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LIKE PION FORM FACTOR. -

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FORM FACTOR⁽⁺⁾. -

ABSTRACT. -

We study an unitarization method in connection with the recently proposed Veneziano-like models for the pion electromagnetic form factor.

Using as input the model proposed by Suura, good agreement is obtained with the experimental data in the time-like region.

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I. - INTRODUCTION. -.

The structure of the electromagnetic form factors in the framework of the Veneziano model has been recently studied by several authors^(1,2). By use of current-field identities, current algebra and off shell extension of the Veneziano representation the expression

$$(I. 1) \quad G(t) = P(t) \frac{\Gamma[1 - \alpha_{\mathcal{P}}(t)]}{\Gamma\left[\frac{1}{2}n - \alpha_{\mathcal{P}}(t)\right]}$$

has been obtained for the isovector electromagnetic form factor. Here $\alpha_{\mathcal{P}}(t)$ is the linear \mathcal{P} Regge trajectory, n a positive odd integer and $P(t)$ a polynomial in t . The various off-mass-shell extrapolations used by different authors only affect the detailed form of $P(t)$. We remark that the fact that n in Eq. (I. 1) is an odd integer is a consequence of the quantization condition for Regge trajectories⁽³⁾.

However all these models fail to satisfy a natural symmetry requirement. Namely that two chirally conjugate sources, the vector and axial-vector currents V_{μ}^a and A_{μ}^a , are coupled in a more or less symmetrical way to all the daughter poles present in the Veneziano representation and which have the correct quantum numbers. The representation given in Eq. (I. 1) satisfy this requirement for V_{μ} , but only the π and the A_1 , out of all the possible 0^- and 1^+ particles have been coupled to A_{μ} .

These symmetry problems have been recently studied by Sura⁽⁴⁾ in connection with the pion form factor. The requirements of the symmetry discussed above lead to the following expression for the pion form factor

$$(I. 2) \quad F_{\pi}(t) = \frac{\Gamma\left[\frac{1}{2} - \frac{1}{2}\alpha_{\mathcal{P}}(t)\right]}{\Gamma\left[\frac{5}{4} - \frac{1}{2}\alpha_{\mathcal{P}}(t)\right]} c$$

The obvious disease of all these representation is in their lack of unitarity. In the following Sections we will study the problem of the partial unitarization (the meaning of this will be clear below) of the above expression for the pion form factor. However, before doing that, let us discuss in general some features of Eq. (I. 1) and (I. 2).

Eq. (I. 1), with $n = 7$ and $P(t) = \text{const.}$, reproduces quite well⁽⁵⁾ the data on the electromagnetic form factors of the nucleon, up to a momentum transfer $-t \approx 25 \text{ GeV}^2$. Moreover, as pointed out by Freund⁽⁶⁾, Eq. (I. 1) gives some very interesting predictions on the hadron mass spectrum. However, as discussed above, in the derivation of Eq. (I. 1) a strong asymmetry is introduced between the currents A_{μ} and V_{μ} .

On the other hand Eq. (I. 2) correctly embodies the asymmetry between A_π and V_π . However we remark that for $t \rightarrow \infty$, $\text{argt} \neq 0$, Eq. (I. 2) gives $F_\pi(t) \sim t^{-3/4}$, while composite models of elementary particles seems to imply $F_\pi(t) \leq t^{-1}$ asymptotically⁽⁷⁾. Finally it has been pointed out that an expression of the kind

$$(I. 3) \quad G(t) = \text{const.} \frac{\Gamma\left[\frac{1}{2} - \frac{1}{2} \alpha_\rho(t)\right]}{\Gamma\left[\frac{5}{4} - \frac{1}{2} \alpha_\rho(t)\right]}$$

cannot account for the behaviour of the nucleon form factor⁽⁸⁾.

In the following we will discuss the unitarization of the pion form factor starting from the two possible representation (I. 2) or

$$(I. 4) \quad F_\pi(t) = \text{const.} \frac{\Gamma\left[1 - \alpha_\rho(t)\right]}{\Gamma\left[\frac{5}{2} - \alpha_\rho(t)\right]}$$

The general idea of the following discussion will be that if a Veneziano-like form factor gives a reasonably good representation of the experimental data in some region, it cannot be too different, in an average sense, from the correct unitary one.

Starting from this idea in Sec. II we derive a singular inhomogeneous integral equation for the pion form factor, which can be solved following the Omnès method⁽⁹⁾. The resulting form factor exactly satisfies elastic unitarity: in fact elastic unitarity is imposed up to $t \approx 1 \text{ GeV}^2$.

In Sec. III we discuss the results of Sec. II, using as input Eq. (I. 2) and (I. 4). It turns out that starting from Eq. (I. 2) a good agreement with the experimental data is obtained.

For completeness in the Appendix we show how the solution of the integral equation derived in Sec. II is obtained.

II. - THE INTEGRAL EQUATION. -

Let us begin this Section with a more detailed exposition of the unitarization method sketched in the Introduction. Suppose that we know that the function

$$(II. 1) \quad f(t) = c_0 \sum_{n=0}^{\infty} \frac{\gamma_n}{t - t_n},$$

where $f(t)$ can be either (I. 2) or (I. 4), gives an approximate representation of the form factor. Eq. (II. 1) is obviously non-unitary: however an

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optimistic point of view is that, in view of its phenomenological successes, $f(t)$ is in fact close, in some average sense, to the correct, unitary form factor. We can therefore try to use Eq. (II. 1) as a starting point, but imposing the exact unitarity requirements in a limited region.

In more details, we have from (II. 1)

$$(II. 2) \quad \text{Im } f(t) = c_0 \sum_{n=0}^{\infty} \pi \gamma_n \delta(t - t_n)$$

The use of an unsubtracted dispersion relation gives back Eq. (II. 1) from (II. 2).

Unitarity now tells us that

$$(II. 3) \quad \text{Im } F_{\pi}(t) = e^{-i\delta_{11}} \sin \delta_{11}(t) F_{\pi}(t) \theta(t - 4m_{\pi}^2) + \text{inelastic contributions}$$

where δ_{11} is the $J=1, I=1$ phase shift for elastic $\pi-\pi$ scattering.

We will use the elastic unitarity relation up to some t_1 ($\simeq 1 \text{ GeV}^2$): this is suggested by the experimental data and generally accepted. For $t > t_1$ we will assume that both the elastic and inelastic contributions to Eq. (II. 3) are well approximated by Eq. (II. 2) from which the first term (corresponding to the φ meson in our case) has been removed.

We therefore have

$$(II. 4) \quad \text{Im } F_{\pi}(t) = e^{-i\delta_{11}(t)} \sin \delta_{11}(t) F_{\pi}(t) \theta(t - 4m_{\pi}^2) \theta(t_1 - t) + c \sum_{n=1}^{\infty} \pi \gamma_n \delta(t - t_n)$$

It is now clear that the whole procedure is meaningful only if Eq. (II. 1) is really a good approximation to the correct unitary expression. This is essentially the same philosophy used in an N/D attempt⁽¹⁰⁾ to unitarize the Veneziano amplitude.

Using now (II. 4) in an unsubtracted dispersion relation we get the singular inhomogeneous integral equation

$$(II. 5) \quad F_{\pi}(t) = c g(t) + \frac{1}{\pi} \int_{4m_{\pi}^2}^{t_1} \frac{e^{-i\delta_{11}(x)} \sin \delta_{11}(x) F_{\pi}(x)}{x - t - i\epsilon} dx$$

Here $g(t) = \frac{1}{c_0} g(t) - \frac{\gamma_0}{t - t_0}$: in (II. 1) the constant c_0 is fixed by the normalization condition $f(0) = 1$ and c is fixed by the analogous condition for $F_{\pi}(t)$. We expect that if (II. 1) is a good starting point the renormalization effect due to unitarity corrections will be small and $c \simeq c_0$. This is confirmed "a posteriori".

In Section III we will use a particular model for the phase shift $\delta_{11}(t)$, which reproduces quite well the available experimental data. With such a parametrization $\delta_{11}(t) = 0 \pmod{\pi}$ for some $t = t_0$.

We will choose $t_1 = t_0$: in this way the discontinuity (II. 4) is continuous both at $t = 4m^2/\pi$ and $t = t_1$. We will fix the phase shift determination in such a way that $\delta_{11}(4m^2/\pi) = -\pi$ and $\delta_{11}(t_1) = 0$: obviously the results do not depend on this choice.

Once $F_\pi(t)$ has been determined from (II. 5) in the interval $4m^2/\pi \leq t \leq t_1$, then it can be obtained in the whole complex plane by using Eq. (II. 5) again.

It will be shown in the Appendix that the most general solution of Eq. (II. 5) is

$$(II. 6) \quad F_\pi(t) = c \left[G_\pi(t) + \frac{\psi(t)}{(t - 4m^2/\pi)^n (t - t_1)^m} \exp[\rho(t)] \right] \exp[i\delta_{11}(t)]$$

where

$$(II. 7) \quad G_\pi(t) = g(t) \cos \delta_{11}(t) + \exp[\rho(t)] \times \\ \times \frac{P}{\pi} \int_{4m^2/\pi}^{t_1} \frac{\sin \delta_{11}(x) g(x) \exp[-\rho(x)]}{x - t} dx$$

and

$$(II. 8) \quad \rho(t) = \frac{P}{\pi} \int_{4m^2/\pi}^{t_1} \frac{\delta_{11}(x)}{x - t} dx$$

$\psi(t)$ is an essentially arbitrary function of t , regular and non zero at $t = 4m^2/\pi$ and $t = t_1$, and n and m are a priori arbitrary integers.

We will now show how it is possible to remove this arbitrariness. We require that $F_\pi(t)$ has no poles at $t = 4m^2/\pi$ nor at $t = t_1$. Since $\exp[\rho(t)]$ has a simple zero at $t = 4m^2/\pi$ and is finite at $t = t_1$ the only possible choices are: $n = 0$ or 1 and $m = 0$: the function $\psi(t)$ is still unspecified. We will assume now that $F_\pi(t)$ has no singularities for $|t| < \infty$, besides the elastic unitarity cut and the input poles. Therefore $\psi(t)$ must be a polynomial in t . Moreover we will impose, in the spirit of our approach, that the asymptotic behaviour of $F_\pi(t)$ is the same of that of $f(t)$, Eq. (II. 1).

Since $\exp[\rho(t)] \rightarrow 1$ for $t \rightarrow \infty$ we are left with the only possibility $n = 1$ and $\psi(t) = \text{const.} = k$.

The constant k is fixed by the above requirements, namely:

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$$(II. 9) \quad k = \gamma_0 + \frac{1}{\pi} \int_{4m_\pi^2}^{t_1} \sin \delta_{11}(x) G_\pi(x) dx$$

The constant c is now fixed by the normalization condition, and is given by

$$(II. 10) \quad c = \left[g(0) + \frac{1}{\pi} \int_{4m_\pi^2}^{t_1} \frac{\sin \delta_{11}(x) G_\pi(x)}{x} dx - \frac{k}{4m_\pi^2} \exp [\rho(0)] \right]^{-1}$$

III. - APPLICATIONS AND RESULTS. -

We will now specify the form used for the phase-shift $\delta_{11}(t)$. We assume for the unitary $J=1, I=1$ pion-pion scattering amplitude the form suggested by Lovelace⁽¹¹⁾ and collaborators⁽¹²⁾:

$$(III. 1) \quad f_{J=1}^{I=1}(t) = \frac{V_{11}(t)}{1 + h(t) V_{11}(t)}$$

where $h(t)$ is a modified Chew-Mandelstam form⁽¹²⁾

$$(III. 2) \quad h(t) = -i \frac{q}{\sqrt{t}} - \frac{2m_\pi^2}{q\sqrt{t}} \log \left[\frac{\sqrt{t} + 2q}{2m_\pi} \right], \quad (q = \sqrt{\frac{t}{4} - m_\pi^2})$$

$$(III. 3) \quad V_{11}(t) = \frac{1}{2} \int_{-1}^1 \cos \theta d(\cos \theta) [V(t,s) - V(t,u)]$$

and

$$(III. 4) \quad V(s,t) = -\gamma \frac{\Gamma(1 - \alpha_\rho(s)) \Gamma(1 - \alpha_\rho(t))}{\Gamma(1 - \alpha_\rho(s) - \alpha_\rho(t))}$$

($\alpha_\rho(t) = at+b$) and γ is fixed in term of the mass and width of the ρ .

The phase-shifts deduced from (III. 1) are in good agreement with the experimental data⁽¹³⁾.

We believe however that our results do not strongly depend on the detailed structure assumed for $\delta_{11}(t)$, provided that $\delta_{11}(t) = 0 \pmod{\pi}$ at $t = 4m_\pi^2$ and $t = t_1$, and that it correctly reproduces the ρ resonance.

III. a) - Suura Model. -

We take here

$$(III. 5) \quad f(t) = c_0 \frac{\Gamma\left(\frac{1}{2} - \frac{1}{2} \alpha_{\mathcal{S}}(t)\right)}{\Gamma\left(\frac{5}{4} - \frac{1}{2} \alpha_{\mathcal{S}}(t)\right)}$$

c_0 is given (in the limit $m_{\pi} = 0$) by $c_0 = 1/\Gamma(1/4)$. The trajectory $\alpha_{\mathcal{S}}(t)$ is constrained to satisfy the Adler condition $\alpha_{\mathcal{S}}(m_{\pi}^2) = 1/2$.

Using (III. 5) as input in (II. 6) we obtain the results shown in Fig. 1: the agreement with the experimental data^(14 + 16) is quite good. Obviously at this point the only free parameters in the model are the mass and the width of the \mathcal{S} meson.

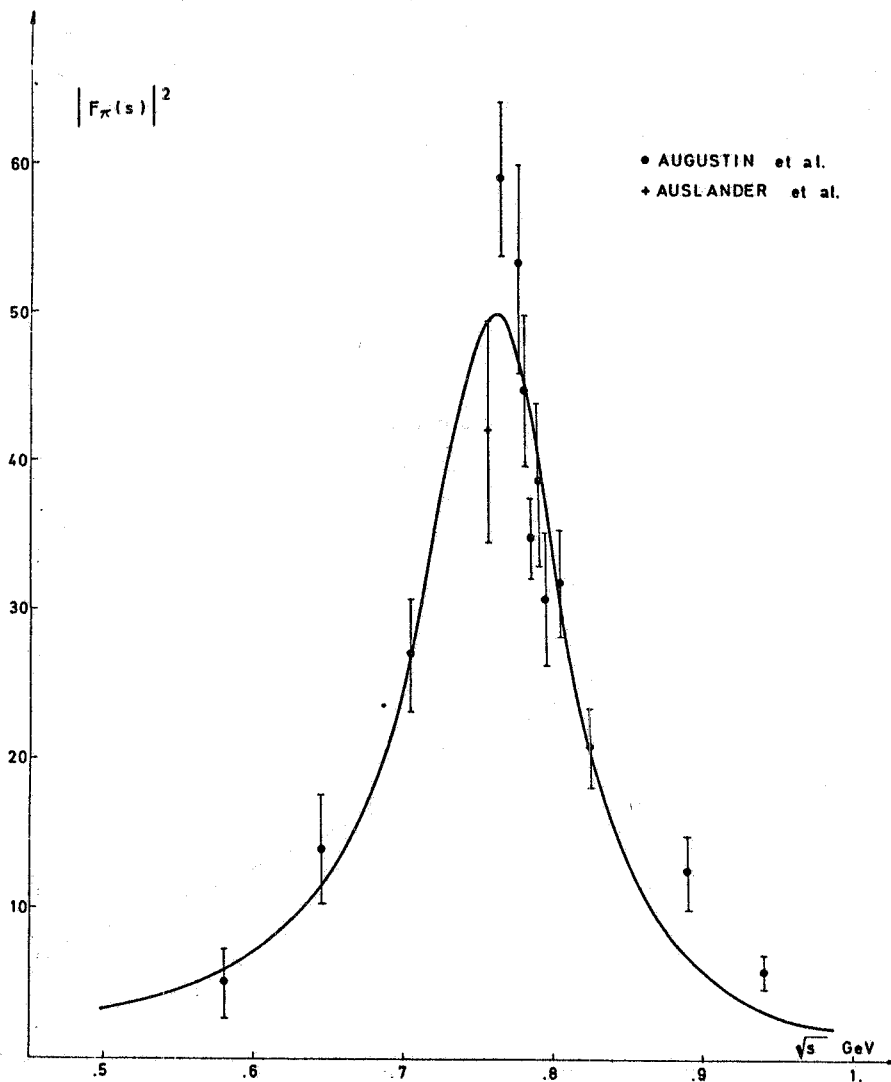


FIG. 1 - The predictions of the unitarized version of Eq. (I. 2) compared with the data of Ref. (14 + 16) (Time-like region).

The curve shown in Fig. 1 has been obtained with $m_\rho = 765$ MeV and $\Gamma_\rho = 110$ MeV. It is presumably possible to obtain a better fit with small variations of the ρ mass and width: we did not attempt such a detailed fit since we are here interested only in the general features of the various models.

Note that, if a small ω - ρ interference (which is obviously outside the present model) exists, as suggested by the recent Orsay results⁽¹⁶⁾, one could probably obtain a better fit in the peak region (in this connection compare our Fig. 1 with Fig. 2 of Ref. 16).

As anticipated in Sec. II the unitarity renormalization effects turn out to be rather small: with the value of m_ρ and Γ_ρ given above, and $m_\pi = 139$ MeV, we find $c_0/c = 1.05$.

The predictions of the model in the space-like region are presented in Fig. 2: the experimental information^(17, 18) however is very meagre in this region. In the figure we plotted the unitarized form of (III. 5). It is interesting to note that the results obtained directly from (III. 5) and from its unitarized version are practically coincident in the space-like region: the effect of unitarity is to increase by a negligible amount the radius

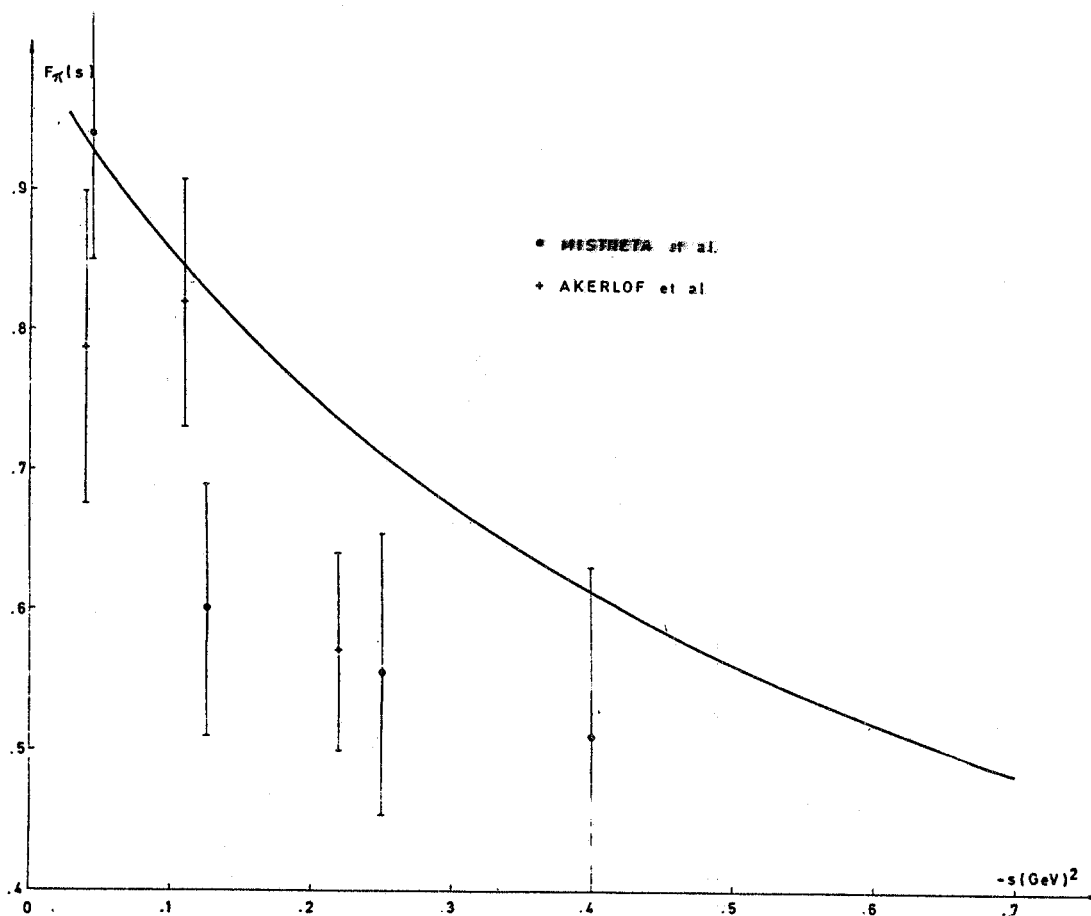


FIG. 2 - The predictions of the unitarized version of Eq. (I. 2) compared with the data of Ref. (17, 18) (Space-like region)

calculated from (III. 5). These results seem to confirm "a posteriori" the validity of our procedure, as well as of our starting point, Eq. (III. 5).

III. b) - Other models. -

We take here⁽²⁾

$$(III. 6) \quad f(t) = c_0 \frac{\Gamma[1 - \alpha_{\mathcal{F}}(t)]}{\Gamma[3 - \alpha_{\mathcal{F}}(m_{\pi}^2) - \alpha_{\mathcal{F}}(t)]} = c_0 \frac{\Gamma[1 - \alpha_{\mathcal{F}}(t)]}{\Gamma[\frac{5}{2} - \alpha_{\mathcal{F}}(t)]}$$

the last equality following from the Adler self-consistency condition⁽³⁾.

In the limit $m_{\pi} = 0$, $c_0 = 1/\sqrt{\pi}$. If we now use (III. 6) in (II. 6) the results are very bad: the resulting peak at the \mathcal{F} mass is by far too

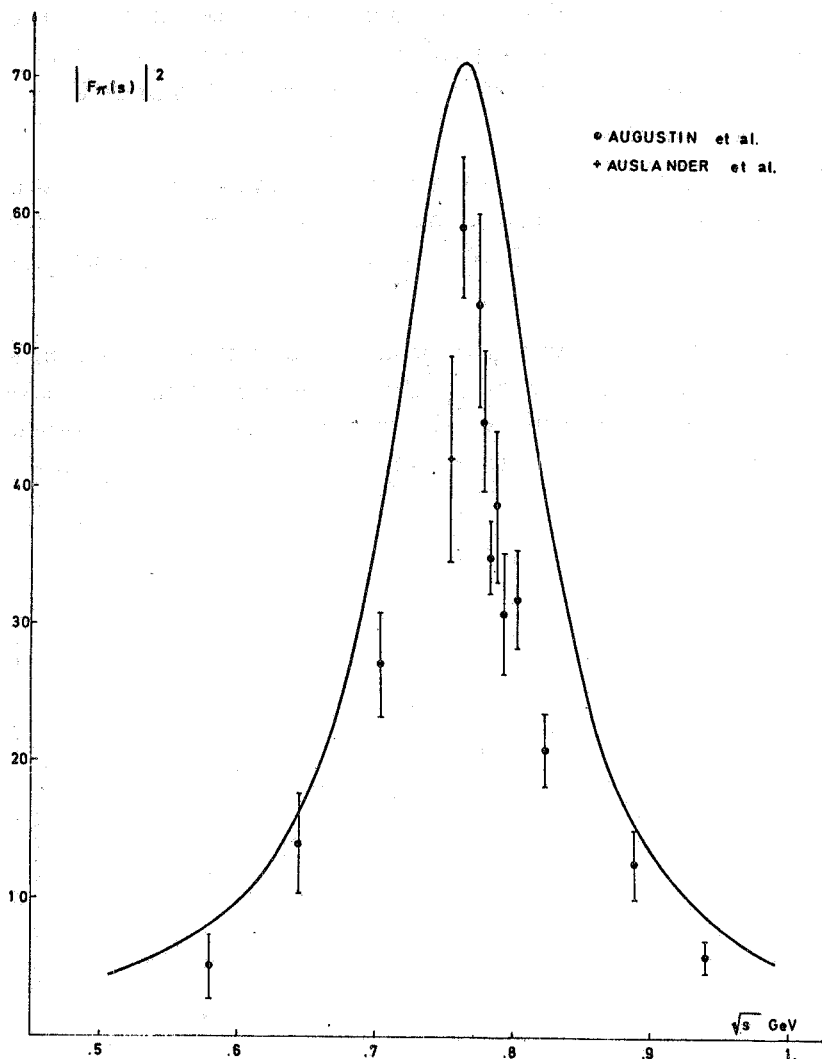


FIG. 3 - The predictions of the unitarized version of Eq. (I. 3) compared with the data of Ref. (14 + 16) (Time-like region).

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high. The results improve if we relax the condition $\alpha_S(m_\pi^2) = 1/2$, but even in this case are not very good.

We plotted in Fig. 3 the results obtained with $\alpha_S(m_\pi^2) = 0.6$, $m_S = 765$ MeV and $\Gamma_S = 120$ MeV.

We considered the possibility of satellite contributions in (III. 6): we added a satellite term and repeated the calculation, but we did not find any simple way of improving the results with this method. Of course by adding a larger number of satellites (and a corresponding number of parameters) the results could improve, but their meaning is certainly unclear in this case.

IV. - CONCLUSIONS. -

There are at present two classes of models for the pion form factor in the framework of the Veneziano representation. Models of the first class embody the natural requirement of symmetry in the couplings of the vector and axial vector currents. Models of the second class fail to satisfy these requirements.

We have presented here an unitarization method that, starting from the narrow width approximation, gives a form factor that satisfies elastic unitarity up to $t_1 \simeq 1$ (GeV)² and contains zero width resonances for larger t .

It turns out that using as input in our unitarization scheme models of the first class one obtain good agreement with the experimental data. On the other hand if models of the second class are used as input, at least in their simpler form, they cannot account for the experimental results.

APPENDIX. -

For completeness we present here the general solution of the integral equation

$$(A. 1) \quad \psi(x) = g(x) + \frac{1}{\pi} \int_a^b \frac{\exp[-i \delta_{11}(y)] \sin \delta_{11}(y) \psi(y)}{y - x - i \epsilon} dy$$

following the method of Omnès⁽⁹⁾. In (A. 1) $g(x)$ is a function continuous in the interval $[a, b]$ and $\sin \delta_{11}(a) = \sin \delta_{11}(b) = 0$.

We define the function

$$(A. 2) \quad F(z) = \frac{1}{2\pi i} \int_a^b \frac{\exp[-i \delta_{11}(y)] \sin \delta_{11}(y) \psi(y)}{y - z} dy$$

With this definition Eq. (A. 1) reads

$$(A. 3) \quad \begin{aligned} & \exp[-2i \delta_{11}(x)] F(x + i \epsilon) - F(x - i \epsilon) = \\ & = g(x) \exp[-i \delta_{11}(x)] \sin \delta_{11}(x) \end{aligned}$$

for $a < x < b$.

We now put

$$(A. 4) \quad F(z) = \phi(z) \Omega(z)$$

where the function $\Omega(z)$ is defined by the condition

$$(A. 5) \quad \exp[-2i \delta_{11}(x)] \Omega(x + i \epsilon) - \Omega(x - i \epsilon) = 0$$

Taking the logarithm of (A. 5) we find its solution to be

$$(A. 6) \quad \Omega(z) = \exp \left[\frac{1}{\pi} \int_a^b \frac{\delta_{11}(y)}{y - z} dy \right]$$

Eq. (A. 3) reads now

$$(A. 7) \quad \phi(x + i \epsilon) - \phi(x - i \epsilon) = g(x) \sin \delta_{11}(x) \exp[-\delta(x)]$$

with $a < x < b$ and

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$$(A. 8) \quad \mathcal{S}(x) = \frac{P}{\pi} \int_a^b \frac{\delta_{11}(y)}{y-x} dy$$

The solution of (A. 7) is trivial and we finally obtain

$$(A. 9) \quad \psi(x) = \left[g(x) \cos \delta_{11}(x) + \frac{1}{\pi} \exp[\mathcal{S}(x)] \times \right. \\ \left. \times P \int_a^b \frac{g(y) \sin \delta_{11}(y) \exp[-\mathcal{S}(y)]}{y-x} dy \right] \exp[i \delta_{11}(x)]$$

The general solution of Eq. (A. 1) can be obtained by adding to (A. 9) the general solution of the associated homogeneous equation

$$(A. 10) \quad \varphi_0(x) = \frac{1}{\pi} \int_a^b \frac{\exp[-i \delta'_{11}(y)] \sin \delta_{11}(y) \varphi_0(y)}{y-x-i\varepsilon} dy$$

Following the procedure described above, we can define

$$(A. 11) \quad F_0(z) = \phi_0(z) - \Omega(z)$$

where now $\phi_0(z)$ satisfies, for $a < x < b$

$$(A. 12) \quad \phi_0(x+i\varepsilon) - \phi_0(x-i\varepsilon) = 0$$

This relation shows that $\phi_0(z)$ is analytic in the interval (a, b) except eventually at the points a and b where it can have poles (we are excluding here essential singularities). The general solution of (A. 1) is therefore

$$(A. 13) \quad \varphi'(x) = \varphi(x) + \frac{\psi(x)}{(x-a)^n(x-b)^m} \exp[\mathcal{S}(x) + i \delta_{11}(x)]$$

where the function $\psi(x)$, regular and non zero at $x=a$ and $x=b$, is essentially arbitrary.

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