}} k (1 - \beta_p \beta_w)$$

and

(A.4c)
$$D_1' = - \frac{2 i e \lambda_o}{m \, a^{m+1}} k (\beta_w - \beta_p) .$$

Imposing, at last, $E_\psi = 0$ at $r = b$, it results

(A.4d)
$$(D_1 + D_1') b^{m-1} + D_1' \frac{a^{2m}}{b^{m+1}} = 0 ,$$

having made use of eq. (A.4a).

Inserting the above equations (A.4) in the equations (A.3), we have for E_r and H_ψ at $r = b$,

(A.5a)
$$E_r = \frac{4 e \lambda_o \gamma}{a^2} \left(\frac{a}{b}\right)^{m+1} \cos m \, \psi ,$$

(A.5b)
$$H_\psi = \frac{4 e \lambda_o \gamma}{a^2} \left(\frac{a}{b}\right)^{m+1} \beta_p \sin m \, \psi .$$

We define the induced surface charge σ and the induced line density current I in the following way,

(A.6a)
$$\sigma = - \frac{E_r}{4 \pi} = \sigma_o \, g \cos m \psi \, e^{-i(kz - \omega t)} ,$$

(A.6b)
$$I = - \frac{H_\psi \, c}{4 \pi} = \beta c \sigma ,$$

where

(A.7)
$$\beta = \beta_p ,$$

(A.8a)
$$\sigma_o = - \frac{e \lambda_o \gamma_o}{\pi a^2} ,$$

and

(A.8b)
$$g = \left(\frac{a}{b}\right)^{m+1} .$$

APPENDIX 2. - Derivation of the $\mathfrak{S}_0$ and g - factor for the zero-th mode transverse beam motion: m = 0.

The sources of the field, in the limit $k \ll b\gamma$, in this case are

(A.9a)
$$\rho_1 = \frac{e\lambda_0 \gamma}{\pi a^2}\left[\delta((r-a) - \frac{2}{a}\mathcal{H}(a-r)\right]$$

and

(A.9b)
$$\vec{J}_1 = \frac{e\lambda_0 \gamma}{\pi a^2}\begin{pmatrix} i(\omega - kv)\frac{r}{a}\mathcal{H}(a-r) \\ \\ 0 \\ \\ v\left[\delta(r-a) - \frac{2}{a}\mathcal{H}(a-r)\right] \end{pmatrix} .$$

It is immediately verified that

$$E_\varphi = H_r = H_z = 0 ,$$

anywhere inside the vacuum chamber.

It can been verified that

$$H_\varphi = \beta_w E_r \quad \text{for} \quad r > a, \quad \text{and} \quad E_r = \beta_w H_\varphi \quad \text{for} \quad r < a,$$

so that it is sufficient to limit ourselves to the E_r-component. We have for it

(A.10)
$$E_r = \gamma\left\{C_1 r\,\mathcal{H}(a-r) + C_2 r^{-1}\mathcal{H}(r-a)\right\} ,$$

having used the fact that E_z, given by

$$ikE_z = \frac{E_r}{r} + \frac{\partial E_r}{\partial r} ,$$

is zero at r = b.

From

$$ik(1-\beta_w^2)E_r = \frac{4\pi}{c}\beta_w J_{1r} ,$$

relation valid in the region r < a, we obtain

(A.11)
$$C_1 = \frac{4e\lambda_0}{a^3}\beta_w\frac{\beta_w - \beta_p}{1-\beta_w^2} .$$

Imposing, then, the discontinuity of E_r at r = a, which is given by the surface perturbed charge, we have:

$$(A.12) \qquad \frac{C_2}{a^2} = C_1 + \frac{4e\lambda_o}{a^3} \quad .$$

From (A.11) and (A.12) inserted in (A.10), we have for E_r at $r = b$

$$E_r = \sum \frac{4e\lambda_o}{a^2} (\frac{a}{b}) \frac{1 - \beta_w \beta_p}{1 - \beta_w^2} \quad .$$

Using again the form (A.6) for the induced charge and current densities on the vacuum chamber wall, we obtain that σ_o is still given by eq. (A.8a), while, in the limit $qb \ll 1$,

$$(A.14a) \qquad g = (\frac{a}{b}) \frac{1 - \beta_p \beta_w}{1 - \beta_w^2}$$

and

$$(A.14b) \qquad \beta = \beta_w \quad .$$

APPENDIX 3. - Derivation of the σ_o and g-factor for a longitudinally perturbed beam. -

In this case we must use, as sources of the field,

$$(A.15a) \qquad \rho_1 = \frac{e\lambda}{\pi a^2} \mathcal{H} (a-r)$$

$$(A.15b) \qquad \vec{J}_1 = (0, 0, \frac{\omega}{k} \rho_1),$$

with $\lambda = \lambda_1 e^{-i(kz - \omega t)}$ the perturbed charge per unit length.

For symmetry reasons, and for perfectly conductive vacuum chamber wall, we have

$$E_\varphi = H_r = H_z = 0 ,$$

while the two components E_r and H_φ satisfy respectively the following equations

$$(A.16a) \qquad \nabla^2 E_r - \frac{E_r}{r^2} + \frac{\omega^2}{c^2} E_r = - \frac{4e\lambda}{a^2} \delta (r-a),$$

$$(A.16b) \qquad \nabla^2 H_\varphi - \frac{H_\varphi}{r^2} + \frac{\omega^2}{c^2} H_\varphi = - \beta_w \frac{4e\lambda}{a^2} \delta (r-a);$$

from that we have immediately

$$(A.17) \qquad H_\varphi = \beta_w E_r ,$$

and we can limit ourselves to the E_r-component. This can be expressed in the following way

$$E_r = \lambda \left\{ C_1 \, r \, \mathcal{H}(a-r) + \left[C_1 \, r + C_1' (r - \frac{a^2}{r}) \right] \mathcal{H}(r-a) \right\} ,$$

having used the continuity of E_r at $r = a$.

From $\mathrm{div}\,\vec{E} = 0$, to apply in the region $r > a$, we can compute the E_z-component. Imposing that E_z is zero at $r = b$, we have

$$C_1 + C_1' = 0 ,$$

and E_r becomes

(A. 18)
$$E_r = \lambda C_1 \left\{ r \, \mathcal{H}(a-r) + \frac{a^2}{r} \, \mathcal{H}(r-a) \right\} .$$

We find also that ikE_z is zero for $r > a$ and is given by $(2\lambda C_1 - 4\pi \mathcal{S}_1)$ for $r < a$. The continuity of E_z at $r = a$ yields

$$2\lambda C_1 - 4\pi \mathcal{S}_1 = 0 , \qquad \text{or} \qquad C_1 = \frac{2e}{a^2}$$

Thus, at last, we have, at $r = b$,

$$E_r = 2 e \lambda / b , \qquad\qquad H_\varphi = \beta_w E_r$$

from that

(A. 19)
$$\mathcal{G}_o = - \frac{e \lambda_1}{2\pi b} ,$$

(A. 20a)
$$g = 1 ,$$

and

(A. 20b)
$$\beta = \beta_w .$$

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