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IN STORAGE RINGS. -

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I. - INTRODUCTION.

While longitudinal instabilities of intense electron beams were the first ones predicted theoretically and observed experimentally their theory still remains uncompleted. In particular the collective longitudinal instabilities inside an RF bunched beam have not been investigated.

The recent observations made at ACO and ADONE storage rings showed that the collective longitudinal effects are definitely of practical importance⁽¹⁾. The observed bunch length 2λ was found to be proportional to the beam intensity in some degree (apparently $\sim I^{1/3}$) and decreasing with an energy increase.

We shall consider here excitation of internal coherent synchrotron oscillations as a possible origin of the above mentioned effect. Proper frequencies ν_n of these oscillations at zero beam current must be real multiples of synchrotron oscillation frequency, i. e. $\nu_n = n\Omega$, if one does not take into account Landau damping. One may expect that at sufficiently large currents the space-charge frequency shift would be complex, i. e. that at least some of the

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coherent modes would be unstable. The instability increment should increase with a linear space-charge density, i. e. with I/A . At the same time the well known radiation damping which stabilizes the oscillations does not depend on the bunch length. Thus a certain equilibrium bunch length can exist which has to exceed the natural length determined by radiation fluctuations only.

Several general remarks are to be made on the interaction forces between particles of a bunch. The above mentioned experiments showed that the bunch length depended only on the current in the same bunch and was independent of intensity of other bunches. It makes a good reason for excluding from the beginning all long-distance forces, that is beam-cavity interaction or resistive wake fields related to a decay of eddy currents induced in vacuum chamber walls. We have to take into account short-range forces which are essential at distances small or comparable with the bunch length. These forces may be classified as follows :

- a) A longitudinal electric field induced in a perfectly conducting straight pipe. This field obviously has a head-tail symmetry.
- b) A non-symmetric field related to a finite conductivity of walls. At short distances this field may be of a large value.
- c) Fields of radiation (synchrotron or Cerenkov type) related to the orbit and/or chamber curvature. They are also non-symmetric.

The type c) fields might be of importance for the problem under consideration⁽²⁾ but we shall treat here only the model of a straight orbit in a straight vacuum chamber. It will be shown that under this assumption only type b) fields may be responsible for the excitation of coherent oscillations.

II. - BEAM DYNAMICS.

If ϑ is an angular coordinate along the orbit, $\rho(\vartheta, t)$ a perturbation of a linear space-charge density and $\mathcal{E}(\vartheta, t)$ a longitudinal effective electric field generated by the perturbation, then a general linear integral relation may be written

$$U(\vartheta, \nu) = \frac{\beta c}{2\pi} \oint d\vartheta' \rho(\vartheta', \nu) Z(\vartheta - \vartheta', \nu)$$

(1)

$$U(\vartheta, \nu) = -R \int \mathcal{E}(\vartheta, \nu) d\vartheta$$

Here R is a synchronous orbit radius and βc the synchronous velocity. The function $Z(\vartheta - \vartheta', \nu)$ takes into account all the boundary conditions and may be called a chamber impedance. Note that per-

turbation are supposed to be oscillating $\sim \exp(i\nu t)$ ($\nu \ll \beta c/R$) where the coherent synchrotron oscillation frequency ν is to be determined. It has to be mentioned also that in a stationary case ($\nu = 0$) Z is real even if the chamber walls are not perfectly conducting.

For investigation of internal degrees of freedom of the bunch it is convenient to use Vlasov equation. So let f be a distribution function in a synchrotron oscillation phase plane. The canonically conjugated coordinates in the plane are either $(\Delta p, \varphi)$ or (\mathcal{E}, τ) (see Fig. 1). The value Δp is a momentum deviation from

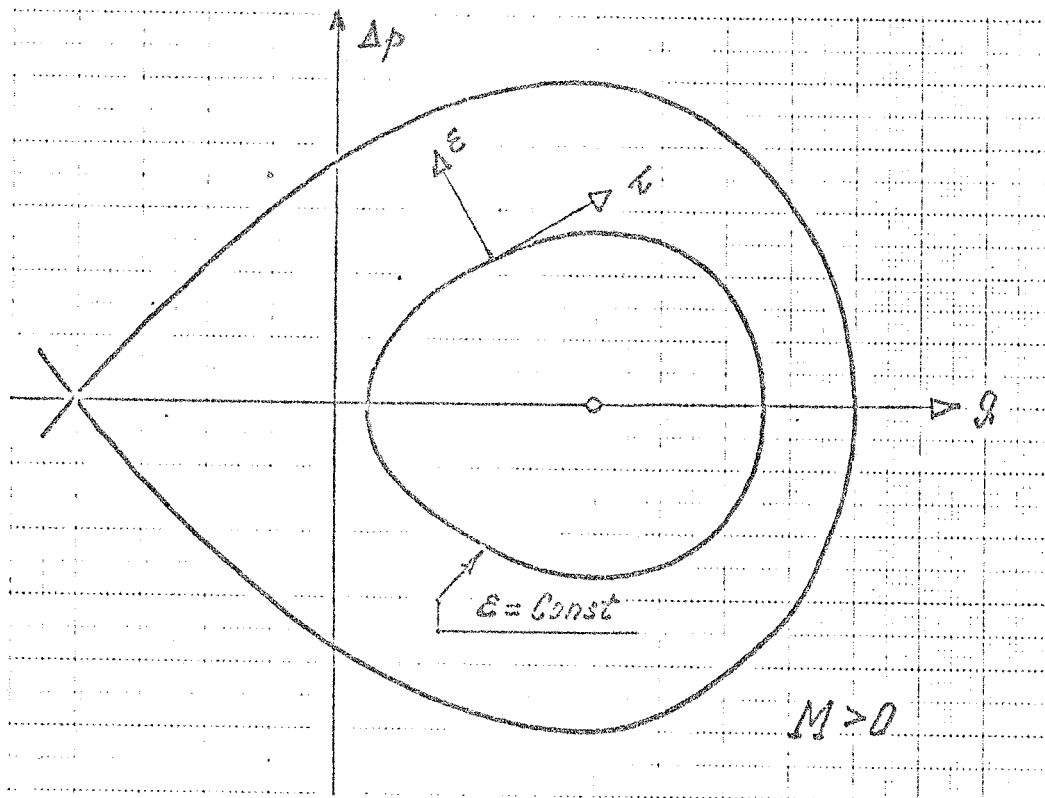


FIG. 1

the equilibrium value, $\mathcal{E} = (\Delta p)^2/2M + e\tilde{U}(\varphi)$ is an energy (Hamiltonian) of synchrotron oscillations and $\tau = MR \int_{\mathcal{E}=\text{const}} (\Delta p(\mathcal{E}, \varphi))^{-1} d\varphi$

is a coordinate along a fixed phase trajectory $\mathcal{E} = \text{const}$.

$e\tilde{U}(\varphi)$ is a stationary (equilibrium) "potential" of synchrotron motion and

$$M = \bar{m}_0 \gamma^3 / (1 - \gamma^2/\gamma_t^2)$$

(2).

$$\gamma_t^2 = \alpha^{-1} \quad (\alpha - \text{momentum compaction factor})$$

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is an effective mass of the longitudinal motion. In a relativistic case when the energy γ is larger than the transition energy γ_t the value of $M = -m_0 \gamma / \alpha$ is negative.

A single particle equations of motion may be written now as^(x)

$$(3a) \quad \dot{\varphi} = \Delta p / MR; \quad \Delta \dot{p} = -\frac{e}{R} \frac{\partial}{\partial \varphi} (\tilde{U} + U)$$

or

$$(3b) \quad \dot{\xi} = -\Delta p \frac{e}{RM} \frac{\partial U}{\partial \varphi} = -\left(\frac{\partial eU}{\partial \tau}\right)_{\xi}; \quad \dot{\tau} = 1 + \left(\frac{\partial \tau}{\partial \xi}\right)_{\varphi} \dot{\xi}$$

Hence Vlasov equation has the form

$$(4) \quad \left[\frac{\partial}{\partial t} + \dot{\xi} \left(\frac{\partial}{\partial \xi} \right)_{\tau} + \dot{\tau} \left(\frac{\partial}{\partial \tau} \right)_{\xi} \right] f = 0$$

Now let us suppose that the perturbations are small enough to linearize eq. (4) with respect to small deviations from the equilibrium distribution $\tilde{f}(\xi)$. Then

$$(5) \quad f \rightarrow \tilde{f}(\xi) + f(\xi, \tau, \nu) e^{i\nu t}$$

$$\left[i\nu + \frac{\partial}{\partial \tau} \right] f(\xi, \tau, \nu) = \frac{d\tilde{f}}{d\xi} \frac{\partial}{\partial \tau} eU(\varphi, \nu)$$

The synchrotron oscillation energy ξ enters into the eq. (5) as parameter only so it can be easily integrated along a fixed phase trajectory $\xi = \text{const}$. Bearing in mind that the solution must be periodic with a period $T = 2\pi / \Omega(\xi)$ we obtain

$$(6) \quad f(\xi, \tau, \nu) + f(\xi, -\tau, \nu) =$$

$$= \frac{d\tilde{f}}{d\xi} (\nu \sin \frac{\nu T}{2})^{-1} \frac{\partial^2}{\partial \tau^2} \int_{\tau}^{\tau+T} eU(\varphi'') \cos \nu (\tau'' - \tau - \frac{T}{2}) d\tau''$$

$\xi = \text{const.}$

(We put $\tau = 0$ at the line $\Delta p = 0$, so $eU(\varphi, \nu)$ is an even and periodic function of τ). Now the linear charge density perturbation is related to the distribution function as follows:

(x) - Radiation damping terms are not included here and will be taken into account in a somewhat artificial way.

$$\begin{aligned}
 \rho(\psi, \nu) &= \frac{2\pi I}{\beta c} \int_{-\infty}^{+\infty} f d(\Delta p) = \\
 (7) \quad &= \frac{2\pi I M}{\beta c} \int_{\psi=\text{const.}} d\varepsilon |\Delta p(\varepsilon, \psi)|^{-1} \left[f(\varepsilon, \tau, \nu) + f(\varepsilon, -\tau, \nu) \right]
 \end{aligned}$$

where I is a total beam current and \tilde{f} is supposed to be normalized for unity in the coordinates $(\Delta p, \psi)$. Combining now the eqs. (1) and (7) we obtain a linear integral equation

$$(8) \quad \rho(\psi, \nu) = I \oint d\psi' \rho(\psi', \nu) K(\psi, \psi', \nu)$$

with the kernel

$$\begin{aligned}
 K(\psi, \psi', \nu) &= \frac{eM}{\nu} \int_{\psi=\text{const.}} \frac{\frac{d\tilde{f}}{d\varepsilon}}{|\Delta p| \sin \frac{\nu T}{2}} \times \\
 (9) \quad &\times \frac{\partial^2}{\partial \tau^2} \int_{\varepsilon=\text{const.}}^{\tau+T} Z(\psi'' - \psi', \nu) \cos \nu \left(\tau'' - \tau - \frac{T}{2} \right) d\tau'' d\varepsilon
 \end{aligned}$$

Note that neglecting the dependence $\Omega(\varepsilon)$ i. e. Landau damping, the kernel $K(\psi, \psi', \nu)$ has poles at $\nu = n\Omega$ (n -integer $\neq 0$). It means that the eigenfrequencies ν at zero current ($I \rightarrow 0$) are multiples of the synchrotron frequency as it follows from simple physical arguments^(x).

Now our problem may be formulated in general form as follows: to find those complex values ν which make a real value I to be an eigenvalue of the non-hermitian kernel K . To make the next step the following additional assumptions have to be accepted:

- The short-range forces play the most important role, so ψ'', ψ' and ψ in eqs. (8), (9) may be considered as belonging to the same bunch.
- The equilibrium distribution in the phase plane is uniform up to some energy ε_0 and is equal to zero for $\varepsilon > \varepsilon_0$. Note that Landau damping is absent in this case and $\frac{d\tilde{f}}{d\varepsilon} = -\frac{NR}{\varepsilon T} \delta(\varepsilon - \varepsilon_0)$.

(x) - To take into account Landau damping one has to bear in mind that the kernel K is defined by the eq. (8) only in the lower half-plane of complex ν and has to be analytically prolonged in the upper half-plane.

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where $N < 1$ is a relative intensity of the bunch.

c) The current per bunch IN is sufficiently small, so the eigenfrequencies are close to their zero-current values:

$$\nu = n\Omega + \Delta\nu \quad ; \quad \Delta\nu \ll \Omega$$

Under these assumption the kernel K may be essentially simplified:

$$K(\varphi, \varphi', \nu) = - \frac{n\Omega^3 eMNR}{\pi^2 \epsilon_0 |\Delta p(\epsilon_0, \varphi)| \Delta\nu} \cos n\Omega\tau \quad \times$$

$$(10) \quad \times \int_0^{T/2} Z(\varphi'' - \varphi', n\Omega) \cos n\Omega\tau'' d\tau''$$

and becomes multiplicative so the eq. (8) can be readily solved:

$$\Delta\nu = - \frac{n\Omega eNI}{\pi^2 |\epsilon_0|} \int_0^{T/2} \int_0^{T/2} \Omega^2 d\tau' d\tau'' Z(\varphi'' - \varphi', n\Omega) \quad \times$$

$$(11) \quad \times \cos n\Omega\tau' \cos n\Omega\tau''$$

One can see from eq. (11) that the imaginary part of ν is proportional to the current per bunch IN and is determined by an imaginary and non-symmetric (for linear synchrotron oscillations) part of the impedance.

III. - STRAIGHT VACUUM CHAMBER IMPEDANCE.

To evaluate the impedance let us note that a longitudinal component of electric field generated by the beam can be represented as a sum of harmonics $\sim \exp i(\omega t - kz)$ where z is a longitudinal coordinate. For an amplitude of k -th harmonic E_k we have:

$$\Delta_{\perp} E_k - x^2 E_k = - \frac{4\pi i x^2}{k} Q \rho_k$$

$$(12) \quad x^2 = k^2 - \omega^2/c^2$$

where ρ_k is an amplitude of the k -th harmonic of linear charge density. Let us suppose further that the current is directed along the z -coordinate and transverse distribution Q is independent of z .

In the case of high electric conductivity of walls the boundary conditions of the problem can be written in the form

$$(13) \quad E_k = - \frac{i\omega}{cx^2} \zeta(\omega) \frac{\partial E_k}{\partial n} \quad \Big| \quad \text{at walls}$$

where $\partial/\partial n$ means derivation along the normal to the wall. The value $\zeta(\omega)$ (surface impedance) is related to the wall conductivity σ by

$$(14) \quad \zeta(\omega) = (1 \pm i) \left[|\omega| / 8\pi\sigma \right]^{1/2} \quad \text{for } \text{Re } \omega \geq 0$$

To obtain the solution of the eq. (12) we shall expand it over eigenfunctions g_μ of the transverse Laplace operator Δ_\perp with the same boundary conditions as (13). Denoting the corresponding eigenvalues as $-\omega_\mu^2/c^2$ (ω_μ^2 are complex because of wall losses) we obtain the longitudinal field averaged over beam cross-section:

$$(15) \quad \bar{E}_k = \frac{4\pi ix^2 c^2}{k} \rho_k \sum_\mu |g_\mu|^2 / (\omega_\mu^2 + x^2 c^2)$$

To obtain from here an expression for $Z(\frac{\omega}{c}, \nu)$ (see the eq. (1)) one has to put $\rho(\nu) = \delta(\nu)$ (i. e. $\rho_k = R/2\pi$), multiply the eq. (15) by $\frac{2\pi}{ik/c} \exp(-ikR\frac{\omega}{c})$ and integrate over k with $\omega = k/c + \nu$.

As a result we have

$$(16) \quad Z\left(\frac{\omega}{c}, \nu\right) = \frac{4\pi cR}{R} \sum_\mu |g_\mu|^2 \int_{-\infty}^{+\infty} \frac{e^{-ikR\frac{\omega}{c}} x^2 dk}{k^2 (\omega_\mu^2 + x^2 c^2)}$$

The imaginary part of ω_μ^2 can be evaluated by the usual perturbation theory methods and is determined by the associated magnetic field at the walls:

$$(17) \quad \text{Im } \omega_\mu^2 = - \frac{R^2 c^2 \sigma |k|^{1/2}}{2x^2 R^{1/2}} \int_{\partial} dC \left| \frac{\partial g_\mu}{\partial n} \right|^2$$

where $\delta = [cR/2\pi\sigma]^{1/2}$ is the skin-depth at the frequency of revolution and the integration is performed over the perimeter of the chamber cross-section. The proportionality of $\text{Im } \omega_\mu^2$ to $|k|^{1/2}$ reflects the above mentioned non-symmetric character of the resistive field. All other parameters of the chamber, e. g. g_μ , $\text{Re } \omega_\mu^2$ may be taken at their ideal values i. e. for $\sigma \rightarrow \infty$.

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Although the investigation of the field structure at small distances is not a purpose of this paper one peculiarity has to be mentioned. At small distances (i. e. at large k-s) the field fall-off is determined by the zeros of the integrand denominator in the eq. (16). A simple analysis shows that the fall-off of the μ -th mode is described by

$$\exp \left[- \frac{i \beta v \gamma^2}{c^2} \frac{r}{\beta} - \left| \frac{r}{\beta} \right| \frac{\gamma}{c} \sqrt{\omega_\mu^2 - v^2 \gamma^2} \right]$$

Usually it gives a very sharp decrease of the field when $\frac{r}{\beta}$ is increasing unless the conditions are satisfied when $\omega_\mu^2 < \gamma^2 v^2$. In this case the whole field pattern changes and that can seriously affect all the results.

IV. - INSTABILITY INCREMENT AND BEAM LENGTH.

To evaluate the instability increment (11) one has to know an explicit dependence $\vartheta(\tau)$ i. e. the phase trajectories for the equilibrium state. We shall suppose the synchrotron oscillations to be linear

$$(18) \quad \vartheta = \text{const} + A \cos \Omega \tau$$

with amplitude A related to energy \mathcal{E}_0 as for a harmonic oscillator

$$(19) \quad \mathcal{E}_0 = A^2 R^2 \Omega^2 M / 2$$

Substituting (16) in (11) and assuming that $\text{Im } \omega_\mu^2 \ll \omega_\mu^2$ we obtain

$$(20) \quad \text{Im } \Delta v = + \frac{8 \pi c n e N I}{|M| R \Omega \beta A^2} \sum_{\mu} \left| \bar{g}_\mu \right|^2 \times$$

$$\times \int_{-\infty}^{+\infty} \frac{x^2 J_0^2(|k R A|) \text{Im } \omega_\mu^2}{k^2 (\omega_\mu^2 + x^2 c^2)} dk$$

One can see from here that only non-symmetric resistive fields contribute the increment and all reactive forces are integrated to zero.

We shall substitute now (17) in (20) and perform integration over k. Note that $\omega_\mu \gg c/R\beta$, so the main part of the integral comes from large k-s. So we may use an asymptotic expression for Bessel functions

$$J_n(kRA) \approx \left[2/\pi |kRA| \right]^{1/2} \cos\left(n\frac{\pi}{2} + \frac{\pi}{4} - |kRA|\right)$$

and substitute the averaged value $1/2$ for $\cos^2\left(n\frac{\pi}{2} + \frac{\pi}{4} - |kRA|\right)$. Then simple integration gives

$$(21) \quad \text{Im } \Delta\nu = - \frac{21 \pi \delta n^2 e N I \beta^2 \gamma^{1/2} \cdot S^{1/4}}{|M| R^2 A^3 c^2 \sqrt{2}} \frac{S^{1/4}}{R^{1/2}} \Lambda$$

where S is a cross-section of the vacuum chamber and

$$(22) \quad \Lambda = \sum_{\mu} \left| \bar{g}_{\mu} \right|^2 \oint dC \left| \frac{\partial g_{\mu}}{\partial n} \right|^2 (\omega_{\mu}/c)^{-11/2} S^{-1/4}$$

The coefficient $S^{1/4}/R^{1/2}$ is inserted in the eq. (21) only for scaling to make the form factor Λ of order of unity. Values of Λ are plotted in Fig. 2 for the case of a cylindrical beam with radius \underline{a} in a cylindrical pipe with radius \underline{d} .

The most essential feature of eq. (21) is the proportionality of the increment to A^{-3} . Hence at large currents the equilibrium bunch length must be proportional to $I^{1/3}$.

To find this equilibrium length one can equalize the radiation damping to the excitation due to the considered space-charge effect and quantum fluctuations. If τ_{rad} is the radiation damping time and A_{rad} the natural half-length of the bunch one gets

$$(23) \quad \dot{A}^2 = 0 = -A^2 \text{Im } \Delta\nu - A^2 \tau_{\text{rad}}^{-1} + A_{\text{rad}}^2 \tau_{\text{rad}}^{-1}$$

The solution of this cubic equation is (see Fig. 3):

$$(24) \quad \frac{A}{A_{\text{rad}}} = \frac{2}{\sqrt{3}} \cos\left(\frac{1}{3} \arccos \frac{3^{3/2} N I}{2 I_0}\right)$$

Where the critical value of current I_0 is given by

$$(25) \quad I_0 = \frac{|M| R^2 c^2 \sqrt{2} A_{\text{rad}}^3 R^{1/2}}{21 \pi \tau_{\text{rad}} \delta n^2 e \beta \gamma^{1/2} \Lambda S^{1/4}}$$

If the current is small the bunch length is determined mainly by fluctuations

$$(26) \quad \frac{A}{A_{\text{rad}}} = 1 + \frac{I N}{2 I_0} + \dots \quad I N \ll I_0$$

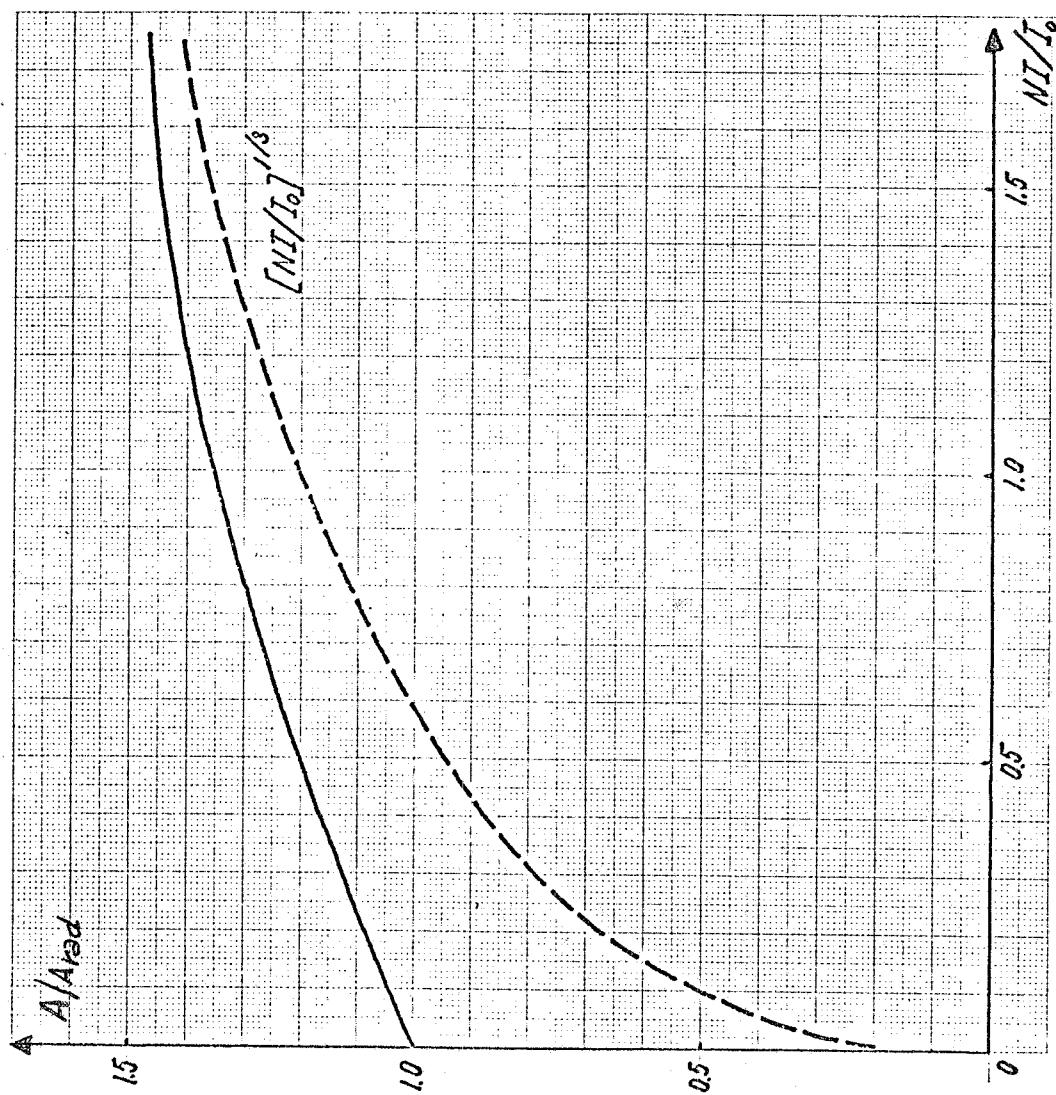


FIG. 3

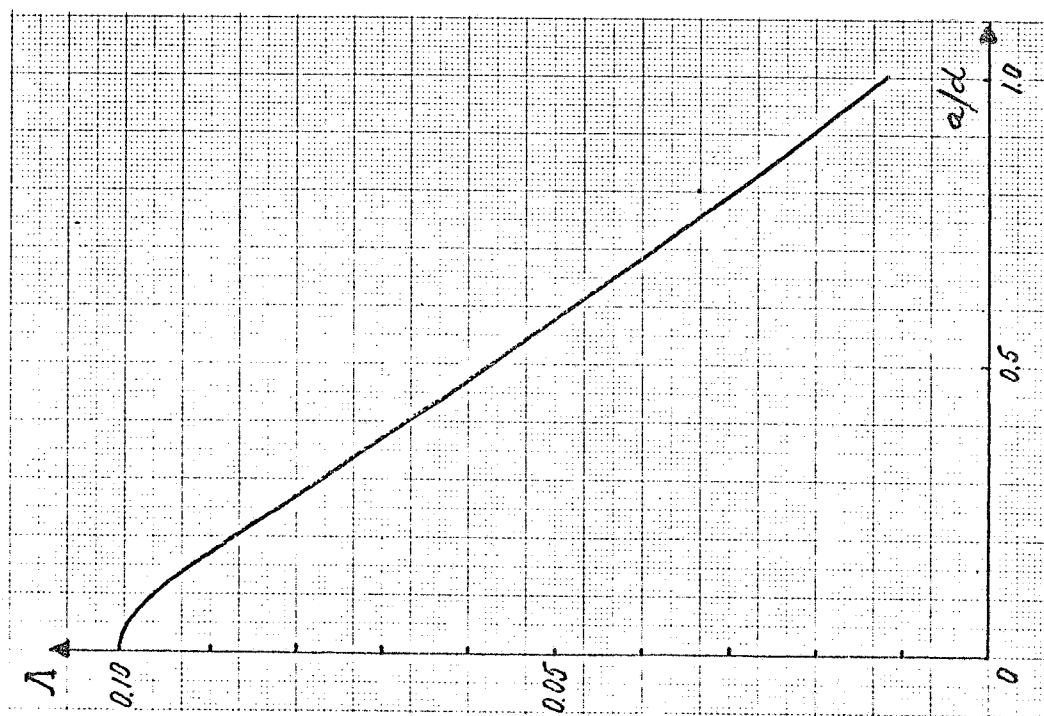


FIG. 2

but in the region $IN \gg I_0$ it increases with current as

$$(27) \quad \frac{A}{A_{\text{rad}}} = \left(\frac{IN}{I_0}\right)^{1/3} + \frac{1}{3} \left(\frac{I_0}{IN}\right)^{1/3} + \dots$$

For ADONE parameters

$$(28) \quad \frac{A}{A_{\text{rad}}} \approx 6.2 \times 10^{-3} \frac{V_{\text{kV}}^{1/2} n^{2/3}}{E_{\text{GeV}}^{8/3}} (IN_{\text{mA}})^{1/3}$$

The value I_0 , aside of machine parameters, depends also on n , the number of the excited mode. Very high modes can not contribute very much to the bunch-length although their increments are large. Obviously some effective n exists which has to be inserted in the eq. (25), (28). Of course this effective n must be larger than one and much less than c/R , because the coherent synchrotron oscillations were supposed to be slow as compared with the period of revolution. Inserting $n=2$ for the lowest (and probably the most dangerous) mode contributing the bunch length we obtain at $V=30$ kV

$$(29) \quad \frac{A}{A_{\text{rad}}} = 6.0 \times 10^{-2} (IN_{\text{mA}})^{1/3} E_{\text{GeV}}^{-8/3}; \quad \frac{A}{A_{\text{rad}}} \gg 1$$

That may be compared with the experimental result

$$(30) \quad \frac{A}{A_{\text{rad}}} \approx 0.31 (IN_{\text{mA}})^{1/3} E_{\text{GeV}}^{-5/3}$$

The eqs. (29) and (30) give the same dependence of the bunch length on current but the energy dependence is stronger than the observed one (The numerical values of (29) and (30) coincide at $E \approx 0.2$ GeV). Moreover, the RF voltage dependence seems to be in contradiction with the experiments. The most probable origin of this discrepancy is an arbitrary choice of number n which is to be calculated from more general theory (taking into account Landau damping, more realistic equilibrium distribution, etc.). Anyway the excitation of coherent synchrotron oscillations considered here has to play an important role in the bunch lengthening effect.

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V. - ACKNOWLEDGEMENTS.

I would like to thank Prof. F. Amman and the Staff of ADONE group for their hospitality and helpful discussions.

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