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P. Di Vecchia and F. Drago : REGGE-POLE FAMILIES AND TOLLER POLES : t = 0

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Regge-Pole Families and Toller Poles: $t=0$†

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We explicitly show, using the factorization theorem, that the $M=0$ and $M=1$ Regge-pole families, deduced from analyticity in unequal-mass scattering processes, are the same as the families generated, in the equal-mass situations, from $M=0$ and $M=1$ Toller poles.

I. INTRODUCTION

During recent years the Regge-pole model has met with considerable success in the phenomenological description of strongly interacting particles, both in connection with their interactions, described by Regge trajectories at negative values of the invariant mass squared, and in connection with their mass-spin classification, described by Regge trajectories at positive values of the mass squared. In some sense this duality is one of the most attractive aspects of the theory.

However, from the oversimplified models used to fit the data in the early days, many theoretical difficulties, connected with the complicated kinematical structure of real particle reactions, had to be overcome. In fact, the interactions that can be studied experimentally always involve particles with spin and, apart from the case of elastic scattering, with different masses.

In the first discussions of the Regge-pole model, the spin complications were considered "inessential" to a better understanding of the theory, and the mass complications were simply ignored.[1] The experience of the following years showed that such ideas were too optimistic and that a detailed study of the kinematical complications was essential. In particular, the study of the problems that arise at the point[2] $t=0$ led to very interesting, and sometimes surprising, results.

In Sec. II, in order to clarify the aim of the present work, we will briefly review some of the previous works on the subject, both analytical and group-theoretical. We will discuss some spin complications that led to the concept of conspiracy, and some mass complications that led to the introduction of the Regge-pole families. The group-theoretical approach to the Regge-pole families is also discussed.

In Sec. III we will clearly establish what our assumptions are: These are essentially the usual S-matrix theory assumptions adapted to the Regge-pole theory, namely, analyticity, crossing symmetry, simplicity, and factorization.

In the subsequent sections we will show how the proper Reggeization of the scattering amplitude naturally leads to the Toller-pole structure of the theory. Since our approach is analytic, we will start from the study of the unequal-mass configurations where the analytic approach works. We will then use the factorization theorem to recover the group-theoretical results in the equal-mass situation.

We will discuss in detail the Toller families with quantum number M equal to zero and 1, and we will restrict ourselves explicitly to configurations in which the equal-mass vertex is provided by an N-$\bar{N}$ system; of course, these are the most interesting situations from the phenomenological point of view. However, our results about the Toller families will be of general validity.

Class-I families will be discussed in Sec. IV, class-II in Sec. V, and class-III in Sec. VI. In Appendix A we give some notations, and in Appendix B a useful identity is proved.

A brief account of our work has already been published[3]; the spinless case has also been discussed independently by Bronzan and Jones.[4]

II. REVIEW OF THE PRESENT SITUATION

Let us first review some spin difficulties which are common to any scattering theory and then the mass difficulties which are peculiar to the Regge-pole theory.[5]

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[1] However, some uneasiness in using the simple-power behavior in backward πN scattering can already be found, for example, in S. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. 126, 2204 (1962).

[2] In the following, we use s as energy variable and t as angular variable. Since we will be interested in the Reggeization of the scattering amplitude, we already use the crossed t-channel terminology.

[3] P. Di Vecchia and F. Drago, Phys. Letters 27B, 387 (1968).

[4] J. B. Bronzan and C. E. Jones, Phys. Rev. Letters 21, 564 (1968).

[5] Or at least of any theory that uses an expansion of the scattering amplitude in terms of the cosine of the crossed-channel scattering angle.

The most difficult problems are, of course, those due to the mixing at the same kinematical point of the two kinds of complications.

We start by briefly discussing some of the properties of the helicity amplitudes that we will need in the following. In fact, the helicity amplitudes not only provide a formalism suitable for a uniform description of arbitrary-spin scattering processes, but also, once formed in the so-called parity-conserving combinations, are particularly suitable for Reggeization.[6] This is the main reason for their popularity in recent years. We will not discuss their merits here, but only some of the difficulties associated with their use.

It was early recognized[6-9] that the parity-conserving helicity amplitudes have kinematical singularities in both the kinematical variables s and t separately. More recently the existence, and the importance, of kinematical constraints among the helicity amplitudes has been rediscovered.[10] In fact, while the existence of this kind of constraint has been known,[11,12] in some particular case, for a long time, their relevance to Regge-pole theory[10,12] has not been fully understood until the present time. These constraints occur at some well-defined kinematical points: at the thresholds, pseudothresholds, and at $t=0$. However, they are very different in nature; in fact, while the threshold (and pseudothreshold) constraints are merely an expression of threshold properties, the $t=0$ constraints reflect a new symmetry of the amplitude at this point, and are particularly important from the point of view of the Regge theory in that they gave rise to the concept of "conspiracy" between different trajectories.[11,13] In the following sections the $t=0$ constraints (Appendix A) will play a central role.

Moreover, other difficulties, peculiar to the Regge-pole model, arise at the point $t=0$ when different-mass particles are involved. In fact, the contribution of a Regge pole to the Sommerfeld-Watson transformed scattering amplitude is introduced through the use of the variable $\cos\theta_t$: This variable, in the unequal-mass case, is bounded by unity for all s when t is in a forward cone, and therefore the conventional Regge representation does not furnish an asymptotic limit in this region. Indeed, any representation $A(s,t)=f(\cos\theta_t,t)$ is suspicious[14] at $t=0$ because the transformation of variables is singular there.

This difficulty can be seen in an alternative way: If one considers the contribution of a single Regge pole for t outside the forward cone, where $\cos\theta_t\to\infty$ as $s\to\infty$, does the usual expansion in powers of s, and then tries to extend the result in the forward cone, one finds that the expression so obtained is singular at $t=0$. The simplest way to avoid these unwanted singularities is to assume[14] that Regge trajectories really occur in families, with definite requirements on the spacing of the members of a family and on the behavior of the residue functions at $t=0$.

However, factorization implies that such families have to be exchanged also in the equal-mass configurations, where analyticity requirements do not impose any additional exchange on a single Regge pole.

On the other hand, the $t=0$ properties of an equal-mass scattering amplitude have been investigated by an altogether different method by various authors[15-19] and similar results were derived. In fact, in the pairwise equal-mass case, at $t=0$ the four-momentum P_μ exchange in the crossed t-channel vanishes identically. Now it can be shown that the scattering amplitude at $P_\mu=0$ is invariant under the little group of the general Poincaré group belonging to $P_\mu=0$, which is isomorphic to the homogeneous Lorentz group $O(3,1)$. Toller[15-17] Reggeized expansions of amplitudes in terms of the representation of the group $O(3,1)$; the simpler compact group $O(4)$ has been used later by Freedman and Wang[18] and by Domokos.[19] It has been found that a Toller pole, that is, a pole in the "four-dimensional angular momentum plane," leads to an infinite family of Regge poles, with defined relations between the trajectories and the residue functions at $t=0$. The families of Regge poles so generated are characterized, apart from the internal quantum numbers, by a Lorentz quantum number M which, for boson trajectories, can take all the integer values.

However, the formalism sketched above applies rigorously only exactly at the point $t=0$ and for equal-mass scattering. Formalisms which permit these limitations to be overcome have been proposed by various authors.[20-22] The most general work in this direction has been done by Cosenza, Sciarrino, and Toller,[23-25] by a generalization of the Lorentz-group formalism. In doing this, however, they were forced to introduce some new and quite strong assumptions.

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[6] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. 133, B145 (1964).
[7] Y. Hara, Phys. Rev. 136, B507 (1964).
[8] L. L. C. Wang, Phys. Rev. 142, 1187 (1966).
[9] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N. Y.) 46, 239 (1968).
[10] M. Gell-Mann and E. Leader, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967).
[11] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. 120, 2250 (1960).
[12] D. V. Volkov and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. 44, 1068 (1963) [English transl.: Soviet Phys.—JETP 17, 720 (1963)].
[13] E. Leader, Phys. Rev. 166, 1599 (1968).
[14] D. Z. Freedman and J. M. Wang, Phys. Rev. 153, 1596 (1967).
[15] M. Toller, Nuovo Cimento 37, 631 (1965); University of Rome Report Nos. 76, 1965, and 84, 1966 (unpublished).
[16] A. Sciarrino and M. Toller, J. Math. Phys. 7, 1670 (1967).
[17] M. Toller, Nuovo Cimento 53A, 671 (1968); 54A, 295 (1968).
[18] D. Z. Freedman and J. M. Wang, Phys. Rev. 160, 1560 (1967).
[19] G. Domokos, Phys. Letters 24B, 293 (1967); Phys. Rev. 159, 1387 (1967).
[20] R. F. Sawyer, Phys. Rev. Letters 18, 1212 (1967); 19, 137 (1967).
[21] R. Delbourgo, A. Salam, and J. Strathdee, Phys. Letters 25B, 230 (1967); Phys. Rev. 164, 1981 (1967).
[22] G. Domokos and G. L. Tindle, Phys. Rev. 165, 1906 (1968).
[23] G. Cosenza, A. Sciarrino, and M. Toller, University of Rome Report No. 158, 1968 (unpublished).
[24] G. Cosenza, A. Sciarrino, and M. Toller, in Proceedings of the Topical Conference on the High-Energy Collisions of Hadrons, CERN, Geneva, 1968, Vol. II, p. 45 (unpublished).
[25] G. Cosenza, A. Sciarrino, and M. Toller, Phys. Letters 27B, 398 (1968).

We are thus faced with the following situation: The analytic approach works in the unequal-mass case, where it requires the existence of an infinite family of daughter trajectories, but fails to give any information in equal-mass case. On the other hand, the group-theoretical approach works well in the equal-mass case, but its extension to the unequal-mass problems meets many difficulties.

In this situation it is highly desirable to bridge the gap between the analytic and the group-theoretical approach and to study whether the analyticity and the group-theoretical families are really the same.

Some results in this direction have already been obtained[26] by showing, under the same assumptions listed in Sec. III, that a classification of the "analytic families" is possible, and in some sense equivalent to the group-theoretical one; moreover, the $t=0$ behavior of the factorized Regge-pole residue functions, satisfying all the kinematical constraints, has been found to be the same of the one derived in Ref. 25 by group-theoretical methods. However, the discussion of Ref. 26 does not eliminate the possibility that an "analyticity family" represents a string of integer-spaced Toller poles rather than a single Toller pole.

The aim of the present paper is to show that the daughters deduced from analyticity in unequal-mass scattering add up to give a single Toller pole in the equal-mass case.

III. ASSUMPTIONS

Our assumptions are the usual ones of S-matrix theory, adapted to the Regge-pole theory and supplemented by a simplicity requirement which has been tacitly used in all the previous analytical works.

We use the following assumptions: (a) analyticity, (b) simplicity, (c) crossing symmetry, and (d) factorization. Their meanings and implications will be discussed below.

Assumption (a) is the fundamental one of S-matrix theory: A scattering amplitude, once its kinematical singularities have been removed, should have only the singularities required by the Mandelstam representation.

In the unequal-unequal (UU) and in the equal-unequal (EU) mass problems, the contribution of a single Regge pole to the scattering amplitude is not an analytic function at $t=0$. This, together with our assumption (b), implies that poles in the angular-momentum plane cannot occur alone, but for every parent pole an infinite family of daughter poles satisfying the condition $\alpha_n(0)=\alpha_0(0)-n$ is needed.[14]

The way in which the $t=0$ singularities of the parent pole are cancelled by the daughter poles is also fixed by our assumption (b). This is an essentially dynamical assumption and it establishes the relations between the parent and the daughter trajectories and residue functions. The simplicity assumption amounts to saying that only the minimum number of daughter trajectories, required in order to save the analyticity of the full amplitude, is present. Also, this assumption is in the spirit of the analytic S-matrix theory.

Moreover, the requirements of analyticity and crossing symmetry impose some constraints on the helicity amplitudes (Appendix A) that must be satisfied by the Regge-pole families.

Assumption (d) is necessary in order to connect the various mass configurations. It amounts to saying that the Regge-pole residue functions factorize, as a consequence of unitarity, not only at the particle pole, but also all along the trajectory.[27–29]

IV. CLASS-I REGGE-POLE FAMILY

This family is defined by the quantum numbers $M=0$, $\sigma=+1$. Its parent Regge trajectory has $P=\xi=\tau$. The quantity ξ is defined in terms of the internal quantum numbers I and G by $\xi=G(-1)^I$; P is the parity, τ is the signature, and $\sigma=P\tau$. We will first discuss the UU case, then the EU case, and finally the nucleon-nucleon case.

A. Unequal-Unequal Mass Case

We consider the s-channel reaction $1+2\to 3+4$ and the t-channel reaction $\bar{4}+2\to 3+\bar{1}$. The notations used are explained in Appendix A. In order to study the class-I family in the UU case, we have to study[26] the amplitude $\bar{f}_{0,0}^{(+)}$, which, for simplicity in writing, we call $f(s,t)$.

The contribution of a single Regge pole to the amplitude $f(s,t)$ is given by

$$f^{1\,\mathrm{pole}}(s,t)=-\frac{2\alpha(t)+1}{\sin\pi\alpha(t)}(1+\tau e^{-i\pi\alpha(t)})\mathcal{P}_\alpha(-z)\beta(t)\,,\quad (4.1)$$

where

$$\mathcal{P}_\alpha(-z)=-\frac{\tan\pi\alpha}{\pi}Q_{-\alpha-1}(-z)=\frac{\Gamma(\alpha+\frac{1}{2})2^\alpha}{\Gamma(\alpha+1)\sqrt{\pi}}(-z)^\alpha$$
$$\times F(-\tfrac{1}{2}\alpha;\tfrac{1}{2}-\tfrac{1}{2}\alpha;\tfrac{1}{2}-\alpha;1/z^2)$$
$$=\frac{\Gamma(\alpha+\frac{1}{2})2^\alpha}{\Gamma(\alpha+1)\sqrt{\pi}}(-z)^\alpha\sum_{k=0}^{\infty}a_k(\alpha)z^{-2k},$$
$$z=\cos\theta_t=\frac{s}{2q_tq_0}+\frac{1}{4q_tq_0t}\times\Big[t\Big(t-\sum_{i=1}^{4}m_i^2\Big)+(m_1^2-m_3^2)(m_2^2-m_4^2)\Big],\quad (4.2)$$
$$a_k(\alpha)=\frac{\Gamma(-\frac{1}{2}\alpha+k)\Gamma(-\frac{1}{2}\alpha+\frac{1}{2}+k)\Gamma(\frac{1}{2}-\alpha)}{\Gamma(-\frac{1}{2}\alpha)\Gamma(-\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\frac{1}{2}-\alpha+k)k!},$$

[26] P. Di Vecchia, F. Drago, and M. L. Paciello, Nuovo Cimento 56, 1185 (1968) and (to be published).

[27] M. Gell-Mann, Phys. Rev. Letters 8, 263 (1962).

[28] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters 8, 343 (1962).

[29] F. Arbab and J. D. Jackson, Phys. Rev. 176, 1796 (1968).

and q_i and q_0 are the initial and final momenta in the c.m. frame of the t channel. We introduce a reduced residue function $\gamma(t)$ through the relation[30]

$$\beta(t)=\gamma(t)(-2q_iq_0/s_0)^{\alpha(t)}. \quad (4.3)$$

The function $\gamma(t)$ is supposed to be analytic at $t=0$,[14] and s_0 is a scale factor. Furthermore, we define

$$\frac{2zq_iq_0}{s_0}=\frac{s}{s_0}+\frac{B(t)}{t}$$

and

$$\frac{2q_iq_0}{s_0}=\frac{D(t)}{t},$$

with

$$B(t)=\frac{1}{2s_0}[t(t-\sum_{i=1}^{4} m_i^2)+(m_1^2-m_3^2)(m_2^2-m_4^2)]$$

and

$$D(t)=\{[t-(m_1+m_3)^2][t-(m_1-m_3)^2][t-(m_2+m_4)^2] \times[t-(m_2-m_4)^2]/4s_0^2\}^{1/2}.$$

With these definitions, one has

$$f^{1\,\text{pole}}(s,t)=g(\alpha)\gamma(t)\sum_{k=0}^{\infty} a_k(\alpha)\left(\frac{s}{s_0}+\frac{B(t)}{t}\right)^{\alpha(t)-2k} \times\left(\frac{D(t)}{t}\right)^{2k}, \quad (4.4)$$

where

$$g(\alpha)=-\frac{2\alpha+1}{\sin\pi\alpha}(1+\tau e^{-i\pi\alpha})\frac{\Gamma(\alpha+\frac{1}{2})2^{\alpha}}{\Gamma(\alpha+1)\sqrt{\pi}}.$$

From (4.4) we see that the contribution of a single Regge pole is not an analytic function at $t=0$. We therefore introduce a family of daughter trajectories $\alpha_n(t)$, with $\alpha_n(0)=\alpha_0(0)-n$, and whose residue functions are singular at $t=0$ and are written as $\gamma_n(t)\times(t_0/t_0)^n$, where t_0 is a scale factor. Our assumption (b) of Sec. III determines the most singular part of the daughters residue functions in terms of the parent residue. The contribution of a whole family of Regge trajectories is therefore given by

$$f(s,t)=\sum_{n=0}^{\infty} g(\alpha_n)\gamma_n(t)\sum_{k=0}^{\infty} a_k(\alpha_n)\sum_{h=0}^{\infty}\left(\frac{D(t)}{t_0}\right)^{2k}\left(\frac{B(t)}{t_0}\right)^{h} \times\left(\frac{t_0}{t}\right)^{2k+n+h}\frac{\Gamma(\alpha_n-2k+1)}{h!\,\Gamma(\alpha_n-2k+1-h)}\left(\frac{s}{s_0}\right)^{\alpha_n-2k-h}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\left(\frac{t_0}{t}\right)^{2k+n+h} a_{h;k;n}(s,t). \quad (4.5)$$

[30] In the general case of $\lambda\neq\mu\neq0$, the reduced residue function is defined by

$$\beta_{\mu,\lambda}^{(\pm)}(t)=K_{\mu,\lambda}^{(\pm)}(t)\gamma_{\mu,\lambda}^{(\pm)}(t)\left(-\frac{2q_iq_0}{s_0}\right)^{\alpha^{(\pm)}(t)-N},$$

where $N=\max\{|\lambda|,|\mu|\}$ and $K_{\mu,\lambda}^{(\pm)}(t)$ is the kinematical Wang factor (Ref. 8) (see Appendix A).

Assuming that the above series can be summed in any order, and defining

$$a_{m-2k-n;k;n}(s,t)=\left(\frac{B(t)}{t_0}\right)^{m}\left(\frac{s}{s_0}\right)^{\alpha_n(t)+n-m} d_{n,k}^{m}(t),$$

with

$$d_{n,k}^{m}(t)=g(\alpha_n)\gamma_n(t)a_k(\alpha_n)\left(\frac{D(t)}{B(t)}\right)^{2k}\left(\frac{B(t)}{t_0}\right)^{-n} \times\frac{\Gamma(\alpha_n-2k+1)}{(m-n-2k)!\,\Gamma(\alpha_n+1-m+n)},$$

one has

$$f(s,t)=\sum_{m=0}^{\infty}\left(\frac{t_0}{t}\right)^{m}\left(\frac{B(t)}{t_0}\right)^{m}\left(\frac{s}{s_0}\right)^{-m} \times\sum_{k=0}^{N(m)}\sum_{n=0}^{m-2k} d_{n,k}^{m}(t)\left(\frac{s}{s_0}\right)^{\alpha_n(t)+n},$$

where

$$N(m)=\tfrac{1}{2}m, \quad \text{if} \quad (-1)^m=1$$
$$=\tfrac{1}{2}(m-1), \quad \text{if} \quad (-1)^m=-1.$$

Since the function $f(s,t)$ has to be analytic at $t=0$, for any s, we must require that

$$\sum_{k=0}^{N(m)}\sum_{n=0}^{m-2k} d_{n,k}^{m}(t)\left(\frac{s}{s_0}\right)^{\alpha_n(t)+n}=O(t^m) \quad \text{for} \quad m\geq1. \quad (4.6)$$

These are the fundamental relations of our approach. From these equations, in fact, not only the quantities $\gamma_n(0)$ can be expressed in terms of $\gamma_0(0)$, but also the behavior of the family for $t\neq0$ can be studied. In this paper we are only concerned with the point $t=0$. We therefore have[31]

$$d_{n,k}^{m}(0)=f_m(\alpha)\zeta^n[2^{n+2k}k!(m-n-2k)! \times\Gamma(\tfrac{1}{2}-\alpha+n+k)]^{-1}, \quad (4.7)$$

with

$$f_m(\alpha)=-\frac{1+\tau e^{-i\pi\alpha}}{\sin2\pi\alpha}\frac{2^{\alpha+1}}{\Gamma(\alpha+1-m)}, \quad (4.8)$$

$$\zeta^n=[2(\alpha-n)+1]\gamma_n(0), \quad (4.9)$$

and the residue functions are given by the system

$$\sum_{k=0}^{N(m)}\sum_{n=0}^{m-2k}\frac{\zeta^n}{2^{n+2k}k!(m-n-2k)!\,\Gamma(\frac{1}{2}-\alpha+n+k)}=0 \quad (4.10)$$

or

$$\sum_{n=0}^{m} I_{mn}\zeta^n=0, \quad (4.11)$$

[31] Here and in the following, α is the intercept of the parent trajectory $\alpha=\alpha_0(0)$. We have also done the following choice for the scale factor t_0: $t_0=D(0)=B(0)=(m_1^2-m_3^2)(m_2^2-m_4^2)/2s_0$.

where

$$I_{mn}=\sum_{k=0}^{N(m-n)}\frac{1}{2^{n+2k}k!(m-n-2k)!\Gamma(\frac{1}{2}-\alpha+n+k)}=\frac{1}{2^{2\alpha+1}\sqrt{\pi}}\frac{1}{(m-n)!}\frac{2^m\Gamma(m-\alpha)}{\Gamma(m+n-2\alpha)}. \quad (4.12)$$

The solution of the system (4.11) is given by

$$\zeta^n=\frac{2(\alpha-n)+1}{2\alpha+1}\frac{(-1)^n}{n!}\frac{\Gamma(n-1-2\alpha)}{\Gamma(-1-2\alpha)}\zeta^{n=0} \quad (4.13)$$

or

$$\gamma_n{}^{(0)}=\frac{(-1)^n}{n!}\frac{\Gamma(n-1-2\alpha)}{\Gamma(-1-2\alpha)}\gamma_0(0). \quad (4.14)$$

In order to show that (4.13) is the solution of the system (4.11), one has to show that

$$\sum_{n=0}^{m}(-1)^n\binom{m}{n}\Big[\prod_{k=0,k\neq n}^{m}(n+k-2\alpha-1)\Big]^{-1}=0.$$

This is explicitly done in Appendix B.

B. Equal-Unequal Mass Case

This case is somewhat simpler than the unequal-unequal one, because of the presence of weaker singularities. This is due to the simpler kinematic: One has $D(t)=[t\tilde{D}(t)]^{1/2}$, with

$$\tilde{D}(t)=(t-4m^2)[t-(m_1+m_3)^2][t-(m_1-m_3)^2]/4s_0{}^2,$$

and $B(t)=t\tilde{B}(t)/s_0$, with $\tilde{B}(t)=\frac{1}{2}(t-\sum m_i{}^2)$.

At the equal-mass vertex the selection rules due to parity and G-parity invariance must be taken into account. In the present case one finds that the parent and the even-daughter trajectories couple to the amplitude $\tilde{f}_{cc,\frac{1}{2}\frac{1}{2}}{}^{(+)}\equiv\tilde{f}_{0,0}{}^{(+)}$, while the odd daughters are decoupled from the N-$\bar{N}$ system.

The contribution of the whole family is therefore given by

$$\tilde{f}_{0,0}{}^{(+)}=\sum_{n=0}^{\infty}g(\alpha_{2n})\gamma_{2n}(t)\sum_{k=0}^{\infty}a_k(\alpha_{2n})\times\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n}(t)-2k}\left(\frac{\tilde{D}(t)}{t_1}\right)^k\left(\frac{t_1}{t}\right)^{n+k}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{t_1}{t}\right)^{n+k}b_{2n;k}(t)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n}(t)-2k}, \quad (4.15)$$

where t_1 is a scale factor and

$$b_{n;k}(t)=g(\alpha_n)\gamma_n(t)a_k(\alpha_n)[\tilde{D}(t)/t_1]^k. \quad (4.16)$$

Assuming that the series (4.15) can be summed in any order, one has

$$\tilde{f}_{0,0}{}^{(+)}=\sum_{m=0}^{\infty}\left(\frac{t_1}{t}\right)^m\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{-2m}\times\sum_{n=0}^{m}b_{2n;m-n}(t)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n}(t)+2n}. \quad (4.17)$$

Therefore, the $t=0$ analyticity conditions turn out to be

$$\sum_{n=0}^{m}b_{2n,m-n}(t)\left[\frac{s+\tilde{B}(t)}{s_0}\right]^{\alpha_{2n}(t)+2n}=O(t^m)\quad\text{for}\quad m\geq1. \quad (4.18)$$

At $t=0$ one has the following system for the residue functions[32]:

$$\sum_{n=0}^{m}\frac{\zeta^{2n}}{(m-n)!\Gamma(\frac{1}{2}-\alpha+m+n)}=0, \quad (4.19)$$

whose solution is

$$\zeta^{2n}=\frac{2(\alpha-2n)+1}{2\alpha+1}\frac{(-1)^n}{n!}\frac{\Gamma(n-\frac{1}{2}-\alpha)}{\Gamma(-\frac{1}{2}-\alpha)}\zeta^{n=0} \quad (4.20)$$

or

$$\gamma_{2n}(0)=\frac{(-1)^n}{n!}\frac{\Gamma(n-\frac{1}{2}-\alpha)}{\Gamma(-\frac{1}{2}-\alpha)}\gamma_0(0), \quad (4.21)$$

as can be seen using the results of Appendix B.

C. Toller Pole

The factorization theorem now provides the bridge necessary in order to study the contribution of the analyticity family to nucleon-nucleon scattering. The factorization theorem can be stated as follows, in terms of the complete residue functions:

$$[\beta_{cc,\frac{1}{2}\frac{1}{2}}{}^{(+)2n}(t)]^2=\beta_{cc,cc}{}^{(+)2n}(t)\beta_{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}{}^{(+)2n}(t). \quad (4.22)$$

Note that the most singular part of the residue functions in the unequal-mass cases that we have obtained previously is enough to determine completely, through (4.22), the finite part at $t=0$ of the residue functions in the equal-mass case. Inserting the expressions (4.14) and (4.21) into (4.22), and remembering the definition of reduced residue functions and the particular choice of the scale factors t_0 and t_1 done in Secs. IV A and IV B, we get, after some rearrangements,

$$\beta_{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}{}^{(+)2n}(0)=\frac{(2n)!}{2^{2n}(n!)^2}\frac{\Gamma(\alpha+1-n)}{\Gamma(\alpha+1)}\times\frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma(\alpha+\frac{3}{2}-n)}\beta_{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}{}^{(+)n=0}(0). \quad (4.23)$$

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[32] The choice $t_1=\tilde{D}(0)=-(m^2/s_0)[(m_1{}^2-m_3{}^2)^2/s_0]$ has been done.

The same result applies to the spinless case.[3,4] We will now compare the expression (4.23) with the results of the group-theoretical approach.

In the simpler $O(4)$ formulation,[18] one finds the following expression for the residue functions of the Regge-pole family generated by a single Toller pole:

$$\beta_{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}^{(+)2n}(0)=\frac{R}{\pi^2}\frac{(\alpha+1)^2}{2(\alpha-2n)+1}|d_{\alpha-2n;0;0}^{\alpha,0}(\tfrac{1}{2}\pi)|^2, \quad (4.24)$$

where R is the residue of the Toller pole and

$$d_{\alpha-2n;0;0}^{\alpha,0}(\tfrac{1}{2}\pi)=\left(\frac{[2(\alpha-2n)+1](2n)!}{(\alpha+1)\Gamma(2\alpha+2-2n)}\right)^{1/2}(2i)^{\alpha-2n}\times(-1)^n\frac{\Gamma(1+\alpha-n)}{\Gamma(1+n)}. \quad (4.25)$$

It is easily seen that the expressions (4.23) and (4.24) are the same. This proves that class-I "analyticity families" are indeed the same as the class-I "group-theoretical" families.

V. CLASS-II REGGE-POLE FAMILY

This family is defined by the quantum numbers $M=0$, $\sigma=-1$. For n even, one has

$$\tau_n=-P_n=-\xi=\tau, \quad (5.1)$$

and for n odd

$$\tau_n=-P_n=\xi=-\tau, \quad (5.2)$$

where ξ is common to all the members of the family and τ is the signature of the parent pole.

The discussion of the amplitude $\tilde{f}_{cc;cc}^{(-)}$ in the UU case goes on in exactly the same way as that in Sec. IV A and we do not repeat it here. In the EU case, however, some new spin complications appear.

A. Equal-Unequal Mass Case

Because of the selection rules at the nucleon vertex, one finds that the parent and the even daughters contribute to the amplitude $\tilde{f}_{cc,\frac{1}{2}-\frac{1}{2}}^{(-)}\equiv\tilde{f}_{0,1}^{(-)}$ (in the λ, μ notation: Appendix A), while the odd daughters contribute $\tilde{f}_{cc,\frac{1}{2}\frac{1}{2}}^{(-)}\equiv\tilde{f}_{0,0}^{(-)}$. The contribution of a single pole to the above amplitudes is given by

$$\tilde{f}_{0,0}^{(-)1\,\text{pole}}=-\frac{2\alpha(t)+1}{\sin\pi\alpha(t)}\mathcal{P}_\alpha(-z)\beta_{00}(t),$$

$$\tilde{f}_{0,1}^{(-)1\,\text{pole}}=-\frac{2\alpha(t)+1}{\sin\pi\alpha(t)}\mathcal{P}_\alpha'(-z)\frac{\beta_{01}(t)}{\{\alpha(t)[\alpha(t)+1]\}^{1/2}}.$$

The contribution of the whole family is therefore given by

$$\tilde{f}_{0,0}^{(-)}=\sum_{n=0}^{\infty}g(\alpha_{2n+1})\gamma_{0,0}^{2n+1}(t)\sum_{k=0}^{\infty}a_k(\alpha_{2n+1})\times\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n+1}(t)-2k}\left(\frac{\tilde{D}(t)}{t_1}\right)^k\left(\frac{t_1}{t}\right)^{n+k}, \quad (5.3)$$

$$\tilde{f}_{0,1}^{(-)}=\sum_{n=0}^{\infty}g(\alpha_{2n})\frac{\gamma_{0,1}^{2n}(t)}{[\alpha_{2n}(\alpha_{2n}+1)]^{1/2}}\times\sum_{k=0}^{\infty}a_k(\alpha_{2n})[\alpha_{2n}(t)-2k]\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n}-2k-1}\times\left(\frac{\tilde{D}(t)}{t_1}\right)^k\left(\frac{t_1}{t}\right)^{n+k}. \quad (5.4)$$

Assuming, as usual, that the series (5.3) and (5.4) can be summed in any order, we have

$$\tilde{f}_{0,0}^{(-)}=\sum_{m=0}^{\infty}\left(\frac{t_1}{t}\right)^m\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{-2m}\times\sum_{n=0}^{m}b_{2n+1;m-n}(t)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n+1}(t)+2n}, \quad (5.5)$$

$$\tilde{f}_{0,1}^{(-)}=\sum_{n=0}^{\infty}\left(\frac{t_1}{t}\right)^m\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{-2m}\times\sum_{n=0}^{m}b_{2n;m-n}(t)\frac{\alpha_{2n}(t)-2(m-n)}{[\alpha_{2n}(\alpha_{2n}+1)]^{1/2}}\times\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n}(t)+2n-1}, \quad (5.6)$$

where $b_{h;k}^{(t)}$ is defined by (4.16) with the appropriate functions. The $t=0$ analyticity conditions are therefore given by

$$\sum_{n=0}^{m}b_{2n+1;m-n}(t)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n+1}(t)+2n}=O(t^m),\quad m\geq1 \quad (5.7)$$

$$\sum_{n=0}^{m}b_{2n;m-n}(t)\frac{\alpha_{2n}(t)-2(m-n)}{[\alpha_{2n}(\alpha_{2n}+1)]^{1/2}}\times\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n}(t)+2n-1}=O(t^m),\quad m\geq1. \quad (5.8)$$

These conditions are to be satisfied together with the constraint (A5),

$$i\tilde{f}_{0,1}^{(-)}-\tilde{f}_{0,0}^{(-)}=O(t),$$

which requires

$$\sum_{n=0}^{m}\left[ib_{2n;m-n}(t)\frac{\alpha_{2n}-2(m-n)}{[\alpha_{2n}(\alpha_{2n}+1)]^{1/2}}\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n}(t)+2n-1}-b_{2n+1;m-n}(t)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n+1}(t)+2n-1}\right]=O(t^{m+1}),\quad m\geq 0. \quad (5.9)$$

At $t=0$, the relations (5.7)-(5.9) take the simpler form

$$\sum_{n=0}^{m}\zeta_{0,0}{}^{2n+1}[(m-n)!\Gamma(\tfrac{3}{2}-\alpha+m+n)]^{-1}=0,\quad m\geq 1 \quad (5.7')$$

$$\sum_{n=0}^{m}\zeta_{0,1}{}^{2n}\{(m-n)!\Gamma(\tfrac{1}{2}-\alpha+m+n)\times[(\alpha-2n)(\alpha-2n+1)]^{1/2}\}^{-1}=0,\quad m\geq 1 \quad (5.8')$$

$$\sum_{n=0}^{m}\left(i\frac{\zeta_{0,1}{}^{2n}}{[(\alpha-2n)(\alpha-2n+1)]^{1/2}(m-n)!\Gamma(\tfrac{1}{2}-\alpha+m+n)}-\frac{1}{2}\frac{\zeta_{0,0}{}^{2n+1}}{(m-n)!\Gamma(\tfrac{3}{2}-\alpha+m+n)}\right)=0,\quad m\geq 0. \quad (5.9')$$

The constraint equation (5.9′) is automatically satisfied for $m\geq 1$ once the analyticity conditions (5.7′) and (5.8′) are taken into account. However, for $m=0$ it gives an additional relation between the parent and first daughter residue functions:

$$\zeta_{0,0}{}^{n=1}=-i\frac{2\alpha-1}{[\alpha(\alpha+1)]^{1/2}}\zeta_{0,1}{}^{n=0} \quad (5.10)$$

or, equivalently,

$$\gamma_{0,0}{}^{n=1}(0)=-i\frac{2\alpha+1}{[\alpha(\alpha+1)]^{1/2}}\gamma_{0,1}{}^{n=0}(0).$$

This relation, which connects the residue functions of Regge trajectories of angular momentum spaced by a unity, opposite signature and parity, but same charge conjugation, is the result of having satisfied the constraint (A5) in the simplest, nontrivial way, namely, by a daughterlike conspiracy.

The solutions of the systems (5.7′) and (5.8′) turn out to be

$$\zeta_{0,1}{}^{2n}=\frac{2(\alpha-2n)+1}{2\alpha+1}\frac{(-1)^n}{n!}\left(\frac{(\alpha-2n)(\alpha-2n+1)}{\alpha(\alpha+1)}\right)^{1/2}\times\frac{\Gamma(n-\frac{1}{2}-\alpha)}{\Gamma(-\frac{1}{2}-\alpha)}\zeta_{0,1}{}^{n=0}, \quad (5.11)$$

$$\zeta_{0,0}{}^{2n+1}=\frac{2(\alpha-2n)-1}{2\alpha-1}\frac{(-1)^n}{n!}\frac{\Gamma(n+\frac{1}{2}-\alpha)}{\Gamma(\frac{1}{2}-\alpha)}\zeta_{0,0}{}^{n=1}, \quad (5.12)$$

or

$$\gamma_{0,1}{}^{2n}(0)=\frac{(-1)^n}{n!}\left(\frac{(\alpha-2n)(\alpha-2n+1)}{\alpha(\alpha+1)}\right)^{1/2}\times\frac{\Gamma(n-\frac{1}{2}-\alpha)}{\Gamma(-\frac{1}{2}-\alpha)}\gamma_{0,1}{}^{n=0}(0), \quad (5.11')$$

$$\gamma_{0,0}{}^{2n+1}(0)=\frac{(-1)^n}{n!}\frac{\Gamma(n+\frac{1}{2}-\alpha)}{\Gamma(\frac{1}{2}-\alpha)}\gamma_{0,0}{}^{n=1}(0). \quad (5.12')$$

These relations are to be considered together with (5.10′). Using these results and those of Sec. IV A, we are now in a position to reconstruct explicitly the class-II Toller pole in nucleon-nucleon scattering.

B. Toller Pole

Using the factorization theorem, we get for the complete residue functions in nucleon-nucleon scattering

$$\beta_{\frac{1}{2}-\frac{1}{2},\frac{1}{2}-\frac{1}{2}}{}^{2n}(0)=\frac{(2n)!}{2^{2n}(n!)^2}\frac{(\alpha-2n)(\alpha-2n+1)}{\alpha(\alpha+1)}\frac{\Gamma(-\alpha)}{\Gamma(n-\alpha)}\times\frac{\Gamma(n-\frac{1}{2}-\alpha)}{\Gamma(-\frac{1}{2}-\alpha)}\beta_{\frac{1}{2}-\frac{1}{2},\frac{1}{2}-\frac{1}{2}}{}^{n=0}(0), \quad (5.13)$$

$$\beta_{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}{}^{2n+1}(0)=\frac{(2n+1)!}{2^{2n}(n!)^2}\frac{\Gamma(-\alpha)}{\Gamma(n-\alpha)}\times\frac{\Gamma(n+\frac{1}{2}-\alpha)}{\Gamma(\frac{1}{2}-\alpha)}\beta_{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}{}^{n=1}(0), \quad (5.14)$$

$$\beta_{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}{}^{n=1}(0)=\frac{2\alpha+1}{\alpha(\alpha+1)}\beta_{\frac{1}{2}-\frac{1}{2},\frac{1}{2}-\frac{1}{2}}{}^{n=0}(0). \quad (5.15)$$

Owing to (5.15), the Volkov-Gribov constraint (Appendix A) is satisfied by a daughterlike conspiracy. We stress that we have not imposed this constraint, but we have found that our results satisfy it.

The results of the $O(4)$ approach are the following:

$$\beta_{\frac{1}{2}-\frac{1}{2},\frac{1}{2}-\frac{1}{2}}{}^{2n}(0)=\frac{2R}{3\pi^2}\frac{(\alpha+1)^2}{2(\alpha-2n)+1}\times|d_{\alpha-2n;1;1}{}^{\alpha,0}(\tfrac{1}{2}\pi)|^2, \quad (5.16)$$

$$\beta_{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}{}^{2n+1}(0)=\frac{R}{3\pi^2}\frac{(\alpha+1)^2}{2(\alpha-2n)-1}\times|d_{\alpha-2n-1;1;0}{}^{\alpha,0}(\tfrac{1}{2}\pi)|^2, \quad (5.17)$$

where R is the residue of the Toller pole and

$$d_{\alpha-2n;1;1}{}^{\alpha,0}(\tfrac{1}{2}\pi)=i\left(\frac{3[2(\alpha-2n)+1](2n)!(\alpha-2n)(\alpha-2n+1)}{2\alpha(\alpha+1)(\alpha+2)\Gamma(2\alpha-2n+2)}\right)^{1/2}(2i)^{\alpha-2n}(-1)^n\frac{\Gamma(1+\alpha-n)}{\Gamma(1+n)}, \tag{5.18}$$

$$d_{\alpha-2n-1;1;0}{}^{\alpha,0}(\tfrac{1}{2}\pi)=-\left(\frac{3[2(\alpha-2n)-1](2n+1)!}{\alpha(\alpha+1)(\alpha+2)\Gamma(2\alpha-2n+1)}\right)^{1/2}(2i)^{\alpha-2n}(-1)^n\frac{\Gamma(1+\alpha-n)}{\Gamma(n+1)}. \tag{5.19}$$

It is easily seen that also in this case the analytic and group-theoretical families are the same.

VI. CLASS-III REGGE-POLE FAMILIES

The discussion of this family is much more involved, essentially because σ is no longer diagonal with M; here for the first time the parity-doubling phenomenon appears. This means that every family contains, in fact, two subfamilies of Regge poles with all the same quantum numbers, but with opposite parity [and with the same α at $t=0$: $\alpha=\alpha^+(0)=\alpha^-(0)$]. The complete assignment of quantum numbers is the following:

$$\begin{aligned}
\sigma&=+1, & n&\text{ even:} & \tau_n&=P_n=\xi=\tau\\
&=+1, & n&\text{ odd:} & \tau_n&=P_n=-\xi=-\tau\\
\sigma&=-1, & n&\text{ even:} & \tau_n&=-P_n=+\xi=+\tau\\
&=-1, & n&\text{ odd:} & \tau_n&=-P_n=-\xi=-\tau.
\end{aligned}$$

We will first discuss the UU case, then the EU case, and finally the nucleon-nucleon case.

A. Unequal-Unequal Mass Case

We have to study the amplitudes $\tilde{f}_{1,1}{}^{(\pm)}$. However, since these amplitudes are dominated by the exchange of definite parity only asymptotically, we must study them simultaneously. The contribution of a parity doublet to the amplitudes in discussion is given by

$$\tilde{f}_{1,1}{}^{(\pm)\text{ doublet}}=\frac{2\alpha^\pm+1}{\sin\pi\alpha^\pm}(1+\tau e^{-i\pi\alpha^\pm})E_{11}{}^{\alpha\pm+}(-z)\beta_{11}{}^{(\pm)}(t)$$

$$-\frac{2\alpha^\mp+1}{\sin\pi\alpha^\mp}(1+\tau e^{-i\pi\alpha^\mp})E_{11}{}^{\alpha\mp-}(-z)\beta_{11}{}^{\mp}(t), \tag{6.1}$$

where

$$E_{1,1}{}^{\alpha+}(z)=[\alpha(\alpha+1)]^{-1}[\mathcal{P}_\alpha'(z)+z\mathcal{P}_\alpha''(z)], \tag{6.2}$$

$$E_{1,1}{}^{\alpha-}(z)=-[\alpha(\alpha+1)]^{-1}\mathcal{P}_\alpha''(z). \tag{6.3}$$

Introducing the reduced residue functions, since[20] $K_{1,1}{}^{(+)}(t)=K_{1,1}{}^{(-)}(t)=t_0/t$, we have, using the notation of Sec. IV,

$$\tilde{f}_{1,1}{}^{(\pm)\text{ doublet}}=\tilde{g}(\alpha^+)\gamma_{1,1}{}^+(t)\sum_{k=0}^{\infty}[\alpha^+(t)-2k]^2a_k(\alpha^+)\left(\frac{s}{s_0}+\frac{B(t)}{t}\right)^{\alpha^+-2k-1}\left(\frac{D(t)}{t}\right)^{2k}-\tilde{g}(\alpha^-)\gamma_{1,1}{}^-(t)\frac{D(t)}{t}$$

$$\times\sum_{k=0}^{\infty}(\alpha^--2k)(\alpha^--2k-1)a_k(\alpha^-)\left(\frac{s}{s_0}+\frac{B(t)}{t}\right)^{\alpha^--2k-2}\left(\frac{D(t)}{t}\right)^{2k}, \tag{6.4}$$

where $\tilde{g}(\alpha^\pm)=-g(\alpha^\pm)/[\alpha^\pm(\alpha^\pm+1)]$, and a similar expression for $\tilde{f}_{1,1}{}^{(-)}$ with the interchange $(+)\leftrightarrow(-)$.

In Ref. 26 we noted that when $M=1$, the parity doublet is necessary in order to satisfy the constraint equations in any mass configuration. Now we want to stress that the parity doubling is also required in order to have an analytic contribution, at $t=0$, to the amplitudes $\tilde{f}_{1,1}{}^{(\pm)}$, as can be seen from an inspection of (6.4).

Introducing the daughter trajectories, with the appropriate singularities factored out from the residue functions (same definition as in Sec. IV), we have

$$\tilde{f}_{1,1}{}^{(+)}=\sum_{n=0}^{\infty}\tilde{g}(\alpha_n{}^+)\gamma_{1,1}{}^{+n}(t)\left(\frac{t_0}{t}\right)^n\sum_{k=0}^{\infty}(\alpha_n{}^+-2k)^2a_k(\alpha_n{}^+)$$

$$\times\left(\frac{D(t)}{t_0}\right)^{2k}\left(\frac{t_0}{t}\right)^{2k}\sum_{h=0}^{\infty}\frac{\Gamma(\alpha_n{}^+-2k)}{h!\,\Gamma(\alpha_n{}^+-2k-h)}\left(\frac{B(t)}{t}\right)^h\left(\frac{s}{s_0}\right)^{\alpha_n{}^+-2k-1-h}-\sum_{n=0}^{\infty}\tilde{g}(\alpha_n{}^-)\gamma_{1,1}{}^{-n}(t)\left(\frac{t_0}{t}\right)^{n+1}$$

$$\times\sum_{k=0}^{\infty}(\alpha_{2n}{}^--2k)(\alpha_{2n}{}^--2k-1)a_k(\alpha_n{}^-)\left(\frac{D(t)}{t_0}\right)^{2k+1}\left(\frac{t_0}{t}\right)^{2k}\sum_{h=0}^{\infty}\frac{\Gamma(\alpha_n{}^--2k-1)}{h!\,\Gamma(\alpha_n{}^--2k-h-1)}\left(\frac{B(t)}{t}\right)^h\left(\frac{s}{s_0}\right)^{\alpha_n{}^--2k-\cdots}$$

and a similar expression for $\tilde{f}_{1,1}{}^{(-)}$.

After some manipulations, we arrive at the following expressions:

$$\tilde{f}_{1,1}^{(+)}=\sum_{m=0}^{\infty}\left(\frac{t_0}{t}\right)^m\left(\frac{B(t)}{t_0}\right)^m\left(\frac{s}{s_0}\right)^{-m-1}\sum_{k=0}^{N(m)}\sum_{n=0}^{m-2k}\tilde{d}_{n;k}^{m+}(t)\times\left(\frac{s}{s_0}\right)^{\alpha_n^+(t)+n}-\sum_{m=0}^{\infty}\left(\frac{t_0}{t}\right)^{m+1}\left(\frac{B(t)}{t_0}\right)^{m+1}\left(\frac{s}{s_0}\right)^{-m-2}\times\sum_{k=0}^{N(m)}\sum_{n=0}^{m-2k}\tilde{c}_{n;k}^{m-}(t)\left(\frac{s}{s_0}\right)^{\alpha_n^-(t)+n},\quad(6.5)$$

$$\tilde{f}_{1,1}^{(-)}=\sum_{m=0}^{\infty}\left(\frac{t_0}{t}\right)^m\left(\frac{(B)t}{t_0}\right)^m\left(\frac{s}{s_0}\right)^{-m-1}\sum_{k=0}^{N(m)}\sum_{n=0}^{m-2k}\tilde{d}_{n;k}^{m-}(t)\times\left(\frac{s}{s_0}\right)^{\alpha_n^-(t)+n}-\sum_{m=0}^{\infty}\left(\frac{t_0}{t}\right)^{m+1}\left(\frac{B(t)}{t_0}\right)^{m+1}\left(\frac{s}{s_0}\right)^{-m-2}\times\sum_{k=0}^{N(m)}\sum_{n=0}^{m-2}\tilde{c}_{n;k}^{m+}(t)\left(\frac{s}{s_0}\right)^{\alpha_n^+(t)+n},\quad(6.6)$$

where

$$\tilde{c}_{n;k}^{m\pm}(t)=\tilde{g}(\alpha_n^{\pm})\gamma_{1,1}^{\pm n}(t)(\alpha_n^{\pm}-2k)(\alpha_n^{\pm}-2k-1)\times a_k(\alpha_n^{\pm})\left(\frac{D(t)}{B(t)}\right)^{2k+1}\left(\frac{B(t)}{t_0}\right)^{-n}\times\frac{\Gamma(\alpha_n^{\pm}-2k-1)}{(m-2k-n)!\Gamma(\alpha_n^{\pm}-m+n-1)},$$

$$\tilde{d}_{n;k}^{m\pm}(t)=\frac{\alpha_n^{\pm}-2k}{\alpha_n^{\pm}-m+n-1}\frac{B(t)}{D(t)}\tilde{c}_{n;k}^{m\pm}(t).$$

From (6.5) and (6.6) it is easy to extract the following $t=0$ analyticity conditions:

$$\sum_{k=0}^{N(m+1)}\sum_{n=0}^{m-2k+1}\tilde{d}_{n;k}^{(m+1)+}(t)\left(\frac{s}{s_0}\right)^{\alpha_n^++n}-\sum_{k=0}^{N(m)}\sum_{n=0}^{m-2k}\tilde{c}_{n;k}^{m-}(t)\left(\frac{s}{s_0}\right)^{\alpha_n^-+n}=O(t^{m+1}),\quad(6.7)$$

$$\sum_{k=0}^{N(m+1)}\sum_{n=0}^{m+1-2k}\tilde{d}_{n;k}^{(m+1)-}(t)\left(\frac{s}{s_0}\right)^{\alpha_n^-+n}-\sum_{k=0}^{N(m)}\sum_{n=0}^{m-2k}\tilde{c}_{n;k}^{m+}(t)\left(\frac{s}{s_0}\right)^{\alpha_n^++n}=O(t^{m+1}),\quad(6.8)$$

which must hold for any $m\geq0$.

In order to satisfy the analyticity requirements completely, we must also take into account the $t=0$ constraint (Appendix A):

$$\tilde{f}_{1,1}^{(+)}(s,t)+\tilde{f}_{1,1}^{(-)}(s,t)=O(t),$$

which, explicitly written, gives rise to

$$\tilde{d}_{0,0}^{0+}(t)(s/s_0)^{\alpha_0^+(t)-1}+\tilde{d}_{0,0}^{0-}(t)(s/s_0)^{\alpha_0^-(t)-1}=O(t),\quad(6.9)$$

$$\sum_{k=0}^{N(m+1)}\sum_{n=0}^{m+1-2k}\left[\tilde{d}_{n;k}^{(m+1)+}(t)\left(\frac{s}{s_0}\right)^{\alpha_n^++n}+\tilde{d}_{n;k}^{(m+1)-}(t)\left(\frac{s}{s_0}\right)^{\alpha_n^-+n}\right]-\sum_{k=0}^{N(m)}\sum_{n=0}^{m-2k}\left[\tilde{c}_{n;k}^{m+}(t)\left(\frac{s}{s_0}\right)^{\alpha_n^++n}+\tilde{c}_{n,k}^{m-}(t)\left(\frac{s}{s_0}\right)^{\alpha_n^-+n}\right]=O(t^{m+2}).\quad(6.10)$$

We now restrict ourselves explicitly to the point $t=0$. Defining

$$a_{n,k}^m=\frac{\alpha-n-2k}{(\alpha-n)(\alpha-n+1)}\times\frac{1}{2^{2k+n}k!\Gamma(\frac{1}{2}-\alpha+n+k)(m-2k-n)!},$$

$$b_{n,k}^m=\frac{a_{n,k}^m}{\alpha-n-2k},$$

we have from (6.7) and (6.8)

$$\sum_{k=0}^{N(m+1)}\sum_{n=0}^{m+1-2k}a_{n,k}^{m+1}\zeta_{1,1}^{+n}-\sum_{k=0}^{N(m)}\sum_{n=0}^{m-2k}b_{n,k}^m\zeta_{1,1}^{-n}=0,\quad(6.7')$$

$$\sum_{k=0}^{N(m+1)}\sum_{n=0}^{m+1-2k}a_{n,k}^{m+1}\zeta_{1,1}^{-n}-\sum_{k=0}^{N(m)}\sum_{n=0}^{m-2k}b_{n,k}^m\zeta_{1,1}^{+n}=0\quad(6.8')$$

and from (6.9)

$$\zeta_{1,1}^{+n=0}+\zeta_{1,1}^{-n=0}=0,\quad(6.9')$$

while (6.10) does not give additional relations between the residue functions.

It is easily seen that (6.7′), (6.8′), and (6.9′) are equivalent to

$$\sum_{k=0}^{N(m+1)}\sum_{n=0}^{m+1-2k}a_{n,k}^{m+1}\zeta_{1,1}^{+n}-\sum_{k=0}^{N(m)}\sum_{n=0}^{m-2k}b_{n,k}^m\zeta_{1,1}^{-n}=0,\quad(6.11)$$

$$\zeta_{1,1}^{+n}+\zeta_{1,1}^{-n}=0.\quad(6.12)$$

We must therefore look for the solutions of the system

$$\sum_{k=0}^{N(m+1)} \sum_{n=0}^{m+1-2k} a_{n,k}{}^{m+1}\zeta_{1,1}{}^{+n} + \sum_{k=0}^{N(m)} \sum_{n=0}^{m-2k} b_{n,k}{}^{m}\zeta_{1,1}{}^{+n}=0 \quad (6.13)$$

that can be written

$$\zeta_{1,1}{}^{+m+1}a_{m+1;0}{}^{m+1}+\sum_{n=0}^{m} J_{mn}\zeta_{1,1}{}^{+n}=0\,, \quad (6.14)$$

where

$$J_{mn}=\sum_{k=0}^{N(m+1-n)} a_{n,k}{}^{m+1}+\sum_{k=0}^{N(m-n)} b_{n,k}{}^{m}$$

$$=-\frac{1}{2^n(m-n+1)!}\frac{\Gamma(m-\alpha)}{\Gamma(\frac{1}{2}(m+n+1)-\alpha)\Gamma(\frac{1}{2}(m+n)+1-\alpha)}.$$

Since $a_{m+1;0}{}^{m+1}=J_{m,m+1}$, we have the final system

$$\sum_{n=0}^{m+1} J_{mn}\zeta_{1,1}{}^{+n}=0\,, \quad (6.15)$$

which must hold for any $m\geq 0$.

The solution of the system (6.15) can be written

$$\zeta_{1,1}{}^{+n}=\frac{2(\alpha-n)+1}{2\alpha+1}\frac{(-1)^n}{n!}\frac{\Gamma(n-1-2\alpha)}{\Gamma(-1-2\alpha)}\zeta_{1,1}{}^{+n=0}. \quad (6.16)$$

The final results for the UU case are therefore the following:

$$\gamma_{1,1}{}^{(\pm)n}(0)=\frac{(-1)^n}{n!}\frac{\Gamma(n-1-2\alpha)}{\Gamma(-1-2\alpha)}\gamma_{1,1}{}^{(\pm)n=0}(0)\,, \quad (6.17)$$

$$\gamma_{1,1}{}^{(+)n=0}(0)=-\gamma_{1,1}{}^{(-)n=0}(0)\,. \quad (6.18)$$

Note that the daughter-to-parent relation (6.17) turns out to be the same as in the spinless case (4.14).

B. Equal-Unequal Mass Case

Owing to the parity and G-parity selection rules, the parent and even daughters of the subfamily with $\sigma=-1$ couple to the amplitudes $F_{1,\frac{1}{2}\frac{1}{2}}{}^{J(-)}\equiv F_{1,0}{}^{J(-)}$ (in the λ,μ notation) and the odd daughters to the amplitudes $F_{1,\frac{1}{2}-\frac{1}{2}}{}^{J(-)}\equiv F_{1,1}{}^{J(-)}$. The parent and the even daughters of the subfamily with $\sigma=+1$ couple to the amplitudes $F_{1,\frac{1}{2}-\frac{1}{2}}{}^{J(+)}\equiv F_{1,1}{}^{J(+)}$, while the $\sigma=+1$ odd daughters are decoupled from the N-$\bar{N}$ system.

The study of the amplitude $\tilde{f}_{1,0}{}^{(-)}$ goes on in the same way as in Sec. V. We will discuss here the amplitudes $\tilde{f}_{1,1}{}^{(\pm)}$.

The contribution of a parity doublet to the amplitudes $\tilde{f}_{1,1}{}^{(\pm)}$ is given in (6.1). In this kinematical situation, however, one has $K_{11}{}^{(+)}(t)=(t_1/t)^{1/2}$ and $K_{11}{}^{(-)}(t)=1$.

The contribution of a parity doublet to the amplitudes $\tilde{f}_{1,1}{}^{(\pm)}$ is therefore given by[33]

$$\tilde{f}_{1,1}{}^{(+)\,\text{doublet}}=\tilde{g}(\alpha^+)\sum_{k=0}^{\infty}(\alpha^+-2k)^2a_k(\alpha^+)\gamma_{1,1}{}^{+}(t)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha^+-1-2k}\left(\frac{\tilde{D}(t)}{t}\right)^{k}-\tilde{g}(\alpha^-)\sum_{k=0}^{\infty}(\alpha^--2k)(\alpha^--2k-1)$$
$$\times a_k(\alpha^-)\gamma_{1,1}{}^{-}(t)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha^--2-2k}\left(\frac{\tilde{D}(t)}{t_1}\right)^{k+1/2}\left(\frac{t_1}{t}\right)^{k}, \quad (6.19)$$

$$\tilde{f}_{1,1}{}^{(-)\,\text{doublet}}=\tilde{g}(\alpha^-)\sum_{k=0}^{\infty}(\alpha^--2k)^2a_k(\alpha^-)\gamma_{11}{}^{-}(t)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha^--2k-1}\left(\frac{\tilde{D}(t)}{t}\right)^{k}-\tilde{g}(\alpha^+)\sum_{k=0}^{\infty}(\alpha^+-2k)(\alpha^+-2k-1)a_k(\alpha^+)$$
$$\times\gamma_{11}{}^{+}(t)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha^+-2k-2}\left(\frac{\tilde{D}(t)}{t_1}\right)^{k+1/2}\left(\frac{t_1}{t}\right)^{k+1}. \quad (6.20)$$

Note at this point that the trajectory α^+ contributes a nonasymptotic term to the amplitude $\tilde{f}_{1,1}{}^{(-)}$ which is singular at $t=0$; such a singularity must be cancelled by the trajectory α^-. This implies that already the reduced residue function of the first daughter of the subfamily with $\sigma=-1$ must have a simple pole at $t=0$. The simplest singularity structure of the daughter residue functions, needed in order to cancel the singularities, can be determined by direct inspection of (6.19) and (6.20). Therefore, the contribution of the whole family to the amplitudes under study can be written

$$\tilde{f}_{1,1}{}^{(+)}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left[\left(\frac{t_1}{t}\right)^{n+k}(\alpha_{2n}{}^{+}-2k)^2\tilde{b}_{2n,k}{}^{+}(t)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n}{}^{+}-2k-1}-\left(\frac{t_1}{t}\right)^{n+k+1}\left(\frac{\tilde{D}(t)}{t_1}\right)^{1/2}(\alpha_{2n+1}{}^{-}-2k)(\alpha_{2n+1}{}^{-}-2k-1)\right.$$
$$\left.\times\tilde{b}_{2n+1,k}{}^{-}(t)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n+1}{}^{-}-2k-2}\right], \quad (6.2\ldots)$$

---

[33] The reader is reminded that, owing to the P and G selection rules, here α^+ is an even daughter of the subfamily with $\sigma=+1$, say, the parent trajectory, and α^- is an odd daughter of the subfamily with $\sigma=-1$, say, the first daughter.

$$\tilde{f}_{1,1}^{(-)}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left[\left(\frac{t_1}{t}\right)^{n+k+1}(\alpha_{2n+1}^{-}-2k)^2\tilde{b}_{2n+1,k}^{-}(t)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n+1}^{-}-2k-1}-\left(\frac{t_1}{t}\right)^{n+k+1}\tilde{b}_{2n,k}^{+}(t)\left(\frac{\tilde{D}(t)}{t_1}\right)^{1/2}(\alpha_{2n}^{+}-2k)\right.$$
$$\left.\times(\alpha_{2n}^{+}-2k-1)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n}^{+}-2k-2}\right], \quad (6.22)$$

where $\tilde{b}_{n,k}^{(\pm)}$ is defined as in (4.16) with the appropriate residue function and with $\tilde{g}(\alpha_n^{\pm})$ in place of $g(\alpha_n)$.

From these expressions we get the following $t=0$ analyticity conditions:

$$\sum_{n=0}^{m+1}\tilde{b}_{2n,m+1-n}^{+}(t)(\alpha_{2n}^{+}-2m+2n-2)^2\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n}^{+}+2n-1}-\sum_{n=0}^{m}\left(\frac{\tilde{D}(t)}{t_1}\right)^{1/2}\tilde{b}_{2n+1;m-n}^{-}(t)(\alpha_{2n+1}^{-}-2m+2n)$$
$$\times(\alpha_{2n+1}^{-}-2m+2n-1)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n+1}^{-}+2n}=O(t^{m+1}), \quad (6.23)$$

$$\sum_{n=0}^{m}\tilde{b}_{2n+1;m-n}^{-}(t)(\alpha_{2n+1}^{-}-2m+2n)^2\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n+1}^{-}+2n}-\sum_{n=0}^{m}\tilde{b}_{2n,m-n}^{+}(t)\left(\frac{\tilde{D}(t)}{t_1}\right)^{1/2}(\alpha_{2n}^{+}-2m+2n)$$
$$\times(\alpha_{2n}^{+}-2m+2n-1)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n}^{+}+2n-1}=O(t^{m+1}), \quad (6.24)$$

for any $m\geq 0$.

In order to satisfy the analyticity requirements completely, we must take into account the $t=0$ constraint (Appendix A)

$$i\tilde{f}_{1,1}^{(+)}-\tilde{f}_{1,0}^{(-)}=O(t),$$

which, explicitly written, gives rise to

$$i\tilde{b}_{0,0}^{+}(t)[\alpha_0^{+}(t)]^2\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_0^{+}(t)-1}-b_{0,0}^{-}(t)\frac{\alpha_0^{-}(t)}{[\alpha_0^{-}(\alpha_0^{-}+1)]^{1/2}}\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_0^{-}(t)-1}=O(t), \quad (6.25)$$

$$\left[\sum_{n=0}^{m+1}(\alpha_{2n}^{+}-2m+2n-2)^2\tilde{b}_{2n,m+1-n}^{+}(t)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n}+2n-1^{+}}-\sum_{n=0}^{m}\left(\frac{\tilde{D}(t)}{t_1}\right)^{1/2}(\alpha_{2n+1}^{-}-2m+2n)(\alpha_{2n+1}^{-}-2m+2n-1)\right.$$
$$\left.\times\tilde{b}_{2n+1,m-n}^{-}(t)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n+1}^{-}+2n}\right]-\sum_{n=0}^{m}\frac{\alpha_{2n}^{-}-2(m+1-n)}{[\alpha_{2n}^{-}(\alpha_{2n}^{-}+1)]^{1/2}}b_{2n,m+1-n}^{-}(t)\left(\frac{s+\tilde{B}(t)}{s_0}\right)^{\alpha_{2n}^{-}+2n-1}=O(t^{m+1}), \quad (6.26)$$

for any $m\geq 0$.

We now explicitly restrict ourselves to the point $t=0$. Then from (6.23) and (6.24) we get

$$\sum_{n=0}^{m+1}\frac{\alpha-2m-2}{(\alpha-2n)(\alpha-2n+1)}\frac{\zeta_{1,1}^{+2n}}{2(m-n+1)!\Gamma(\frac{3}{2}-\alpha+n+m)}-\sum_{n=0}^{m}\frac{1}{(\alpha-2n-1)(\alpha-2n)}\frac{\zeta_{1,1}^{-2n+1}}{(m-n)!\Gamma(\frac{3}{2}-\alpha+n+m)}=0, \quad (6.23')$$

$$\sum_{n=0}^{m}\frac{\alpha-2m-1}{(\alpha-2n-1)(\alpha-2n)}\frac{\zeta_{1,1}^{-2n+1}}{2(m-n)!\Gamma(\frac{3}{2}-\alpha+m+n)}-\sum_{n=0}^{m}\frac{1}{(\alpha-2n)(\alpha-2n+1)}\frac{\zeta_{1,1}^{+2n}}{(m-n)!\Gamma(\frac{1}{2}-\alpha+m+n)}=0. \quad (6.24')$$

From (6.23′) and (6.24′) we get, after some manipulations,

$$\sum_{n=0}^{m+1}\frac{\zeta_{1,1}^{+2n}}{(m+1-n)!\Gamma(\frac{3}{2}-\alpha+m+n)}=0, \quad m\geq 0. \quad (6.27)$$

The solution of this system is

$$\zeta_{1,1}^{+2n}=\frac{2(\alpha-2n)+1}{2\alpha+1}\frac{(-1)^n}{n!}\frac{\Gamma(n-\frac{1}{2}-\alpha)}{\Gamma(-\frac{1}{2}-\alpha)}\zeta_{1,1}^{+n=0} \quad (6.28)$$

or

$$\gamma_{1,1}^{(+)2n}(0)=\frac{(-1)^n}{n!}\frac{\Gamma(n-\frac{1}{2}-\alpha)}{\Gamma(-\frac{1}{2}-\alpha)}\gamma_{1,1}^{(+)n=0}(0). \quad (6.29)$$

Going back to the determination of the $\zeta_{1,1}^{-n+1}$, we find that (6.24′) is equivalent to the following two relations:

$$\gamma_{1,1}^{(-)n=1}(0)=-\frac{2\alpha+1}{\alpha+1}\gamma_{1,1}^{(+)n=0}(0), \quad (6.30)$$

$$\sum_{n=0}^{m+1}\frac{\alpha-2m-3}{(\alpha-2n-1)(\alpha-2n)}\frac{\zeta_{1,1}{}^{-2n+1}}{2(m+1-n)!\Gamma(\frac{5}{2}-\alpha+n+m)}$$

$$-\sum_{n=0}^{m+1}\frac{1}{(\alpha-2n)(\alpha-2n+1)}$$

$$\times\frac{\zeta_{1,1}{}^{+2n}}{(m+1-n)!\Gamma(\frac{3}{2}-\alpha+m+n)}=0. \quad (6.31)$$

Using (6.31) and (6.23′), we obtain

$$\sum_{n=0}^{m+1}\frac{\zeta_{1,1}{}^{-2n+1}}{(m+1-n)!\Gamma(\frac{5}{2}-\alpha+n+m)}=0,\quad m\geq 0. \quad (6.32)$$

The solution of this system is

$$\zeta_{1,1}{}^{-2n+1}=\frac{2(\alpha-2n)-1}{2\alpha-1}\frac{(-1)^n}{n!}\frac{\Gamma(n+\frac{1}{2}-\alpha)}{\Gamma(\frac{1}{2}-\alpha)}\zeta_{1,1}{}^{-n=1} \quad (6.33)$$

or

$$\gamma_{1,1}{}^{(-)2n+1}(0)=\frac{(-1)^n}{n!}\frac{\Gamma(n+\frac{1}{2}-\alpha)}{\Gamma(\frac{1}{2}-\alpha)}\gamma_{1,1}{}^{(-)n=1}(0). \quad (6.34)$$

Moreover, from the constraint (6.23), we obtain

$$\gamma_{1,0}{}^{(-)n=0}(0)=i\left(\frac{\alpha}{\alpha+1}\right)^{1/2}\gamma_{1,1}{}^{(+)n=0}(0), \quad (6.35)$$

while (6.26) does not give additional relations between the residue functions.

From the study of the amplitude $\tilde{f}_{1,0}{}^{(-)}$ (Sec. V), we obtain

$$\gamma_{1,0}{}^{(-)2n}(0)=\frac{(-1)^n}{n!}\frac{\Gamma(n-\frac{1}{2}-\alpha)}{\Gamma(-\frac{1}{2}-\alpha)}\gamma_{1,0}{}^{(-)n=0}(0)$$

$$\times\left(\frac{(\alpha-2n)(\alpha-2n+1)}{\alpha(\alpha+1)}\right)^{1/2}. \quad (6.36)$$

This completes our work in the $EU, M=1$, case. We see, from (6.29), (6.30), and (6.34)-(6.36), that at $t=0$ the residue functions of all the members of the class-III family can be expressed in terms of only one quantity.

We are now in a position to reconstruct the class-III Toller pole in nucleon-nucleon scattering.

C. Toller Pole

Using the factorization theorem and the results of the two previous subsections, and remembering the choice of the scale factors t_0 and t_1, we get the following results for the nucleon-nucleon scattering: From (6.29) and (6.17),

$$\beta_{\frac12-\frac12,\frac12-\frac12}{}^{(+)2n}(0)=\frac{(2n)!}{2^{2n}(n!)^2}\frac{\Gamma(-\alpha)}{\Gamma(n-\alpha)}$$

$$\times\frac{\Gamma(n-\frac{1}{2}-\alpha)}{\Gamma(-\frac{1}{2}-\alpha)}\beta_{\frac12-\frac12,\frac12-\frac12}{}^{(+)n=0}(0), \quad (6.37)$$

from (6.36) and (6.17),

$$\beta_{\frac12\frac12,\frac12\frac12}{}^{(-)2n}(0)=\frac{(2n)!}{2^{2n}(n!)^2}\frac{\Gamma(-\alpha)}{\Gamma(n-\alpha)}\frac{\Gamma(n-\frac{1}{2}-\alpha)}{\Gamma(-\frac{1}{2}-\alpha)}$$

$$\times\frac{(\alpha-2n)(\alpha-2n+1)}{\alpha(\alpha+1)}\beta_{\frac12\frac12,\frac12\frac12}{}^{(-)n=0}(0), \quad (6.38)$$

and from (6.34) and (6.17),

$$\beta_{\frac12-\frac12,\frac12-\frac12}{}^{(-)2n+1}(0)=\frac{(2n+1)!}{2^{2n}(n!)^2}\frac{\Gamma(-\alpha)}{\Gamma(n-\alpha)}$$

$$\times\frac{\Gamma(n+\frac{1}{2}-\alpha)}{\Gamma(\frac{1}{2}-\alpha)}\beta_{\frac12-\frac12,\frac12-\frac12}{}^{(-)n=1}(0). \quad (6.39)$$

Moreover, we have two relations between the residue functions of the subfamilies with $\sigma=+1$ and $\sigma=-1$ which connect, respectively, the even members of the subfamily with $\sigma=+1$ to the even members of the subfamily with $\sigma=-1$, and the even members of the subfamily with $\sigma=+1$ to the odd members of the subfamily with $\sigma=-1$. These relations are derived from (6.34), (6.30), and (6.17), and are

$$\beta_{\frac12\frac12,\frac12\frac12}{}^{(-)n=0}(0)=\frac{\alpha}{\alpha+1}\beta_{\frac12-\frac12,\frac12-\frac12}{}^{(+)n=0}(0), \quad (6.40)$$

$$\beta_{\frac12-\frac12,\frac12-\frac12}{}^{(-)n=1}(0)=\frac{2\alpha+1}{(\alpha+1)^2}\beta_{\frac12-\frac12,\frac12-\frac12}{}^{(+)n=0}(0). \quad (6.41)$$

Relation (6.40) is exactly what is needed in order to satisfy the Volkov-Gribov constraint (Appendix A) by a parity doublet conspiracy.

According to (6.37)-(6.41), all the $t=0$ residue functions of the members of the $M=1$ "analyticity family" can be determined in terms of only one parameter.

Let us see which are the results of the $O(4)$ formulation. In this formulation, the $t=0$ residue functions of all the Regge poles associated with an $M=1$ Toller pole can be expressed in terms of the residue function of the Toller pole R through the relations

$$\beta_{\frac12\frac12,\frac12\frac12}{}^{(-)2n}(0)=\frac{2\alpha(\alpha+2)R}{3\pi^2[2(\alpha-2n)+1]}$$

$$\times|d_{\alpha-2n;1;0}{}^{\alpha,1}(\tfrac{1}{2}\pi)|^2, \quad (6.42)$$

$$\beta_{\frac12-\frac12,\frac12-\frac12}{}^{(-)2n+1}(0)=\frac{4\alpha(\alpha+2)R}{3\pi^2[2(\alpha-2n)-1]}$$

$$\times|d_{\alpha-2n-1;1;1}{}^{\alpha,1}(\tfrac{1}{2}\pi)|^2, \quad (6.43)$$

$$\beta_{\frac12-\frac12,\frac12-\frac12}{}^{(+)2n}(0)=\frac{4\alpha(\alpha+2)R}{3\pi^2[2(\alpha-2n)+1]}$$

$$\times|d_{\alpha-2n;1;1}{}^{\alpha,1}|^2, \quad (6.44)$$

where

$$d_{\alpha-2n;1;0}{}^{\alpha,1}(\tfrac{1}{2}\pi)=i\left(\frac{3[2(\alpha-2n)+1](2n)!(\alpha-2n)(\alpha-2n+1)}{2(\alpha+1)(\alpha+2)\alpha\Gamma(2\alpha+2-2n)}\right)^{1/2}(2i)^{\alpha-2n}(-1)^n\frac{\Gamma(\alpha+1-n)}{\Gamma(1+n)}, \tag{6.45}$$

$$d_{\alpha-2n-1;1;1}{}^{\alpha,1}(\tfrac{1}{2}\pi)=-i\left(\frac{3[2(\alpha-2n)-1](2n+1)!}{\alpha(\alpha+1)(\alpha+2)\Gamma(2\alpha+1-2n)}\right)^{1/2}(2i)^{\alpha-2n-1}(-1)^n\frac{\Gamma(\alpha+1-n)}{\Gamma(1+n)}, \tag{6.46}$$

$$d_{\alpha-2n;1;1}{}^{\alpha,1}(\tfrac{1}{2}\pi)=\frac{1}{2}\left(\frac{3[2(\alpha-2n)+1](2n)!(\alpha+1)}{\alpha(\alpha+2)\Gamma(2\alpha+2-2n)}\right)^{1/2}(2i)^{\alpha-2n}(-1)^n\frac{\Gamma(\alpha+1-n)}{\Gamma(1+n)}. \tag{6.47}$$

Using (6.45)–(6.47), it is easily seen that the group-theoretical relations (6.42)–(6.44) are equivalent to the analytic relations (6.37)–(6.41).

The proof that the $M=1$ Regge-pole families are the same as the Regge-pole families derived from $M=1$ Toller poles in equal-equal mass scattering is therefore completed.

VII. CONCLUSIONS

We have built a formalism that enables us to study the behavior near $t=0$ of the Regge-pole families whose existence is required from analyticity in the unequal-mass scattering problems. In this paper we limited ourselves to the study of the point $t=0$. Using the factorization theorem as a bridge from the unequal- to the equal-mass kinematics, we have shown explicitly that the $M=0$ and $M=1$ analyticity families[26] are the same as the families deduced from the $M=0$ and $M=1$ Toller poles. In the following, therefore, we will not need to distinguish between analytical and group-theoretical families and we will call them Toller families.

Although we have limited ourselves to the discussion of the nucleon-nucleon system, our results about the $M=0$ and $M=1$ Toller families are, of course, of general validity.

Our approach permits the study of the behavior of the Toller families for $t\neq 0$. This is a particularly interesting problem from the complementary point of view, according to which a Regge trajectory is associated to a whole string of resonances, provided that its real part reaches the "physical" angular-momentum region for positive values of t.

The possibility that some of the recently discovered resonances could be associated with Toller families stimulated a study of the properties of these families for $t\neq 0$. The results of this study will be the subject of a forthcoming paper. (In this connection, see Ref. 3 and the references quoted therein.)

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APPENDIX A

We use the customary notation $f_{cd,ab}$ to denote a t-channel helicity amplitude for the reaction $a+b\to c+d$. Helicity amplitudes free from kinematical singularities in s are defined by[6]

$$\bar{f}_{cd,ab}=(\sqrt{2}\sin\tfrac{1}{2}\theta)^{-|\lambda-\mu|}(\sqrt{2}\cos\tfrac{1}{2}\theta)^{-|\lambda+\mu|}f_{cd,ab}, \tag{A1}$$

where $\lambda=a-b$, $\mu=c-d$, and θ is the scattering angle in the t channel. When no confusion can arise, we use the notation $f_{\mu,\lambda}$.

The amplitudes are now formed in "parity-conserving" combinations,[6] which are suitable for Reggeization and whose kinematical t singularities can be easily factored out[8]:

$$\bar{f}_{\lambda,\mu}{}^{(\pm)}(s,t)=K_{\mu,\lambda}{}^{(\pm)}(t)\tilde{f}_{\mu,\lambda}{}^{(\pm)}(s,t)=\bar{f}_{\mu,\lambda}\pm\rho\bar{f}_{-\mu,\lambda}, \tag{A2}$$

where $\rho=\sigma_c\sigma_d(-1)^{\lambda+N}$, $N=\max\{|\lambda|,|\mu|\}$, σ is the natural parity $[=P(-1)^J]$, and $K_{\mu,\lambda}{}^{(\pm)}(t)$ is a known factor containing the kinematical singularities at $t=0$.[34] The $\bar{f}_{\mu,\lambda}{}^{(\pm)}$ have the following partial-wave expansion:

$$\bar{f}_{\lambda,\mu}{}^{(\pm)}=\sum(2J+1)\times(e_{\lambda,\mu}{}^{J+}(z)F_{cd;ab}{}^{J\pm}+e_{\lambda\mu}{}^{J-}(z)F_{cd,ab}{}^{J\mp}), \tag{A3}$$

where the $e_{\lambda\mu}{}^{J\pm}$ functions are defined in Ref. 6, $z=\cos\theta$, and can b eReggeized following the method of Ref. 6.

In the present paper we limit ourselves to the study of t-channel reactions of the kind $\bar{N}+N\to J+\bar{S}$ and the others related to these through factorization. Here N is a nucleon, and J (S) is a particle with spin J (S) and mass m_J (m_S), where $m_J\neq m_S\neq$ (nucleon mass).

Analyticity and crossing symmetry impose the following constraints on the amplitudes[26,35,36]:

(a) *EU case.*

$$i\bar{f}_{cd,\frac{1}{2}-\frac{1}{2}}{}^{(+)}-\bar{f}_{cd,\frac{1}{2}\frac{1}{2}}{}^{(-)}=O(t) \tag{A4}$$

for any c and d satisfying the inequality $c\neq d$, and

$$i\bar{f}_{cc,\frac{1}{2}-\frac{1}{2}}{}^{(-)}-\bar{f}_{cc,\frac{1}{2}\frac{1}{2}}{}^{(-)}=O(t) \tag{A5}$$

for any c.[37]

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[34] We have not factored out the kinematical singularities at the thresholds and pseudothresholds $t_{ij}{}^{\pm}=(m_i\pm m_j)^2$, since they are not relevant for the present discussion.

[35] P. Di Vecchia, F. Drago, and M. L. Paciello, Frascati Internal Report No. LNF 68/5, 1968 (unpublished).

[36] J. D. Stack, Phys. Rev. 171, 1666 (1968).

[37] Note that the relations (A5) differ by a factor $\frac{1}{2}$ in front of the amplitude $\bar{f}_{cc,\frac{1}{2}-\frac{1}{2}}{}^{(-)}$ from the corresponding one given in Refs. 26, 29, and 36, because of the different definition of the amplitudes $\bar{f}_{cd,ab}$, Eq. (A1).

(*b*) *UU case.*

$$\tilde{f}_{cd,ab}{}^{(+)}+\tilde{f}_{cd,ab}{}^{(-)}=O(t^m) \tag{A6}$$

if $|\lambda-\mu|<|\lambda+\mu|$, and

$$\tilde{f}_{cd,ab}{}^{(+)}-\tilde{f}_{cd,ab}{}^{(-)}=(t^m) \tag{A7}$$

if $|\lambda-\mu|>|\lambda+\mu|$, where $m=\min\{|\lambda|,|\mu|\}$.

(*c*) *EE case.* This constraint is not imposed in our discussion, but our results satisfy it automatically. It is[11,12]

$$\tilde{f}_{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}{}^{(-)}-2\tilde{f}_{\frac{1}{2}-\frac{1}{2},\frac{1}{2}-\frac{1}{2}}{}^{(+)}-\tilde{f}_{\frac{1}{2}-\frac{1}{2},\frac{1}{2}-\frac{1}{2}}{}^{(-)}=O(t). \tag{A8}$$

These constraints will play a crucial role in our discussion, imposing strong restrictions on the trajectories and on the residue functions at $t=0$.

APPENDIX B

In this Appendix we will prove the identity

$$\sum_{n=0}^{m}\binom{m}{n}(-1)^n\bigg/\prod_{h=0,h\neq n}^{m}(x+n+h)=0. \tag{B1}$$

We start by showing that

$$\sum_{n=0}^{k}\binom{m}{n}(-1)^n\bigg/\prod_{h=0,h\neq n}^{m}(x+n+h)=\binom{m-1}{k}(-1)^k\bigg/\prod_{h=1}^{m}(x+h+k) \tag{B2}$$

for any integral k less or equal to $m-1$. Relation (B2) is satisfied for $k=1$, as can be easily checked. Assuming that it is satisfied for $k-1$, we can show that it is satisfied for k. In fact, one has

$$\begin{aligned}\sum_{=0}^{k}\binom{m}{n}(-1)^n\bigg/\prod_{h=0,h\neq n}^{m}(x+n+h)&=\sum_{n=0}^{k-1}\binom{m}{n}(-1)^n\bigg/\prod_{h=0,h\neq n}^{m}(x+n+h)+\binom{m}{k}(-1)^k\bigg/\prod_{h=0,h\neq k}^{m}(x+k+h)\\&=(-1)^k\left[-\binom{m-1}{k-1}(x+k+m)+\binom{m}{k}(x+2k)\right]\bigg/\prod_{h=0}^{m}(x+h+k)\\&=(-1)^k\left[\binom{m-1}{k}(x+k)-(m-k)\binom{m}{k}+m\binom{m-1}{k}\right]\bigg/\prod_{h=0}^{m}(x+h+k)\\&=\binom{m-1}{k}(-1)^k\bigg/\prod_{h=1}^{m}(x+h+k),\end{aligned}$$

which is the required result.

Using (B2), with $k=m-1$, we can evaluate the sum in (B1):

$$\begin{aligned}\sum_{n=0}^{m}\binom{m}{n}(-1)^n\bigg/\prod_{h=0,h\neq n}^{m}(x+n+h)&=\sum_{n=0}^{m-1}\binom{m}{n}(-1)^n\bigg/\prod_{h=0,h\neq n}^{m}(x+n+h)+(-1)^m/\sum_{h=0}^{m-1}(x+m+h)\\&=(-1)^{m-1}/\prod_{h=1}^{m}(x+h+m-1)+(-1)^m/\prod_{h=0}^{m-1}(x+m+h)\\&=(-1)^{m-1}\left[1/\prod_{h=0}^{m-1}(x+h+m)-1/\prod_{h=0}^{m-1}(x+m+h)\right]\end{aligned}$$