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C. Pellegrini : ON A NEW INSTABILITY IN ELECTRON-POSITRON STORAGE RINGS (THE HEAD-TAIL EFFECT).

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1. - The instabilities of relativistic electron and positron beams of storage rings are essentially due, at least in the range of currents achieved up to now, to the electromagnetic interaction between the beam itself and the material structures surrounding the beam, like vacuum chamber, clearing field electrodes, radio frequency cavities and so on. This interaction produces an exchange of momentum between the revolution motion and the oscillation modes around the revolution path, which, in some circumstances, can lead to an increase of oscillation amplitudes and hence to beam loss.

In general the mechanism is as follows: a particle of the beam, going through one of these structures, produces on it a signal proportional to its oscillation amplitude; the electromagnetic field thus produced acts on the particles which go through the same structure at subsequent times. Hence on these particles acts a force proportional to the induced signal delayed in time by a quantity proportional to the particle distance. If the phase relationship in the oscillation of the particles is appropriate, this will produce an increase of oscillation amplitudes.

It is convenient to consider in different ways the two cases in which the induced signal decays in a time longer or shorter than the revolution period. We will call the effects of the first type "multiturn instabilities", and the effects of the second type "single turn instabilities".

An important difference exists between a multiturn and a single turn effect. Let us consider, in the multiturn case, a bunch producing a signal in a point of the machine; this signal acts on the same bunch after one revolution, thus introducing a feed-back mechanism which is clearly absent in the single turn effect. Neverthe less a feed-back effect can be introduced in the single turn case by the synchrotron oscillations.

Let us consider, for simplicity, a bunch consisting of two particles only. The "head" particle, when oscillating, produces a signal which then acts on the "tail" particle and gives rise to a forced oscillation. After one half of a period of the synchrotron oscillations, the head and tail particles have exchanged theyr positions giving clearly rise to a regenerative effect.

A classical example of multiturn effect is the resistive wall instability(1). A peculiar characteristic of the multiturn effects, for the case of transverse oscillations, is their strong dependence on the betatron wave number Q. For instance in the case when there is only one bunch in the beam, and for the resistive wall, the motion is stable if

$$n < Q < n + 1/2 ,$$

and unstable if

$$n + 1/2 < Q < n + 1 ,$$

n being an integer number.

For a multiturn effect different from the resistive wall instability we can have an exchange of the stable and unstable region, but a similar rule still applies.

The instability observed in the Adone and ACO storage rings during the last years do not show any strong dependence on the betatron wave number, and cannot be described as a multiturn effect.

In this paper we want to show that they might be due to single turn effects excited by electrodes or some structure present in the machine.

In order to simplify the treatment we will completely neglect the multiturn effects, and limit ourselves to a discussion of single turn or "head-tail" instabilities.

The effect of "head-tail" instabilities in proton synchrotrons, are being considered by H. Hereward, P. Morton and K. Schindl. Hence here we will only consider the case of electrons positrons sto rage rings, assuming also to have a single beam in the machine. Furthermore since the head-tail effect can act only within a bunch, we assume also that in the beam there is only one bunch.

In section 2 we will write the general equations of motion and reduce these equations to a linear system of homogeneous equations. We consider only the case of transverse radial or vertical betatron oscillations. Couplings between these oscillations are neglected. The analysis is hence made for a one dimensional transverse oscillation and should be valid equally well for both radial and vertical betatron modes.

In section 3 we first show that for a storage ring in which the change in betatron wave number with energy is zero, the "head-tail" effect does not introduce instabilities. Afterwards we reduce our problem to the solution of an infinite set of linear homogeneous integral equations, whose eigenvalues determine the characteristics of the bunch motion.

In section 4, 5 we discuss two approximate solutions of these integral equations, limiting ourselves to the case in which only resonant types of forces, as due for instance to electrodes, act on the beam.

Space charge forces or forces due to images on the vacuum chamber wall are neglected, altough they can, in some cases, be important. Nevertheless, since usually in a storage ring, resonant forces can play a dominant role and are responsible for the worst instabilities, we believe that it is reasonable, at least in a first approximation, to neglect non resonant types of forces.

In section 6 we make some numerical estimates of the rise time and frequency shift due to the head-tail effect. These show that unmatched clearing electrodes or low Q resonant cavities, can introduce strong instabilities, with thresholds near to those observed in Adone.

The forces due to the electrodes or RF cavities are evaluated in Appendix.

2. - Let s_l, z_l, be the longitudinal and transverse coordinate of the "l-th" particle, s_l being defined assuming the synchronous particles as origin of the coordinates. We introduce also the quantity

$$\sigma_l = s_l/v ,$$

v being the average particle velocity, which measures the distance in time between the particle "l" and the synchronous one.

Then the equations of motion can be written

(1) $$\ddot{z}_1(t) + \nu_1^2 z_1(t) = \sum_{k \neq 1} F_{k1}(t) ,$$

(2) $$\ddot{\sigma}_1(t) + \omega_s^2 \sigma_1(t) = 0 .$$

The quantity ν_1 is the betatron frequency of the "1-th" particle and ω_s is the synchrotron frequency, which we assume equal for all particles, and much smaller than ν_1.

The quantity $F_{k1}(t)$ represents the forces due to particle "K" and acting on particle "1". The relevant part of this force, that is the part proportional to the displacement of particle "K", can be written as

(3) $$F_{k1}(t) = z_k(t + \sigma_1(t) - \sigma_k(t)) \times \\ \times \sum_{n=-\infty}^{+\infty} e^{in\omega_0(t+\sigma_1(t))} \varsigma_n(\sigma_k(t) - \sigma_1(t)) ,$$

ω_o being the revolution frequency.

The sum over n takes into account the fact that the part of the machine generating the force, for instance an electrode, reappears periodically at each revolution.

It can easily be shown that, assuming $F_{k1}(t)$ to represent a small perturbation, only the term n = 0 of (3) is relevant in the analysis of the motion. Hence in what follows we will consider only this term.

The betatron frequency ν_1 depends on the betatron oscillation amplitudes and on the energy displacement from the synchronous particle. It is convenient to factorize the last dependence, writing ν_1 as

$$\nu_1 = \nu_{o1} \left[1 + \left(\frac{E}{\nu_{o1}} \frac{\Delta \nu}{\Delta E} \right) \frac{\Delta E}{E} \right],$$

where ν_{o1} depends now on the betatron amplitude only.

Writing ν as

$$\nu = Q \omega_o ,$$

where Q is the betatron wave number and ω_o is the revolution frequency, the change in ν with energy can be written, for relativistic beams, as

$$\frac{E}{\nu}\frac{\Delta\nu}{\Delta E} = \frac{E}{Q}\frac{\Delta Q}{\Delta E} - \alpha ,$$

where α is the momentum compaction factor. On the other hand, above transition and for relativistic beams,

$$\dot{\sigma}_1 = -\alpha\frac{\Delta E}{E} ,$$

so that we can write

(4) $$\nu_1 \simeq \nu_{o1}(1 + \frac{1}{2}\mathcal{D}\dot{\sigma}_1) ,$$

with

(5) $$\mathcal{D} = 2(1 - \frac{1}{\alpha}\frac{E}{Q}\frac{\Delta Q}{\Delta E}) .$$

Assuming $\mathcal{D}\dot{\sigma}_1 \ll 1$, we can also write

$$\nu_1^2 \simeq \nu_{o1}^2(1 + \mathcal{D}\dot{\sigma}_1) .$$

Equation (1) becomes now

(1')
$$\ddot{z}_1(t) + \nu_{o1}^2(1 + \mathcal{D}\dot{\sigma}_1)\, z_1(t) = \sum_{k\neq 1} z_k(t + \sigma_1(t) - \sigma_k(t))\, \rho_o(\sigma_k(t) - \sigma_1(t)) .$$

If the force on the r. h. s. of (1') is zero, a solution of (1'), to first order in σ_1, is

$$z_1(t) = \xi_1\, e^{i\nu_{o1}(t + \frac{1}{2}\mathcal{D}\sigma_1(t))}$$

When the collective force is different from zero, we look for a solution of (1') of the form

(6) $$z_1(t) = \xi_1(t)\, e^{i\nu t + \frac{1}{2}\nu_{o1}\mathcal{D}\sigma_1(t)} ,$$

where the function $\xi_1(t)$ is periodic with period equal to that of the synchrotron oscillations, and ν is the collective oscillation frequency.

Let us introduce the quantity

(7) $$\delta_l = (\nu - \nu_{ol})/\nu_o ,$$

where ν_o is the average value of the ν_{ol}. Since $\sum_k F_{kl}$ is a small perturbation, we can assume $\delta_l \ll 1$.

Using (6) and since $\omega_s \ll \nu_o$, equation (1') becomes, neglecting terms like $\ddot{\xi}_l$, $\dot{\sigma}_l \dot{\xi}_l$ and so on,

(8) $$\dot{\xi}_l(t) + i\nu_o \delta_l \xi_l = - \frac{i}{2\nu_o} \sum_{k \neq l}' \rho_o(\sigma_k(t) - \sigma_l(t)) \times$$
$$\times\ \xi_k(t) \exp\left\{ i \frac{\vartheta}{2} (\nu_{ok}\sigma_k - \nu_{ol}\sigma_l) - i\nu(\sigma_k - \sigma_l) \right\} .$$

Since the term on the r. h. s. of (8) represents a small perturbation, we can approximate in it the quantity ν with the average value, ν_o, of ν_{ol}. The quantity $\nu_{ok}\sigma_k - \nu_{ol}\sigma_l$ can also be approximated with $\nu_o(\sigma_k - \sigma_l)$ if the betatron frequency spread is not too large.

Using this approximations, the solution of (8) can be written as

(9) $$\xi_l(t) = - \frac{i}{2\nu_o} \int^t dt\, e^{i\nu_o \delta_l (t'-t)} \times$$
$$\times \sum_{k \neq l} \rho_o(\sigma_k(t') - \sigma_l(t')) \xi_k(t')\, e^{-i\bar{\nu}_o(\sigma_k(t') - \sigma_l(t'))} ,$$

with

(10) $$\bar{\nu}_o = \nu_o (1 - \vartheta/2) .$$

The two functions ξ_l and $\rho_o(\sigma_k - \sigma_l) \exp\{-i\bar{\nu}_o(\sigma_k - \sigma_l)\}$, being both periodic with period equal to that of the synchrotron oscillations, can be written as

(11) $$\xi_l(t) = \sum_n C_n^l\, e^{in\omega_s t} ,$$

(12) $$\varrho_o(\sigma_k - \sigma_l)\, e^{-i\bar{\nu}_o(\sigma_k - \sigma_l)} = \sum_n H_n^{kl}\, e^{in\omega_s t} .$$

Substituting (11), (12) in (9) and assuming the appropriate boundary conditions for $\xi_l(t)$ at the lower integration limit, we obtain

(13) $$C_n^l = -\frac{1}{2\nu_o(\nu_o\delta_l + n\omega_s)} \sum_{k\neq l} \sum_{q=-\infty}^{+\infty} H_{n-q}^{kl}\, C_q^k .$$

We can now apply perturbation theory to simplify (13). Since in the limit of $\varrho(\sigma_k - \sigma_l) \to 0$, ξ_l becomes a constant, equation (13) yields, up to second order in the H's,

(14) $$C_o^l + \frac{1}{2\nu_o^2\delta_l}\left\{ \sum_{k\neq l} H_o^{kl}\, C_o^k - \sum_{k,j\neq l} \sum_{q\neq 0} \frac{H_q^{jk}\, H_{-q}^{kl}}{2\nu_{ok}(\nu_o\delta_k + q\omega_s)}\, C_o^j \right\} = 0 .$$

But for the case in which the condition

(15) $$\nu_{ok}\delta_k + q\,\omega_s \simeq 0$$

is satisfied, the second order terms in (14) can be neglected. In the following section we will consider only this case.

3. - Neglecting second order terms, equation (14) is written as

(14') $$C_o^l + \frac{1}{2\nu_o^2\delta_l} \sum_{k\neq l} H_o^{kl}\, C_o^k = 0 .$$

This homogeneous system of linear equations determine the eigenvalues, ν, of our problem, at least in principle.

The matrix H_o^{kl} is symmetric and non-hermitian; its elements are given by :

(16) $$H_o^{kl} = \frac{\omega_s}{2\pi}\int_0^{2\pi/\omega_s} \rho\left[\sigma_k(t) - \sigma_l(t)\right] e^{-i\bar{\nu}_o\left[\sigma_k(t) - \sigma_l(t)\right]} dt$$

It is interesting to notice that if $\bar{\nu}_o = 0$, then the matrix H_o^{kl} is real and symmetric; hence the eigenvalues of (14') are real and the motion is stable. From (10) and (5) it follows that the condition $\bar{\nu}_o = 0$ corresponds to

$$\mathcal{D} = 2$$

or

(17) $$\frac{1}{\alpha}\frac{E}{Q}\frac{\Delta Q}{\Delta E} = 0 .$$

This means that for a machine in which the change in betaron wave number with energy is zero, there is no "head-tail" effect.

In general, from this result one can expect that the effect under discussion should be more important for strong focusing rings than for weak focusing ones.

In the general case, $\bar{\nu}_o \neq 0$, introducing the Fourier transform of the function $\rho(\sigma)$,

(18) $$\mathcal{F}(\omega) = \frac{1}{2\pi}\int \rho(\sigma)\, 1^{-i\omega\sigma}\, d\sigma ,$$

and writing $\sigma_k(t)$ as

(19) $$\sigma_k(t) = A_k \cos(\omega_s t + \varphi_k) ,$$

(16) becomes

(20) $$H_o^{kl} = H_o^{lk} = \int d\omega\, \mathcal{F}(\omega - \bar{\nu}_o) \times$$
$$\times\; J_o\left[\omega\left(A_k^2 + A_l^2 - 2A_kA_l\cos(\varphi_k - \varphi_l)\right)^{1/2}\right] .$$

As it is seen from (20), H_o^{kl} is periodic in $\varphi_k - \varphi_l$ and can be written as

(21) $$H_o^{kl} = \sum_{m=0}^{\infty} F_m(A_k, A_l)\cos m(\varphi_k - \varphi_l) ,$$

where

(22) $$F_m(A_k, A_l) = \frac{1}{2\pi}\int_0^{2\pi} H_o^{kl} \cos m(\varphi_k - \varphi_l)\, d(\varphi_k - \varphi_l)\ .$$

It is now convenient to introduce the longitudinal density distribution function, and rewrite (14') considering the bunch as a continuous system.

The longitudinal density distribution function is assumed to be stationary and is written as

$$n(A)A\, dA\, d\varphi$$

with the normalization condition

$$\int_0^{\infty}\int_0^{2\pi} n(A)\, A\, dA\, d\varphi = N\ ,$$

N being the number of particles in the bunch.

We define also the two quantities

$$\Gamma(A_l, \varphi_l)\ , \qquad \Lambda^{-1}$$

obtained by summing respectively C_o^l and δ_l^{-1} over all the particles in the surface element $A_l dA_l d\varphi_l$ and dividing by the number of particles contained in the same area. $\Gamma(A_l, \varphi_l)$ represents the transverse center of mass in the point A_l, φ_l. The quantity Λ can be written explicitely introducing the distribution of betatron amplitudes. Assuming this distribution to be indipendent from A, φ and writing it as $f(\xi)d\xi$ we have

(23) $$\frac{1}{\Lambda} = \frac{1}{2\nu_o}\int \frac{f(\xi)\, d\xi}{\nu - \nu(\xi)}\ .$$

In a self consistent calculation $f(\xi)$ should be determined considering the effect on it of the collective force. This self consistent calculation is outside the aim of this work and, in a first approximation, we will assume $f(\xi)$ equal to the unperturbed distribution function.

Considering the bunch as a continous system and using Γ, Λ, equation (14') can now be written as

(24) $$\Gamma(A_l, \varphi_l) + \Lambda^{-1}\int A_k\, dA_k\, d\varphi_k\, n(A_k)\, \Gamma(A_k, \varphi_k) H_o^{kl} = 0$$

A partial diagonalization of (24) can be obtained by assuming

(25) $$\Gamma(A_1, \varphi_1) = \sum_n D_n(A_1)\, 1^{-in\varphi_1} .$$

Then (24) becomes, using (21),

(26) $$D_n(A_1) + \frac{2\pi}{\Lambda} \int A_k\, dA_k\, n(A_k)\, D_n(A_k)\, F_n(A_k, A_1) = 0 .$$

The solution of equation (26) will give us the eigenvalues and eigen functions of our problem.

4. - In general it does not seem possible to obtain an analitical solution of equation (26) and it will be necessary to use numerical methods to get the eigenvalues of our problem. Nevertheless it is possible to make an approximate evaluations of these eigenvalues in some simple cases.

Let us assume that the collective force acting on the beam is due to some resonant structure present in the machine, like resonant cavities or electrodes, or that the force can be analized as a sum of resonant contributions. Then the function $\mathcal{Y}(\omega)$ can be written as

(27) $$\mathcal{Y}(\omega) = \frac{\eta_\alpha}{\omega - \omega_\alpha} - \frac{\eta_\alpha^*}{\omega + \omega_\alpha^*}$$

where η_α^* and ω_α^* are the complex conjugate quantities of η_α, ω_α and $\mathrm{Im}\, \omega_\alpha > 0$.

Then from (20) it follows that

(28) $$H_o^{kl} = \pi i\, \eta_\alpha \left\{ J_o\left[(\omega_\alpha + \bar{\nu}_o) d_{kl}\right] + i S_o\left[(\omega_\alpha + \bar{\nu}_o) d_{kl}\right] \right\} - \pi i\, \eta_\alpha^* \left\{ J_o\left[(\bar{\nu}_o - \omega_\alpha^*) d_{kl}\right] + i S_o\left[(\nu_o - \omega_\alpha^*) d_{kl}\right] \right\} ,$$

where

$$d_{kl} = \left\{ A_k^2 + A_1^2 - 2 A_k A_1 \cos(\varphi_k - \varphi_1) \right\}^{1/2}$$

and $S_o(z)$ is the Struwe sunction of order zero.

When ω_α and $\bar{\nu}_o$ are such that

(29)
$$\left|\omega_\alpha\right| d_{kl} \ll 1$$
$$\bar{\nu}_o d_{kl} \ll 1$$

the expression for H_o^{kl} becomes, to first order in d_{kl},

(30) $$H_o^{kl} = -2\pi \operatorname{Im} \eta_\alpha - 4\left\{\operatorname{Re}(\omega_\alpha \eta_\alpha) + i\,\bar{\nu}_o \operatorname{Im} \eta_\alpha\right\} d_{kl} \quad .$$

Having assumed $\left|\omega_\alpha\right| d_{kl} \ll 1$, one has that the term of (30) proportional to $\operatorname{Re}(\omega_\alpha \eta_\alpha)$ is important only when $\operatorname{Re} \eta_\alpha \gg \operatorname{Im} \eta_\alpha$.

In the following part of this work, we will limit surselves to consider only the simple case in which condition (29) is satisfied and the term of (30) proportional to $\operatorname{Re}(\omega_\alpha \eta_\alpha)$ can be neglected, hence assuming

(31) $$H_o^{kl} = -2\pi \operatorname{Im} \eta_\alpha \left(1 + \frac{2i}{\pi} \bar{\nu}_o d_{kl}\right) .$$

The eigenvalues of our problem can be obtained very easely if we consider a simple model, such that all the particles lie on the same invariant, of amplitude A^*, in synchrotron phase space.

Then one has from (31), (22), that

(32) $$F_m(A^*, A^*) = -2\pi \operatorname{Im} \eta_\alpha \left\{\delta_{m,o} - \frac{8i\,\bar{\nu}_o A^*}{\pi^2} \frac{1}{4m^2 - 1}\right\} .$$

Since $A\,n(A)$ is now given by

$$A\,n(A) = N\,\delta(A - A^*)$$

the eigenvalues are easily obtained from (26); namely

$$\Lambda_m = -2\pi N F_m(A^*, A^*) ,$$

or

(33) $$\Lambda_m = 4\pi^2 N \operatorname{Im} \eta_\alpha \left\{\delta_{m,o} - \frac{8i\,\bar{\nu}_o A^*}{\pi^2} \frac{1}{4m^2 - 1}\right\} .$$

N being the total number of particles in the bunch. The eigenvalue Λ_o has a large real part. The Im Λ_m has opposite signs for m = 0 or m $\neq$ 0 so that, when the frequency spread is not large enough, there is always at least one unstable mode. The rise time, in the case when all the particles have the same betatron frequency, is given, for m = 0, by

$$\frac{1}{\tau} = 8(2-\vartheta)\, N A^{*} \,\mathrm{Im}\, \eta_\alpha \; . \tag{34}$$

5. - In this section we will try to get the eigenvalues and eigenmodes of our problems, making use of the following approxi mations :

a) the longitudinal distribution function is assumed to be a step function

$$n(A) = \frac{N}{2\Delta^2} \qquad \text{for} \quad 0 \leq A \leq \Delta ,$$

$$n(A) = 0 \qquad \text{for} \quad A > \Delta ,$$

the quantity 2Δ being the bunch length;

b) only the two modes m = 0, m = 1 are considered (we notice that these two modes should be the most important ones, as it can be seen, for instance, from (33)); for these two modes we approximate the Kernal $F_m(A_k, A_l)$, substituting in (22) $H_o^{kl} \cos m(\varphi_k - \varphi_l)$ with its average value; then one has:

$$F_o(A_k, A_l) = \begin{cases} -2\pi \,\mathrm{Im}\, \eta_\alpha \left(1 + \frac{2i}{\pi} \bar{\nu}_o A_l\right), & \text{if } A_k < A_l , \\ -2\pi \,\mathrm{Im}\, \eta_\alpha \left(1 + \frac{2i}{\pi} \bar{\nu}_o A_k\right), & \text{if } A_l < A_k , \end{cases} \tag{35}$$

$$F_1(A_k, A_l) = \begin{cases} 2i \,\mathrm{Im}\, \eta_\alpha \, \bar{\nu}_o A_k , & \text{if } A_k < A_l , \\ 2i \,\mathrm{Im}\, \eta_\alpha \, \bar{\nu}_o A_l , & \text{if } A_l < A_k . \end{cases} \tag{36}$$

The integral equation (26), for the two modes m = 0 and 1, becomes, using (35), (36),

(37)
$$D_o(A) - \frac{2\pi^2 N\,\mathrm{Im}\,\eta_\alpha}{\Delta^2 \Lambda}\int_0^\Delta \bar{A} D_o(\bar{A})d\bar{A} - 4\pi\,\mathrm{Im}\,\eta_\alpha \frac{i\bar{\nu}_o N}{\Delta^2\Lambda} \times$$
$$\times \left\{ A\int_0^A \bar{A} D_o(\bar{A})d\bar{A} + \int_A^\Delta \bar{A}^2 D_o(\bar{A})d\bar{A} \right\} = 0\,,$$

(38)
$$D_1(A) + \frac{2\pi\,\mathrm{Im}\,\eta_\alpha i\bar{\nu}_o N}{\Delta^2\Lambda} \times$$
$$\times \left\{ \int_0^A \bar{A}^2 D_1(\bar{A})d\bar{A} + A\int_A^\Delta \bar{A} D_1(\bar{A})d\bar{A} \right\} = 0\,.$$

Equations (37), (38) can also be written as differential equations with suitable boundary conditions. Performing the transformation of variable

$$y = A/\Delta\,,$$

$$\psi_i(y) \rightarrow D_i(A)\,, \qquad i = 0,1\,,$$

the differential equations are

(39)
$$\frac{\partial^2\psi_i(y)}{\partial y^2} + \lambda_i\, y\, \psi_i(y) = 0, \qquad i = 1,2\,,$$

with the boundary conditions

(40)
$$\psi_o(1) = \left.\frac{\partial\psi_o(y)}{\partial y}\right|_{y=1} \left(1 + \frac{\pi}{2i\nu_o\Delta}\right),$$

$$\left.\frac{\partial\psi_o(y)}{\partial y}\right|_{y=0} = 0\,,$$

$$\psi_1(0) = 0\,,$$

$$\left.\frac{\partial\psi_1(y)}{\partial y}\right|_{y=1} = 0\,.$$

The quantities λ_o, λ_1 are defined as

$$\lambda_o = -4\pi \,\mathrm{Im}\, \eta_\alpha \frac{i\bar{\nu}_o \Delta N}{\Lambda} ,$$

(41)

$$\lambda_1 = -2\pi \,\mathrm{Im}\, \eta_\alpha \frac{i\bar{\nu}_o \Delta N}{\Lambda} .$$

The solutions of (39), (40) are given by

(42) $$\psi_o(y) = \text{const} \times y^{1/2} J_{-1/3}\left(\tfrac{2}{3} \lambda_o^{1/2} y^{3/2}\right) ,$$

(43) $$\psi_1(y) = \text{const} \times y^{1/2} J_{1/3}\left(\tfrac{2}{3} \lambda_1^{1/2} y^{3/2}\right) ,$$

with the conditions

(44) $$J_{-1/3}\left(\tfrac{2}{3} \lambda_o^{1/2}\right) + \lambda_o^{1/2}\left(1 + \frac{\pi}{2i\bar{\nu}_o\Delta}\right) J_{2/3}\left(\tfrac{2}{3} \lambda_o^{1/2}\right) = 0$$

for m = 0, and

(45) $$\lambda_1^{1/2} J_{-2/3}\left(\tfrac{2}{3} \lambda_1^{1/2}\right) = 0$$

for m = 1.

The values of λ_o, λ_1, corresponding to the zeros of the two last equations determine the eigenvalues Λ of our problem.

Due to the oscillatory behaviour of the functions J, there are an infinite number of zeros for (44), (45), which we can call λ_o^k, λ_1^k. The corresponding Λ will be denoted by Λ_k^m, the index m referring to the normal mode in ψ and K to that for A.

We want to make now an approximate evaluation of the zeros of (44). Since $\bar{\nu}_o \Delta \ll 1$, these zeros are given, neglecting terms of second order in $\bar{\nu}_o\Delta$, by those of

$$J_{2/3}\left(\tfrac{2}{3}\sqrt{\lambda_o}\right) = 0 ,$$

i. e.,

$$\tfrac{2}{3}\sqrt{\lambda_o^k} \simeq \left(K + \tfrac{1}{14}\right)\pi , \qquad K = 1, 2, \ldots ,$$

The corresponding Λ's are

(46) $$\Lambda_k^o = -\frac{16}{9} i \frac{\bar{\nu}_o \Delta N \operatorname{Im} \eta_\alpha}{(k+\frac{1}{14})^2 \pi}, \qquad k = 1, 2, \ldots,$$

In addition there is another zero for $\lambda_o \ll 1$, given by

$$\lambda_o^o = -\frac{4i\bar{\nu}_o \Delta}{\pi + 2i\bar{\nu}_o \Delta}$$

or

(47) $$\Lambda_o^o = \pi^2 N \operatorname{Im} \eta_\alpha (1 + \frac{2}{\pi} i \bar{\nu}_o \Delta).$$

In the case $m = 1$ the zeros of (45) are given approximately by

$$\frac{2}{3} \lambda_1^{1/2} \simeq (k + \frac{5}{12})\pi, \qquad k = 0, 1, \ldots,$$

or

(48) $$\Lambda_k^1 \simeq -\frac{8 \operatorname{Im} \eta_\alpha i \bar{\nu}_o \Delta N}{9\pi (k + \frac{5}{12})^2}, \qquad k = 0, 1, \ldots,$$

As it can be seen from (46), (47), (48), in the case $m = 1$ all the eigenvalues have the imaginary part of Λ of the same sign ($\operatorname{Im} \Lambda_k^1 < 0$, for $\operatorname{Im} \eta_\alpha > 0$) while in the case $m = 0$ the first eigenvalue has $\operatorname{Im} \Lambda > 0$, and the other eigenvalues have $\operatorname{Im} \Lambda < 0$ when $\operatorname{Im} \eta_\alpha > 0$ and the opposite for $\operatorname{Im} \eta_\alpha < 0$.

Notice also that Λ_k^m goes rapidly to zero for increasing K, so that only the first eigenvalues are of practical interest.

To close this section we want to avaluate the transverse center of mass amplitude, Z_k^m, integrated over all the bunch length, for the various modes m, K.

This quantity is of interest since it is related to the possibility of using an external feed-back system to stabilize the beam and can be easily observed.

From (6), (25) we obtain

$$Z_k^m(t) = \frac{2\pi}{N}(-i)^m 1^{i\nu t + im\omega_s t} \int_0^\infty A\,dA\,n(A)\,D_m^k(A)\,J_m(\nu_o \vartheta A/2).$$

For the model discussed above, assuming

$$\nu_o \mathcal{D} \Delta \ll 1 \ ,$$

and considering only the modes such that

$$\left| (\lambda_k^m)^{1/2} \right| \ll 1 \ ,$$

one has approximately

$$Z_k^o(t) \simeq \frac{\pi}{2(-\frac{1}{3})!} 1^{i\nu t} \left[\frac{(\lambda_k^o)^{1/2}}{3} \right]^{-1/3} \ ,$$

$$Z_k^1(t) \simeq - \frac{i\pi}{16(\frac{1}{3})!} 1^{i(\nu + \omega_s)t} \left[\frac{1}{3} (\lambda_k^1)^{1/2} \right]^{1/3} \nu_o \mathcal{D} \Delta \ .$$

This shows that, under the conditions assumed above, the integrated center of mass amplitude is smaller for $m = 1$ than for $m = 0$. In any case this quantity is very dependent on machine parameters and can give rise to a variety of situations for different rings.

6. - We will now make some order of magnitude estimates of the head-tail effect assuming as possible sources of the collective force some of the possible elements usually present in a storage ring, like clearing electrodes, RF cavities or discontinuities in the vacuum chamber cross section. The forces due to these elements are discussed in Appendix A.

a) Perfectly matched clearing field electrode. From (A-13), (A-23) the function $\rho(\sigma)$ can be approximately written as

$$\rho(\sigma) \simeq - \frac{2\pi v Z_o c^2 r_e}{\gamma d^2 L} \theta(\sigma) \ ,$$

with $\theta(\sigma) = 0$ if $\sigma < 0$, $\theta(\sigma) = 1$ if $\sigma > 0$, and where Z_o = characteristic inpedance; d = distance between the electrodes, L = = storage ring circumference, γ = particle energy in rest mass unit, r_e = classical radius of the electron.

Then

$$\mathcal{F}(\omega) = \frac{i}{\omega} \frac{Z_o v c^2 r_e}{\gamma d^2 L} \ .$$

Comparing with (27) we have

$$\omega_\alpha = 0 ,$$

$$\mathrm{Im}\, \eta_\alpha = \frac{Z_o v c^2 r_e}{2\gamma d^2 L} .$$

We then have from (47)

$$\Lambda_o^o = \frac{\pi^2 Z_o v c^2 N r_e}{2\gamma d^2 L} \left\{ 1 + \frac{i 2 \bar{\nu}_o \Delta}{\pi} \right\} .$$

Here and in the following we will denote as usual by U and V the real and imaginary part of Λ.

In the Adone case, and assuming

$L = 10^4$ cm, $\gamma = 10^3$, $d = 5$ cm, $Z_o = 20\Omega\ (2 \times 10^{-11}$ cgs), $\nu_o =$
$= 2\pi \times 10^7\ \mathrm{sec}^{-1}$, $\Delta = 10^{-9}$ sec ,

we have

$$U_o^o \simeq 3\ N\ \mathrm{sec}^{-2} ,$$

$$V_o^o \simeq 0.12\ N (1 - \frac{\mathcal{D}}{2})\ \mathrm{sec}^{-2} .$$

For the Adone case $\mathcal{D}$ is negative and large,

$$\mathcal{D} \simeq -30 ,$$

so that the approximation $\bar{\nu}_o \Delta \ll 1$ is not strictly valid. If nevertheless we estimate V_o^o using the above formula, we obtain,

$$V_o^o \simeq 1.8\ N\ \mathrm{sec}^{-2} .$$

Hence the motion is stable ($V > 0$).

For the modes $m = 0$, $k = 1$ and $m = 1$, $k = 0$, we have, from (46), (48)

$$\Lambda_1^o \cong -i\ 0.17 N\ \mathrm{sec}^{-2} ,$$

$$\Lambda_o^1 \cong -i\ 0.34\ N\ \mathrm{sec}^{-2} ,$$

so that these modes are unstable. We can evaluate the rise time for these modes using the relationship

$$\frac{1}{\tau} = \frac{V}{2\nu_o} \ .$$

Introducing the current per bunch I, which in the Adone case is related to N by

$$N = \frac{2}{3} 10^9 \, I \quad (mA) \ ,$$

one has

$$\tau_1^o = - \frac{1.1}{I} \ sec/mA \ ,$$

$$\tau_o^1 = - \frac{0.55}{I} \ sec/mA \ .$$

The real frequency shift for this modes is introduced only by neglected terms, like space-charge or image forces. Since this frequency shift is small, this mode should be easy to stabilize.

b) Grounded or floating clearing electrodes. From (A-25) we have, for an electrode of length l,

$$\omega_\alpha = h \, \pi \, c/l \ , \quad \text{with h an integer number, and}$$

$$\eta = \frac{Z_o c^4 r_e}{\gamma d^2 v L} \left(\frac{\nu_o l}{v} \right)^2 \frac{(-1)^h}{h} \ .$$

Since $Im \, \eta = 0$, we have $V = 0$ for all modes so that no instability can arise.

Of course grounded or floating electrodes might give very strong multiturn instabilities.

c) Clearing electrode closed on resistance and capacitance. In this case the quantities ω_α, η_α, have been evaluated with the use of a computer. Assuming the impedances to be formed by a resistance and a capacitance in series and also

$$R = Z_o = 20 \Omega ,$$

$$C = 500 \ pF \ ,$$

one has that the first pole of $\mathcal{F}(\omega)$ is

$$\omega_\alpha \simeq (1 + 1.76 \, i) \times 10^8 \ sec^{-1} \ ,$$

and

$$Re \, \eta_\alpha \simeq Im \, \eta_\alpha \simeq 16 \ sec^{-2} \ .$$

In this case one has, using Adone parameters, and (47)

$$U_o^o \simeq 160 \text{ N sec}^{-2} \ ,$$

$$V_o^o \simeq 96 \text{ N sec}^{-2} \ .$$

Since $V_o^o > 0$ this mode is stable with a damping time

$$\tau_o^o \simeq \frac{10^7}{7.6\,N} \text{ sec} = \frac{2 \times 10^{-3}}{I} \text{ sec/mA} \ .$$

As before we have instability for the mode 1, 0 and 0, 1, with a rise time, in absence of frequency spread, given by

$$\tau_1^o \simeq -2.2 \times 10^{-2}/I \quad \text{sec/mA} \ ,$$

$$\tau_o^1 \simeq -1.1 \times 10^{-2}/I \quad \text{sec/mA} \ .$$

To evaluate the rise time and the threshold of the instability in the case of an arbitrary distribution of betatron frequencies, one should solve equation (23), obtaining ν as a function of Λ.

We do not want to consider here the details of this calculation and we will limit ourselves to perform a simple order of magnitude estimate of the threshold, by comparing the width of the distribution of betatron frequencies, with the quantities $U/2\nu_o$ and $V/2\nu_o$.

Let us consider the case m = 1, K = 0, which is the most dangerous. The electrode gives

$$\frac{U_o^1}{2\nu_o^2} = 0 \ ,$$

$$\frac{V_o^1}{2\nu_o^2} = -2 \times 10^{-15} \text{ N} \ .$$

Assuming that the contribution to U_o^1 from neglected effects is such that U_o^1 remains smaller or of the same order than V_o^1, we can use, in order to evaluate the threshold, the relationship

$$\frac{\xi^2}{\nu} \frac{\Delta \nu}{\Delta \xi^2} \simeq \frac{V_o^1}{2\nu_o^2} \ ,$$

where ξ is the betatron oscillation amplitude.

Since, for Adone, $\frac{1}{\nu} \frac{\Delta\nu}{\Delta\xi^2} \simeq 3 \times 10^{-4}\ \mathrm{cm}^{-2}$, we have, for $\xi = 1$ mm,

$$N_{th} \simeq 1.5 \times 10^9$$

corresponding to a current of about 2 mA per bunch. The observed threshold in Adone for radial instability was about 200 μA, but the number of electrodes, in the initial design, was very high, about 50. These electrodes were terminated only in one point, at about 1/3 of their length. The termination was designed to avoid multi-turn effects, but did not provide a good matching of the electrode at frequencies of the order of the characteristic frequencies of the plate.

Since we use here the results of Appendix A, hence considering the less dangerous case of an electrode terminated at both ends, our results do not strictly apply to the Adone case. Nevertheless we believe that the above crude estimate shows that the electrodes could well be responsible for the Adone single beam instabilities. Notice also that we obtain a threshold current inversely proportional to the bunch length, as was observed in Adone.

In general we can say that the presence in a storage ring of unmatched electrodes is a source of strong instabilities, generated through the head-tail effect.

d) We consider now the effect of a resonant cavity. From (B-4) we have

$$\mathcal{F}(\omega) = -R\left\{\frac{1}{\omega - \omega_r - i\Gamma} - \frac{1}{\omega + \omega_r - i\Gamma}\right\} -$$

$$- \frac{iR}{4Q^2}\left\{\frac{1}{\omega - \omega_r - i\Gamma} + \frac{1}{\omega + \omega_r - i\Gamma}\right\}$$

with

$$R \cong \frac{Z_o c^3 r_e}{2\gamma d^2 L} .$$

One has

$$\omega_\alpha = \omega_r \left(1 + \frac{i}{2Q}\right) ,$$

$$\eta = -R\left(1 + \frac{i}{4Q^2}\right) .$$

We consider here only the most dangerous case of Q of the order of one.

Using (47) one obtains

$$U_o^o = -\frac{\pi^2 R N}{4Q^2} ,$$

$$V_o^o = -\frac{\pi \bar{\nu}_o \Delta R N}{2Q^2} ,$$

which shows that the mode m = 0, k = 0 is now unstable. All the other modes are stable and will not be considered here. In order to make an order of magnitude estimate we assume $\omega_r = 2\pi \times 10^8\ sec^{-1}$, $Z_o = 10^3 \Omega$, Q = 1. Then one has

$$R \simeq 15\ sec^{-2} ,$$

$$U_o^o \simeq -40 N\ sec^{-2} ,$$

$$V_o^o \simeq -24 N\ sec^{-2} .$$

The rise time for the unstable mode, in the absence of frequency spread, is

$$\tau_o^o \approx -5 \times 10^6/N\ sec = -7.5 \times 10^{-3}/I\ sec/mA ,$$

and the real frequency shift, $\Delta\nu/\nu$, is

$$\frac{U_o^o}{2\nu_o^2} \simeq -5 \times 10^{-15} N .$$

Evaluating the threshold in presence of Landau damping as in case "c" of this section, but using U instead of V since now $U_o^o > V_o^o$, one obtains

$$N_{th} \sim \frac{3}{5} 10^9$$

corresponding, for Adone, to about one milliampére.

e) At last we want to make some remarks on the effect on the beam stability, through the head-tail mechanism, of the finite conductivity of the vacuum chamber. Strictly speaking this case is not included in the present calculation. since the force is not of the type assumed in (27)[1]. Nevertheless we can make an order of magnitude estimate assuming for the force due to the finite conductivity the approximate expression obtained by evaluating the wall impedance

$$Z(\omega) = (1 - i)\left(\frac{\omega}{8\pi\chi}\right)^{1/2} ,$$

where χ is the conductivity, at the cut off frequency

$$\omega = c/d ,$$

d being the vacuum chamber radius. Then one has(1)

$$\mathcal{F}(\omega) \simeq \frac{i}{\omega} \frac{4 r_e c^2}{2\pi\gamma d^3} (1 - i)\left(\frac{c}{8\pi\chi d}\right)^{1/2} ,$$

and

$$\omega_\alpha = 0 .$$

This approximate expression of $\mathcal{F}(\omega)$ should be good enough, in the case of bunches not much longer than the vacuum chamber radius, as to allow an order of magnitude estimate of the effect. Using the above expression of $\mathcal{F}(\omega)$, one has that, when $\varnothing < -2$, the mode m = 0, k = 0 is stable, while the modes m = 0, k = 1 and m = 1, k = 0 are unstable.

For these last modes we have

$$V_0^1 \simeq 2 V_1^0 ,$$

$$V_1^0 \simeq - \frac{32}{9\pi^2} \frac{\bar{\nu}_0 \Delta N r_e c^2}{\gamma d^3} \left(\frac{c}{8\pi d\chi}\right)^{1/2} .$$

Again the rise time, in absence of frequency spread, is given by

$$\tau_1^0 \simeq - \frac{4}{3} 10^9/N \ \text{sec} \simeq -2/I \ \text{sec/mA} ,$$

$$\tau_0^1 \simeq - \frac{2}{3} \frac{10^9}{N} \text{sec} \simeq -1/I \ \text{sec/mA} .$$

It is interesting to notice that the "multi turn" resistive wall effect gives a rise time for Adone, with three bunches, of the order of

$$\tau \simeq -4/I \ (\text{sec/mA}) .$$

f) The effect of discontinuities in the vacuum chamber wall on the beam stability has also been evaluated and found to be usually negligible, being small, for instance, as compared to the resistive wall effect.

APPENDIX A -- The forces due to electrodes.

We consider first the case of clearing field or pick-up electrodes. The results reported in this Appendix were essentially obtained by J. J. Laslett[2] and by A. Ruggiero, V. Vaccaro and P. Strolin[3] and are reported here only for the convenience of the reader.

Let us assume to have in the ring an electrode made up of two plates at a transverse distance d and of length l. We assume also that the plates are closed at their ends on two impedances Z_1, Z_2.

Following Laslett[2], we consider each plate and the neighbouring vacuum chamber as a transmission line of characteristic impedance $Z_o = (cC)^{-1}$, c being the light velocity and C the capacitance per unit length.

Then, calling λ_I and I_I the charge and current density induced on the plate and V and A the scalar and vector potenzial on the plate, the trasmission line equations can be written as[2]

$$\text{(A-1)} \quad \begin{aligned} \frac{1}{c}\frac{\partial V}{\partial t} + \frac{\partial A}{\partial s} &= -Z_o\left\{\frac{\partial \lambda_I}{\partial t} + \frac{\partial I_I}{\partial s}\right\}, \\ \frac{\partial V}{\partial z} + \frac{1}{C}\frac{\partial A}{\partial t} &= 0, \end{aligned}$$

where s is the longitudinal coordinate.

Equations (A-1) must be solved considering also the appropriate boundary conditions at the ends.

The force on a particle of charge 1, due to the electrodes, can then be written, to a good approximation, as

$$\text{(A-2)} \quad F = -\frac{e}{d}(V - \beta A).$$

Since we are interested in transverse forces, we will consider only that part of λ_I, I_I proportional to the transverse displacement.

Considering an alectron "K" which moves in the longitudinal direction according to the law

$$\text{(A-3)} \quad s = v(t + \sigma_k)$$

and whose transverse displacement is represented by

$$z = z_k(t)$$

the induced charge density can be written as

(A-4) $$\lambda_I^{(k)} = \mathcal{E}\, e z_k(t)\, \delta(s - vt - v\sigma_k)\ ,$$

where the quantity $\mathcal{E}$ is a geometrical factor which, to a first approximation can be assumed to be of the order of $1/d$.

The induced current density is related to λ_I by

(A-5) $$I_I = v\lambda_I\ .$$

Introducing the Fourier transform $\tilde{V}(\omega, k)$, $\tilde{A}(\omega, k)$, $\tilde{\lambda}_I(\omega, k)$, defined in general by the relationship

$$f(s,t) = \int d\omega\, dk\, \tilde{f}(\omega, k)\, e^{i(\omega t - ks)}\ ,$$

the solution of (A-1) can be written as

(A-6)
$$\tilde{V}(\omega, k) = - Z_o \frac{\omega}{c} \frac{\omega - kv}{(\frac{\omega}{c})^2 - k^2} \tilde{\lambda}_I(\omega, k) + $$
$$+ a(\omega)\,\delta(\frac{\omega}{c} - k) + b(\omega)\,\delta(\frac{\omega}{c} + k)\ ,$$
$$\tilde{A}(\omega, k) = - Z_o k \frac{\omega - kv}{(\frac{\omega}{c})^2 - k^2} \tilde{\lambda}_I(\omega, k) + $$
$$+ a(\omega)\,\delta(\frac{\omega}{c} - k) - b(\omega)\,\delta(\frac{\omega}{c} + k)\ .$$

The boundary conditions, assuming the plate to be between $s = 0$ and $s = l$, are

(A-7)
$$\int dk \left\{ \frac{\tilde{V}(\omega, k)}{Z_1(\omega)} + \frac{\tilde{A}(\omega, k)}{Z_o} + v\tilde{\lambda}_I(\omega, k) \right\} = 0\ ,$$
$$\int dk\, e^{-ikl} \left\{ - \frac{\tilde{V}(\omega, k)}{Z_2(\omega)} + \frac{\tilde{A}(\omega, k)}{Z_o} + v\tilde{\lambda}_I(\omega, k) \right\} = 0\ .$$

Using (A-6), (A-7) the force can be written as

(A-8) $$F(s,t) = - \frac{2Z_o ce}{d} \int d\omega\, dk\, e^{i(\omega t - ks)} \tilde{\lambda}_I(\omega, k)\, g(\omega, k, s)\ .$$

Using the notations,

(A-9) $$\beta_w = \frac{\omega}{kc}\ ,\ r_1(\omega) = \frac{Z_1(\omega)}{Z_o}\ ,\ r_2(\omega) = \frac{Z_2(\omega)}{Z_o}\ ,\ \beta = \frac{v}{c}$$

the function $g(\omega, k, s)$ is

(A-10)
$$g(\omega, k, s) = \frac{(\beta - \beta_w)^2}{(1 - \beta_w^2)} + \frac{\beta_w(1+\beta)}{(1-\beta_w^2)D(\omega)}\, e^{+i(\frac{\omega}{c}+k)s} \times$$
$$\times \left\{ (1-r_2)\left[(\beta\beta_w - 1)r_1 + \beta - \beta_w\right] e^{-i\omega l/c} + \right.$$
$$\left. + (1+r_1)\left[(\beta\beta_w - 1)r_2 - \beta + \beta_w\right] e^{-ikl} \right\} - \frac{\beta_w(1-\beta)}{(1-\beta_w^2)D(\omega)} \times$$
$$\times\, e^{-i(\frac{\omega}{c}-k)s} \left\{ (1+r_2)\left[(\beta\beta_w - 1)r_1 + \beta - \beta_w\right] e^{i\omega l/c} + \right.$$
$$\left. + (1-r_1)\left[(\beta\beta_w - 1)r_2 - \beta + \beta_w\right] e^{-ikl} \right\} ,$$

and

(A-11) $$D(\omega) = (1-r_1)(1-r_2)\, e^{-i\omega l/c} - (1+r_1)(1+r_2)\, e^{i\omega l/c} .$$

Using (A-4), (A-8) and assuming

(A-12) $$Z_k(t) = \frac{1}{2}\left\{ \xi_k e^{i\nu t} + \xi_k^{*} e^{-i\nu t} \right\} ,$$

the force acting on particle "1" of longitudinal coordinate

$$s = v(t + \sigma_1)$$

is

(A-13)
$$F_{1k}(t) = -\frac{2Z_o ce^2}{d^2}\, Z_k(t + \sigma_1 - \sigma_k) \int d\omega\, e^{i\omega(\sigma_k - \sigma_1)} \times$$
$$\times\ g(\omega, \frac{\omega - \nu}{v}, s_1) .$$

Considering the circular structure of the machine, we must generalize this result to the case of an infinite number of electrodes separated by a distance L.

The effect of the "n-th" electrode is simply obtained from (A-13) by substituting in $g(\omega, (\omega - \nu)/v, s_1)$, s_1 with $s_1 - nL$. The integral on the r. h. s. of (A-13) hence represents, in the case of an infinite number of electrodes, a periodic function of period L, which can be written as

$$\text{(A-14)} \qquad \sum_n e^{in\omega_o(t+\sigma_1)} \int d\omega \, e^{i\omega(\sigma_k - \sigma_1)} g_n(\omega) \;,$$

where

$$\text{(A-15)} \qquad g_n(\omega) = \frac{1}{L}\int_0^L g\left(\omega, \frac{\omega - \nu}{v}, s_1\right) e^{2\pi i n s_1/L} ds_1 \;.$$

We are essentially interested in the term $n = 0$ of F_{kl} since this is the part oscillating on the betatron frequency ν. This term is given by

$$\text{(A-16)} \qquad F_{kl}(t) = -\frac{2Z_o ce^2}{d^2} Z_k(t + \sigma_k - \sigma_1)\, \rho_o(\sigma_k - \sigma_1) \;,$$

where

$$\text{(A-17)} \qquad \rho_o(\sigma_k - \sigma_1) = \int d\omega \, e^{i\omega(\sigma_k - \sigma_1)} g_o(\omega) \;.$$

Evaluating g_o from (A-15), and remembering that the force is different from zero only for $0 \leq s \leq 1$, we obtain

$$Lg_o(\omega) = \frac{(\beta - \beta_w)^2}{(1-\beta_w^2)} + \frac{\beta_w}{(1-\beta_w^2) D(\omega)} \times$$

$$\times \left[(\beta\beta_w - 1)r_1 + \beta - \beta_w\right] \left\{ (1+\beta)(1-r_2)e^{-i\omega l/c} \times \right.$$

$$\text{(A-18)} \qquad \times \frac{e^{il[\omega(1+\beta)-\nu]/v} - 1}{il[\omega(1+\beta)-\nu]/v} - (1-\beta)(1+r_2)e^{i\omega l/c} \times$$

$$\left. \times \frac{e^{il[\omega(1-\beta)-\nu]/v} - 1}{il[\omega(1-\beta)-\nu]/v} \right\} + \frac{\beta_w}{(1-\beta_w^2) D(\omega)} \left[(\beta\beta_w - 1)r_2 - \right.$$

$$\left. - \beta + \beta_w\right] \left\{ (1+\beta)(1+r_1)\, e^{i\omega l/c} \, \frac{1 - e^{-il[\omega(1+\beta)-\nu]/v}}{il[\omega(1+\beta)-\nu]/v} - \right.$$

$$-(1-\beta)(1-r_1)e^{-i\omega l/c}\ \frac{1-e^{-il[\omega(1-\beta)-\nu]/v}}{il[\omega(1-\beta)-\nu]/v} \quad .$$

A few general properties of the function $\varrho_o(\sigma_k-\sigma_l)$ can now be easily established. Since all the zeros of $D(\omega)$ are in the half-plane $\mathrm{Im}\,\omega > 0$, by choosing the integration path as shown in fig. 1, it is possible to see that

(A-19) $$\varrho_o(\sigma_k-\sigma_l) = 0 , \qquad \text{if } \sigma_k-\sigma_l < -l(1-\beta) .$$

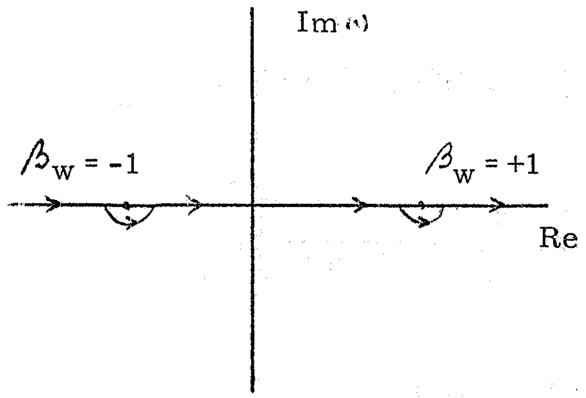

FIG. 1

Excluding the case of a matched line ($r_1 = r_2 = 1$) one has also that $\varrho_o(\sigma_k-\sigma_l)$ is determined only by the zeros of $D(\omega)$ if

(A-20) $$\sigma_k - \sigma_l > l(1-\beta) .$$

Hence in the case of an unmatched electrode and for ultrarelativistic particles one can neglect the interval $-l(1-\beta) \mapsto l(1-\beta)$, and assume that the force is only determined by the zero's of $D(\omega)$.

Calling ω_α the generic solutions of $D(\omega) = 0$ and η_α tha residue of $g_o(\omega)$ in $\omega = \omega_\alpha$, it is then possible to write $g_o(\omega)$ in the form (27), namely

(A-21) $$g_o(\omega) = \sum_\alpha \left\{ \frac{\eta_\alpha}{\omega-\omega_\alpha} - \frac{\eta_\alpha^*}{\omega+\omega_\alpha^*} \right\} .$$

To write (A-21) we used the fact that if ω_α is a zero, also $-\omega_\alpha^*$ is a zero.

The evaluation of the quantities ω_α, η_α will require in general the use of a computer.

Simple formulas for $\varrho_o(\sigma_k-\sigma_l)$ and $g_o(\omega)$ can instead be obtained in the simple cases of $r_1 = r_2 = 1$, or $r_1 = r_2 = 0$, or $r_1 = r_2 \to \infty$.

In the case of matched electrodes we obtain

$$\rho_o(\sigma_k - \sigma_l) = 0 \,, \qquad \text{if } \sigma_k - \sigma_l < -l(1-\beta)$$
$$\qquad\qquad\qquad \text{or if } \sigma_k - \sigma_l > l(1+\beta) \,.$$

In the interval

$$-l(1-\beta) < \sigma_k - \sigma_l < l(1+\beta)$$

one obtains

$$\rho_o(\sigma_k - \sigma_l) = \frac{\pi l}{L} e^{i\nu(\sigma_k - \sigma_l)/(1+\beta)} \times$$

(A-22)
$$\times \left\{ \frac{\beta v}{l} - i\nu\beta \left[1 - \frac{\sigma_k - \sigma_l}{(1+\beta)l} \right] \right\} .$$

If the conditions

$$l \gg \text{bunch length} ,$$

$$\frac{\nu l}{v} \ll 1 ,$$

are satisfied, one can approximate (A-22) with

(A-23)
$$\rho_o(\sigma_k - \sigma_l) \simeq \frac{\pi\beta v}{L} , \qquad \text{for } \sigma_k - \sigma_l > 0 .$$
$$\rho_o(\sigma_k - \sigma_l) = 0 , \qquad \text{for } \sigma_k - \sigma_l < 0 .$$

In the case of grounded ($r_1 = r_2 = 0$) or floating ($r_1 = r_2 \to \infty$) electrodes, one has that the zeros of $D(\omega)$ are

(A-24)
$$\omega_k = k\pi c/l ,$$

with k a positive or negative integer different from zero.

A simple expression for $g_o(\omega)$ is obtained by assuming that

$$k\pi c/l \gg \nu .$$

Then one has, near a zero ω_k, that

(A-25)
$$g_o(\omega) \simeq - \frac{(-1)^k}{\omega - k\pi c/l} \frac{c}{2L} \frac{1-\cos(\nu l/v)}{k\pi} - \frac{(-1)^k}{\omega + k\pi c/l} \frac{c}{2L} \frac{1-\cos(\nu l/v)}{k\pi} ,$$

valid for both the floating or grounded electrodes.

APPENDIX B - Force due to a resonant cavity.

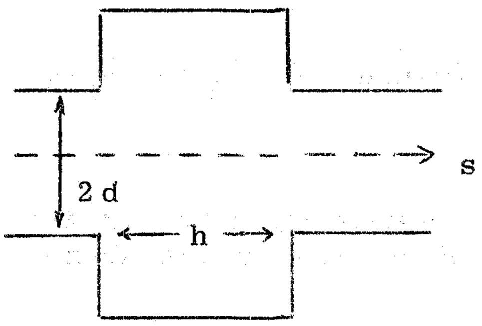

FIG. 2

We consider a cavity as shown in fig. 2, and assume the cavity to resonate on a single mode. Calling ω_r its resonant frequancy, Γ its decay time related to the loss factor Q by $\Gamma = \omega_r/2Q$, and Z_o its shunt impedance, one has that the voltage induced by particle "K" on the cavity can be written, with good approximation, as

(B-1)
$$V_k(t) = \omega_r Z_o \int^t e^{-\Gamma(t-t')} \left\{ \cos\omega_r(t-t') - \frac{1}{2Q} \sin\omega_r(t-t') \right\} J_k(t')\, dt' ,$$

where J_k is the transverse induced current, defined as

(B-2)
$$J_k(t) = \frac{e Z_k(t)}{d} \sum_n \delta(t - nT + \sigma_k(n)) .$$

The quantity $\sigma_k(n)$ in (B-2) represents the time displacement at the n-th revolution, respect to the synchronous particle, and T is the revolution period.

Using the deflection theorem(4) and (B-1), (B-2), we can write the force acting on particle "1" at the m-th revolution as

$$F_{1k}(t = mT - \sigma_1(m)) = \frac{e^2 v \omega_r^2 Z_o}{d^2 h(\omega_r^2 + \Gamma^2)} \times$$

$$\times \sum_n Z_k(nT - \sigma_k(m)) e^{-\Gamma\left[(m-n)T + \sigma_k(n) - \sigma_1(m)\right]} \times$$

(B-3)

$$\times \left\{ \sin \omega_r \left[(m-n)T + \sigma_k(n) - \sigma_1(m)\right] - \frac{1}{4Q^2} \cos \omega_r \times \right.$$

$$\left. \times \left[(m-n)T + \sigma_k(n) - \sigma_1(m)\right]\right\} \theta\left[(m-n)T + \sigma_k(n) - \sigma_1(m)\right],$$

where $\theta(x)$ is the step function.

Considering only the single turn effect, $m = n$, which is the case of interest for the head-tail effect, (B-3) can be simplified to

$$F_{1k}(t) = Z_k(t + \sigma_1 - \sigma_k)\, \varsigma(\sigma_k - \sigma_1),$$

(B-4)
$$\varsigma(\sigma_k - \sigma_1) = \frac{e^2 v \omega_r^2 Z_o}{d^2 h(\omega_r^2 + \Gamma^2)} e^{-\Gamma(\sigma_k - \sigma_1)} \times$$

$$\times \left\{ \sin \omega_r(\sigma_k - \sigma_1) - \frac{1}{4Q^2} \cos \omega_r(\sigma_k - \sigma_1) \right\} \theta(\sigma_k - \sigma_1).$$

This force is applied only when the longitudinal position of particle "1" is in the interval 0-h, modulus one revolution, and so can be written in the general form (3).

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