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A. Małecki
Instytut Fizyki Jadrowej, Krakow (Poland)

ELECTRON SCATTERING FROM LIGHT NUCLEI. -

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I. - INTRODUCTORY LECTURE: FUNDAMENTALS OF ELECTRON SCATTERING FROM NUCLEI. -

In the two last decades the electron scattering has provided physicists with a rich information about atomic nuclei.

There are some reasons which are in favor of electron scattering as a tool for studying nuclear structure.

The first is that the interaction between the electron and the target nucleus is fairly well known. This is the electromagnetic interaction with the nuclear charge and current densities.

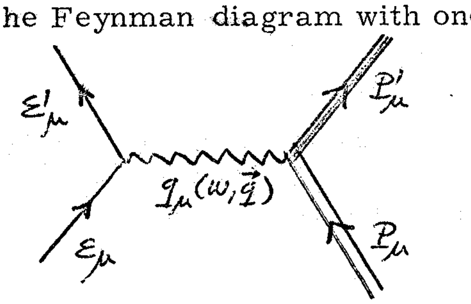
Since the interaction is relatively weak the electron scattering on the target does not greatly disorder the structure of the nucleus, and therefore, the scattering mechanism can be separated from structure effects. With electron scattering one can relate the cross-section to the transition matrix elements of the nuclear charge and current operators, and hence deduce an information about the structure itself. This is in contrast to the situation when working with strongly interacting projectiles, e.g. nucleons, as in that case the reaction mechanism and structure effects become mixed.

The same considerations hold also for processes involving real photons. However, the great advantage of electron scattering over photo excitation consists in the possibility of varying the momentum transfer q to the nucleus. With real photons, for a given energy transfer ω, there is only a single possible momentum transfer $q=\omega$ since the photon mass is zero. On the other hand, for electrons the only restriction is that the four-momentum transfer should be space-like:

$$q \geq \omega$$

Thus with electrons one has the possibility to study the Fourier transforms of the nuclear charge and current densities and to obtain informations also about spatial distribution of these quantities.

Our discussion of electron scattering will be based on the first Born approximation. In this approximation the process is represented by the Feynman diagram with one photon line connected to the nuclear vertex:



Electron scattering

$$q_\mu = \mathcal{E}_\mu - \mathcal{E}'_\mu = P'_\mu - P_\mu$$

The one-photon-exchange is expected to be a good approximation for small Ze^2; Z being charge number. Consequently, our analysis is limited to light nuclei.

Because of our total lack of information on a relativistic wave function for the nucleus we are forced to describe the nucleons as interacting like nonrelativistic Pauli (with two component spinors) particles. This is accomplished by expanding the well known covariant electron-nucleon interaction in powers of $1/M$, the inverse nucleon mass, and retaining terms through order $1/M^2$. The range of validity of this approximation is determined by the three momentum q transferred to the nucleus. One usually[1] considers $q \approx 2.5\,\mathrm{fm}^{-1} \approx 500$ MeV as the maximum q at which the M^{-2} approximation is useful.

The reduction of interaction between electron and relativistic nucleon to two-component form for the nucleon can be carried out by means of the Foldy-Wouthuysen transformation. In order to describe the many--nucleon system one makes assumption that the nucleons in a nucleus do not distort one another, so the nucleon form factors $F_1(q_\mu^2)$ and $F_2(q_\mu^2)$ are the same inside a nucleus as out. The second usual assumption concerns the additivity of the electron-nucleon interaction: the electron-nucleus interaction is obtained by summing over all the nucleons present.

Applying these assumptions one obtains the nuclear charge and current operators (Fourier transforms) with terms through order M^{-2}[1]

$$\hat{Q}(\vec{q}) = f(q_\mu^2) \sum_{j=1}^{A} \left\{ e_j + \frac{e_j - 2\mu_j}{8M^2} \left[-q^2 + 2i\vec{q}(\vec{\sigma}_j \cdot \vec{p}_j) \right] \right\} e^{i\vec{q}\cdot\vec{r}_j}$$

(I.1)

$$\hat{\vec{J}}(\vec{q}) = f(q_\mu^2) \sum_{j=1}^{A} \left\{ \frac{e_j}{2M} \left(\vec{p}_j e^{i\vec{q}\cdot\vec{r}_j} + e^{i\vec{q}\cdot\vec{r}_j} \vec{p}_j \right) + \frac{i\mu_j}{2M} (\vec{\sigma}_j \cdot \vec{q}) e^{i\vec{q}\cdot\vec{r}_j} \right\}$$

where e_j, μ_j are the charge and total magnetic moment (in the nuclear magnetons) for the j-th nucleon; $\vec{r}_j$, $\vec{p}_j$, $1/2\ \vec{\sigma}_j$ are its position, momentum and spin operators, respectively. We have used an approximation concerning the nucleon form factor, namely it was assumed:

$$F_{1p} = F_{2p} = F_{2n} = f(q_\mu^2) \qquad F_{1n} = 0$$

One can add to (I.1) a lot of different corrections coming from the presence of the other nucleons in the nucleus, e.g. many-body currents exchange currents etc. They, however, are expected to be small.

We shall refer to Eq. (I.1) as the McVoy-Van Hove interaction. Notice that the current has not relativistic correction to order M^{-2}. The two terms in $\vec{J}(\vec{q})$ describe the usual convection current, and spin current interactions. The first term in $Q(\vec{q})$ represents the static Coulomb interaction. The last two terms of order M^{-2}, are the Darwin-Foldy and spin-orbit terms. Because the square of the matrix element contains a cross-term between them and the Coulomb term which is of order M^{-2}, they must be included in a consistent calculation of the cross-section through order M^{-2}. For unpolarized nucleons, the spin-orbit term does not contribute to this M^{-2} cross term, so we can drop it at once. On the other hand, the Darwin-Foldy term must be kept for an M^{-2}-order calculation.

The usual analysis of electron scattering from nuclei does not take into account the possibility of meson production which start at $\omega = m_\pi$. The nuclear physics of the production processes is only crudely understood. In order to describe the electron scattering in terms of the usual nuclear physics variables (without taking into account the mesonic degrees of freedom), one must be able to make a separation between the nuclear physics processes and electroproduction processes. This turns out to be possible in practice if the momentum transfer q is not too large. The usually assumed restriction[1] is $q < 2.5\ fm^{-1}$. This is about the value of q at which corrections to the nonrelativistic interaction (I.1) begin to become important.

There are two classes of experiments with electrons as the projectiles: coincidence and non-coincidence experiments.

Coincidence experiments, like the recent study of the (e, e'p) reaction by the Sanità group working Frascati, involve detection of final nuclear products together with the final electron. As the electromagnetic interaction is relatively weak the coincidence experiments require a high intensity beam and a high duty cycle in order to keep the accidental rate low. Therefore these experiments are still scarce; their number should, however, increase in the near future with the increasing number of new accelerators which will have these characteristics.

In these lectures we will confine ourselves to the non-coincidence experiments where one observes only the final electron. This means that one performs experimentally a summation over all final nuclear states compatible with fixed geometry of the scattered electron.

Let us consider the scattering of an electron with incident energy ε through an angle θ to a final state with energy ε' while the nucleus makes a transition from the ground state $|i\rangle$ to the state $|f\rangle$.

The cross section for this process is given, in the first Born approximation, by the following formula(2)(x):

$$\frac{d^2\sigma}{d\Omega' d\varepsilon'} = \frac{e^4 \cos^2\theta/2}{4\,\varepsilon^2 \sin^4\theta/2} \overline{\sum_{|i\rangle}} \sum_{|f\rangle} \delta(\omega - E_f + E_i) \Big[\frac{q_\mu^4}{q^4} Q_{fi}^x Q_{fi} + \tag{I.2}$$

$$+ (\tan^2\frac{\theta}{2} - \frac{q_\mu^2}{2q^2}) (\vec{J}_{fi}^x \cdot \vec{J}_{fi})_\perp \Big]_{LAB.}$$

where $q_\mu^2 = \omega^2 - q^2$, $(\vec{J}_{fi}^x \cdot \vec{J}_{fi})_\perp = \vec{J}_{fi}^x \vec{J}_{fi} - 1/q^2 (\vec{q}\cdot\vec{J}_{fi}^x)(\vec{q}\cdot\vec{J}_{fi})$, $Q_{fi} = \langle f|\hat{Q}(\vec{q})|i\rangle$, $\vec{J}_{fi} = \langle f|\hat{\vec{J}}(\vec{q})|i\rangle$ are the matrix elements between the ground and excited states of the charge and current operators of the target nucleus. In Eq. (I.2) all the quantities are to be taken in the laboratory frame.

Cross-section in (I.2) is a function of ε, ε' and θ. A more convenient choice of kinematic variables is to work with θ, energy loss $\omega = \varepsilon - \varepsilon'$ and momentum q transferred by the electron to the nucleus.

We shall rewrite Eq. (I.2) in the following form:

$$\frac{d^2\sigma}{d\Omega' d\varepsilon'} = \sigma_M \Big[C(\omega, q) \frac{q_\mu^4}{q^4} + (\tan^2\frac{\theta}{2} - \frac{1}{2}\frac{q_\mu^2}{q^2}) T(\omega, q) \tag{I.3}$$

where

$$\sigma_M = \frac{e^4 \cos^2\theta/2}{4\;{}^2 \sin^4\theta/2} f^2 (q_\mu^2)$$

is the Mott (free-proton) cross-section.

We call $C(\omega, q)$ and $T(\omega, q)$ the Coulomb (or longitudinal) and transverse nuclear form factors, respectively.

The terminology is taken over from the multipole analysis of the electromagnetic interaction. $C(q, \omega)$ includes contributions only from the Coulomb multipoles which are absent in the processes involving real photons. $T(q, \omega)$ includes contributions only from the transverse (electric and magnetic) multipoles.

(x) - We use a metric such that $a_\mu = (a_0, \vec{a})$ and $a_\mu b_\mu = a_0 b_0 - \vec{a}\cdot\vec{b}$. The magnitude of the three vector is $a = |\vec{a}|$. We alsouse units $C = \hbar = 1$, $e^2 = 1/137$.

Speaking exactly, the following equations hold[(2)];

$$\frac{1}{2J_i+1}\sum_{M_iM_f} Q^x_{fi}Q_{fi} = \frac{4\pi}{2J_i+1}\sum_{J=0}^{\infty}\left|\langle J_f\|\hat{C}_J(q)\|J_i\rangle\right|^2$$

(I.4)

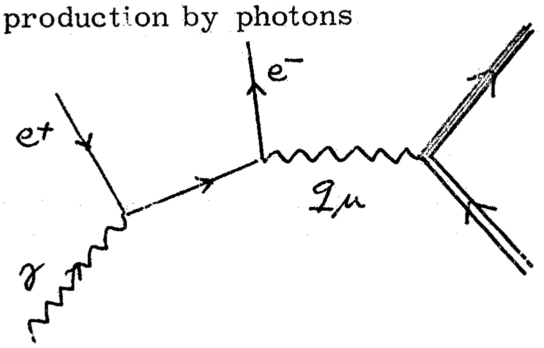
$$\frac{1}{2J_i+1}\sum_{M_im_f} (\vec{J}^x_{fi}\cdot\vec{J}_{fi}) = \frac{4}{2J_i+1}\sum_{J=1}^{\infty}\left\{\left|\langle J_f\|\hat{T}^{el}_J(q)\|J_i\rangle\right|^2+\left|\langle J_f\|\hat{T}^{mag}_J(q)\|J_i\rangle\right|^2\right\}$$

where in the l.h.s. of (I.4) one sums and averages over nuclear orientations; J_i and J_f are the angular momenta of the initial and final states. The multipole operators $\hat{C}_{JM}$ (Coulomb), $\hat{T}^{el}_{JM}$ (transverse electric), T^{mag}_{JM} (transverse magnetic) are irreducible tensor operators of rank J in the nuclear Hilbert space. They are defined in Ref. (2). In Table I are summarized the properties of the multipole operators under some symmetry transformations.

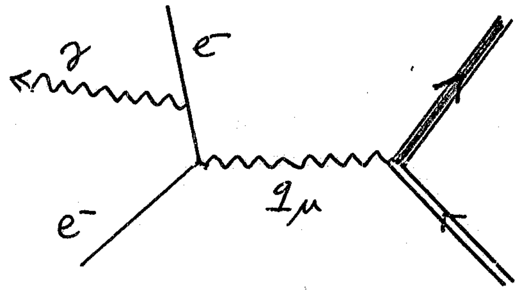
TABLE I

Symmetry property	C_{JM}	T^{el}_{JM}	T^{mag}_{JM}
Parity	$(-1)^J$		$(-1)^{J+1}$
Time reversal	$(-1)^{J_i-J_f+J}$	$(-1)^{J_i-J_f+J+1}$	
Angular momentum conservation	$J_i+J_f > J \geq \lvert J_f-J_i\rvert$ $J>0$	$J>1$	

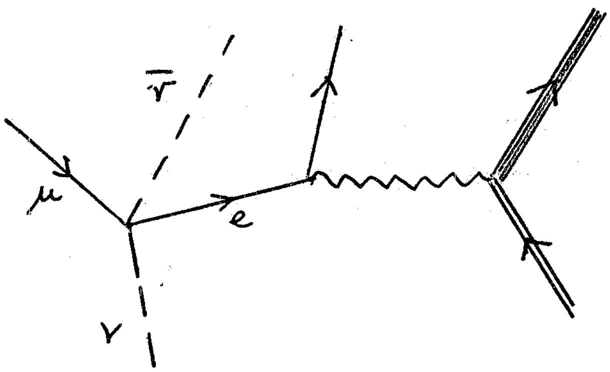
Before going to discussion of Eq. (I.3) let us note that the cross-section formula of that type holds for any process on nuclear target if we describe it in the one-photon-exchange approximation[(3)]. This may be either pair production by photons



or bremsstrahlung



or production of lepton pairs by neutrino



If in the experiment one detects particles in the lepton part of the process only (and thus performs experimentally a summation over all nuclear final states compatible with energy-momentum conservation) the cross-section is given by the formula:

$$\sigma \sim M_1 \left[W_2(\omega, q) + M_2\, W_1(\omega, q) \right]$$

where the $M_1 M_2$ functions are characterized by the process taking place on the target nucleus and the $W_{1,2}(\omega, q)$ functions are always the same. They are called the nuclear form factors and contain all the information on nuclear structure. Thus if one measures W_1, W_2, say in inelastic electron scattering, one knows all the necessary nuclear physics and can eliminate it from other processes. This is especially important in experiments devoted to a detailed study of the lepton vertex, like recent tests of Q. E. D. at small distances.

Let us turn now to Eq. (I. 3) describing the electron-nucleus scattering.

We note that $C(q, \omega)$, or Coulomb form factor, and $T(q, \omega)$, or transverse form factor can be separated experimentally. This could be accomplished by doing experiments at fixed q, and ω and varying θ. Then the plot σ vs $\tan^2 \theta/2$ gives us the two form factors. Another clever way is to do scattering experiments at $\theta = 180^o$ as in this case we are left only with the transverse form factor.

By the way let us point out another advantages of the scattering experiments at $\theta=180^{o}$. For backward scattering one can reach the largest possible momentum transfers, namely $q=2\mathcal{E}-\omega$. At 180^{o} the contribution from the elastic scattering which usually masks the more interesting q dependence of the inelastic terms is much reduced. For nuclei with no magnetic moments in the ground state there is no contribution from elastic scattering at all.

Also in forward scattering, say at small enough values of θ, the contribution of the transverse part is dominant.

On the other hand, at all intermediate angles, say $\theta=30^{o}$ to 120^{o}, the Coulomb terms dominate. The Coulomb form factor usually is larger than the transverse one by an order of magnitude.

The main features of the single (without coincidence) electron scattering are well known. In a typical cross-section curve (see Fig. 1).

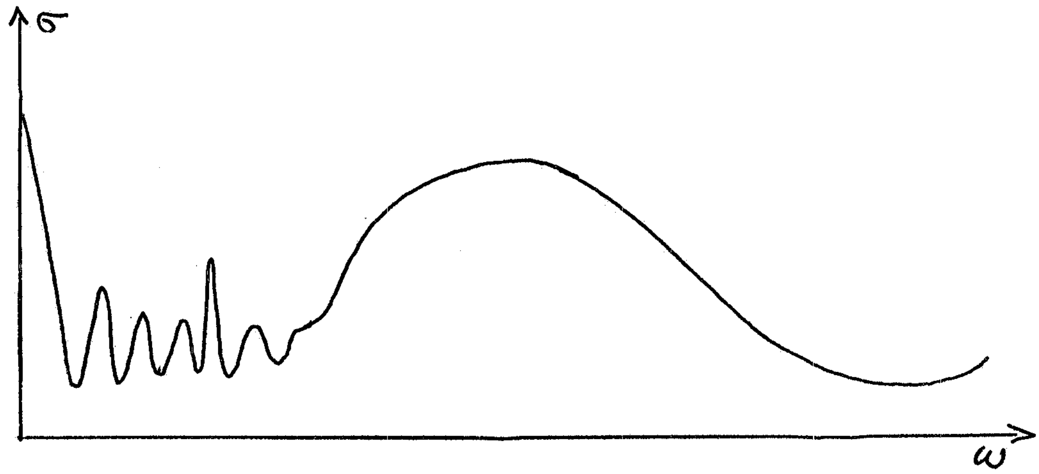


FIG. 1 - Schematic energy spectrum of electrons scattered from a nucleus.

one can distinguish the following parts

a) at $\omega\simeq0$ one has a peak corresponding to the elastic scattering

b) at small energy transfers a number of peaks corresponding to the excitation of discrete nuclear levels are seen

c) at large energy transfers the cross-section follows an almost smooth curve in which only little structure is apparent. This part of the spectrum is referred to as the quasi-elastic peak. It corresponds roughly to direct collisions with the individual nucleons in the nucleus.

II. - SUM RULES FOR ELECTRON-NUCLEUS SCATTERING. -

We shall discuss now the sum rules i.e. the theoretical predictions for the electron scattering cross-section integrated over the energy loss ω : $\int d\omega\, \sigma(q, \vartheta, \omega)\, W(\omega)$, $W(\omega)$ being a weighting factor.

In the experiment one measures the area under the inelastically scattered electron spectrum in Fig. 1.

Theoretically one sums over all final nuclear states in a particular way. It is, therefore, possible to use the closure relation

$$\sum_{|f\rangle} |f\rangle\langle f| = 1$$

in order to eliminate the final states from the resulting expressions and to operate with the ground expectation values of bilinear combinations of charge and current density operators only.

Thus the analysis of the sum rules can give us more information about the ground state of the target nucleus.

Of course, the direct way to study the ground state properties is to measure the elastic cross-section.

However, the sum rules can be expressed through nucleon-nucleon correlation function $\rho(1,2) = \int d^3 3 \ldots d^3 A\, |\psi_i(1, 2 \ldots A)|^2$. It is, therefore, hoped that the sum rules may give some information about dynamical correlations in nuclei. This is in contrast to the situation for elastic scattering which depends only on the single-particle density

$$\rho(1) = \int d^3 2 \ldots d^3 A\, |\psi_i(1 \ldots A)|^2 .$$

Nevertheless, the dynamical correlation effects, may be seen also in the elastic scattering, if the momentum transfers available in the experiment are large enough. At large momentum transfers one can study the high--momentum components of the single-particle wave function which could arise from the strong short-range repulsion in the nucleon-nucleon potential.

Before going on to a discussion of different sum rules for electron scattering, we note two important experimental restrictions on the construction of the sum rule.

Of course, the entire range of ω from O to infinity is never available. The accessible range is seen from the relation $q^2 - \omega^2 = 4\,\varepsilon\,\varepsilon' \sin^2\theta/2$ which tells us that $\omega \leq q$. However, the fact that the contributions to the sum from very large ω's are insignificant, makes the sum rules accessible experimentally provided one works at large enough momentum transfer q.

Another restriction is connected with a separation between the nuclear physics processes and meson production. As the mesonic degrees of freedom will not be taken into account in the sum rules we should also be able to distinguish experimentally the non-mesonic events from the mesonic ones. It terms out to be possible if the momentum transfer is not too large: $q \lesssim 2.5\ fm^{-1}$ (2).

Sum rule for Coulomb scattering.-

We confine now ourserves to the Coulomb past of the electron-nucleus interaction. Moreover, we will keep, for sake of simplicity, the term of zeroth order in 1/M in the Coulomb interaction.

In this approximation the cross-section is given by:

$$\frac{d^2\sigma}{d\Omega' d\varepsilon'} = \sigma_M \frac{q_\mu^4}{q^4} C_o(q,\omega) \tag{II.1}$$

where

$$C_o(q,\omega) = \sum_{|f\rangle} \left| \langle f \left| \sum_{j=1}^{A} e_j \, e^{i\vec{q}\cdot\vec{r}_j} \right| i \rangle \right|^2 \delta\left(\omega - \omega_f - \frac{q^2}{\alpha AM}\right),$$

ω_f being the nuclear excitation energy. It is the usual practice to use for the nuclear states the wave functions of an independent-particle shell model. In this model the nucleons are considered to more in common potential well. The origin of coordinates is taken to be the centre of the potential well and not the centre of mass of the nucleus. This is clearly incompatible with the principle of translational invariance. However, the transformation to the centre-of-mass system of the nucleus can be accomplished by employing the transformation of Gartenhaus and Schwartz[4].

The result of this transformation is that the coordinates $\vec{r}_j$ and momenta $\vec{p}_j$ of the j-th nucleon should be replaced as follows

$$\vec{r}_j \rightarrow \vec{r}_j - \frac{1}{A} \sum_{k=1}^{A} \vec{r}_j = \vec{\rho}_j \tag{II.2}$$

(II. 2) $$\vec{p}_j \rightarrow \vec{p}_j - \frac{1}{A}\sum_{k=1}^{A} \vec{p}_j = \vec{\pi}_j$$

Applying (II. 1) and (II. 2) we obtain the total (elastic + inelastic) sum rule for the Coulomb scattering

(II. 3) $$\int_0^{\infty} d\omega \frac{(\omega, q, \theta)}{\sigma_M} \frac{q^4}{q_\mu^4} = \sum_{|f\rangle} \left|\langle f| \sum_{j=1}^{A} e_j e^{i\vec{q}\cdot\vec{\rho}_j}|i\rangle\right|^2$$

As the elastic scattering contribution is dominating, especially at small q, and makes more interesting q dependence of the inelastic terms, we will define the inelastic sum $C_o^{inel}(q)$ given by the inelastic part of the integral in (II. 3).

Using closure in (II. 3) we assume that the sum in the L. H. S. part is performed experimentally at fixed q and subtracting the elastic contribution we obtain the inelastic Coulomb sum rule:

(II. 4) $$C_o^{inel}(q) = \langle i| \sum_{j,k}^{A} e_j e_k e^{i\vec{q}(\vec{r}_j - \vec{r}_k)}|i\rangle - Z^2 F_{el}^2(q)$$

where

(II. 5) $$F_{el}(q) = \frac{1}{Z} \langle i| \sum_j e_j e^{i\vec{q}\cdot\vec{\rho}_j}|i\rangle$$

is the elastic form factor.

Separating out the diagonal terms (j=k) in the double sum one obtains:

(II. 6) $$C_o^{inel}(q) = Z + \langle i| \sum_{j \neq h}^{A} e_j e_k e^{i\vec{q}(\vec{r}_j - \vec{r}_k)}|i\rangle - Z^2 F_{el}^2(q)$$

The same can be written as follows

(II. 7) $$C_o^{inel}(q) = Z + Z(Z-1) \int d^3r' d^3r'' e^{i\vec{q}(\vec{r}' - \vec{r}'')} \rho(\vec{r}', \vec{r}'') - Z^2 F_{el}^2$$

where

$$\rho(\vec{r}\,',\vec{r}\,'') = \frac{1}{Z(Z-1)} \langle i | \sum_{j \neq k}^{A} e_j e_k \, \delta(\vec{r}_j - \vec{r}\,')\, \delta(\vec{r}_k - \vec{r}\,'') | i \rangle$$

is the proton-proton correlation (two-proton density) function in the ground state.

A system is said to be completely uncorrelated if

$$\rho(1,2) = \rho(1)\,\rho(2),$$

where $\rho(\vec{r})$ is the one nucleon density (the proton density is measured in the elastic electron scattering). It is therefore convenient to subtract off this random part and to define the more useful correlation function

(II.8) $$\tilde{\rho}(1,2) = \rho(1,2) - \rho(1)\,\rho(2)$$

Using (II.8) one can rewrite (II.7) in the following form

(II.9) $$C_o^{inel}(q) = Z(1-F_{el}^2) + Z(Z-1)\int d^3r'\,d^3r''\, e^{i\vec{q}(\vec{r}\,'-\vec{r}\,'')}\,\tilde{\rho}(r',r'')$$

where we have approximated the expression (II.5) for F_{el} by neglecting in it the Gartenhaus-Schwartz correction (thus making an error of order A^{-1}).

There are two-types of correlations which are incorporated into $\tilde{\rho}(1,2)$.

The statistical correlations reflect the fact that the system of nucleons should be described by a wave function in the form of the Slater determinant. In order to account for these correlations it is not sufficiently to dispose the nucleons in different single particle states (thus satisfying the Pauli principle) but one must antysymmetric the wave function in all identical particles.

The more interesting correlation effect is that due to the hard-core repulsion part in the nucleon-nucleon interaction.

It was originally hoped that the sum rules should give some information about such dynamical correlations but it turned out that they are rather insensitive to them.

The effect of the dynamical correlations on the Coulomb scattering could be calculated with some confidence on the basis of various works where one had calculated the two-particle correlation function. It was done by Mc Voy and Van Hove(1) who applied for O^{16} the correlation function of Eden et al. calculated using a realistic force with hard-core radius 0.4 fm. The result $C_{EES}(q)$ is presented in Fig. 2 where it is also compared to the Coulomb sum rule C_{SM} evaluated with the shell model (oscillator potential) ground state wave function:

$$C^{inel}_{SM}(q)=Z\left[1-(2\mu_p-1)\frac{q^2}{4M^2}\right]\left[1-\left(1+\frac{Z-2}{12Z}\frac{q^4}{\alpha^4}\right)\exp\left(-\frac{q^2}{2\alpha^2}\right)\right]- \tag{II.10}$$
$$-Z^2\left[\exp\left(\frac{q^2}{2A\alpha^2}\right)-1\right]\left(1-\frac{Z-2}{6Z}\frac{q^2}{\alpha^2}\right)^2\exp\left(-\frac{q^2}{2\alpha^2}\right)$$

α -being the parameter of the Gaussian factor $\exp(-1/2\,\alpha^2 r^2)$ of the oscillator wave functions (for O^{16} $\alpha = 0.6\ fm^{-1}$ from elastic electron scattering).

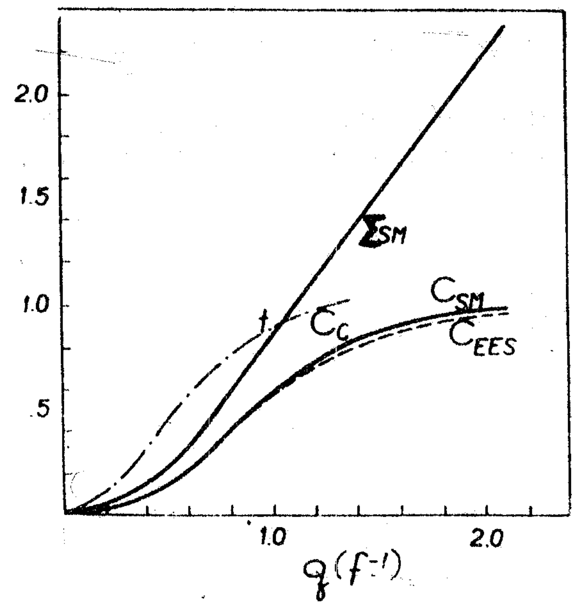


FIG. 2 - The Coulomb inelastic sum rules(1) (divided by Z) for O^{16}: C_c - -no correlations, C_{EES}-statistical correlations and short range hard core effects included, Σ_{SM}-complete (Coulomb plus transverse) sum rule evaluated in Ref. (1). The experimental point from the work of Bishop et al.(13).

The "correlated sum rule does not deviate from $C_{SM}(q)$ by more than 5% at any value of q, so the effect of hard-cores on the Coulomb scattering is very small.

On the other hand, it is evident from Fig. 2 that the statistical correlations have a substantial influence on the Coulomb scattering. The Coulomb sum rule $C_c(q)$ obtained from the classical perfect gas model (shell model without antisymmetrization) differs very much from C_{SM}.

Let us note that $C^{inel}_o(q)$ as given by (II.9) has an important asymptotic behaviour. Because of oscillating integrand the correlation correction vanishes as $q \to \infty$. So does the elastic form factor and we obtain

$$C^{inel}_o(q) = Z$$

for large enough momentum transfers. This means that the cross-section reduces to a sum of free proton cross-sections, without interference. We say say that at large momentum transfers one is dealing with quasi-free (or quasi-elastic) scattering.

Finally let us note that neglecting in (II.9) both statistical and dynamical correlations between nucleons one obtains

$$C_o^{inel}(q) \simeq Z\,(1-F_{el}^2) \tag{II.11}$$

This equation describes the inelastic scattering from the nucleus in which the nucleons move in an average potential well independently of each other.

Transverse sum rule at large momentum transfers. -

We proceed now to discuss the transverse interaction contribution. If one works at $\vartheta = 180^o$ this is the only contribution to the electron scattering

For $\vartheta = 180^o$ one has from (I.3)

$$\frac{d^2\sigma}{d\Omega' d\varepsilon'} = \frac{e^4 f^2 (q_\mu^2)}{4\,\varepsilon^2}\; T(q,\omega) \tag{II.12}$$

with

$$T(q,\omega) = \overline{\sum_{|i\rangle}} \sum_{|f\rangle} \delta(\omega - E_f + E_i)\,(\vec{J}^{\,*}_{fi} \cdot \vec{J}_{fi})_\perp$$

The transverse sum rule is defined as follows (the integration over experimental spectrum is performed at fixed q):

$$\int_o^\infty d\omega\,\sigma(\omega, q, 180^o) \Big/ \frac{e^4 f^2}{4\,\varepsilon^2} = T(q) \tag{II.13}$$

Assuming that the final nuclear states one sums over form a complete set of states and applying the closure relation one gets

$$T(q) = \langle i |\, \hat{\vec{J}}^+ \cdot \hat{\vec{J}} - \frac{1}{q^2}\,(\hat{\vec{J}}^+ \cdot \vec{q})(\hat{\vec{J}} \cdot \vec{q})\, | i \rangle \tag{II.14}$$

Let us point out that although we include in principle the elastic scattering in our transverse sum rule, it turns out to be immaterial for simplest nuclei. When working at 180^{o} the elastic contribution does not vanish only for nuclei with magnetic moments in the ground state.

Using the non relativistic form of the current operator see Eq.(I.1) one obtains

$$T(q) = \frac{q^2}{2M^2} \langle i | \sum_{j,k}^{A} \mu_j \mu_k \sigma_{jx} \sigma_{kx} e^{iq(r_j - r_k)} | i \rangle + \quad \text{(II.15)}$$

$$+ \frac{2}{M^2} \langle i | \sum_{j,k}^{A} e_j e_k p_{jx} p_{kx} e^{iq(r_j - r_k)} | i \rangle$$

where we chose $\vec{q}$ along the Z-axis. As T(q) depends only on the absolute value of $\vec{q}$, we can choose its direction as we please.

Separating out the diagonal terms in (II.15) one gets

$$T(q) = \frac{q^2}{2M^2} (Z\mu_p^2 + N\mu_N^2) + \frac{2}{M^2} \sum_j^{A} \langle i | e_j p_{jx}^2 | i \rangle +$$

$$+ \frac{q^2}{2M^2} \langle i | \sum_{j \neq k}^{A} \mu_j \mu_k \sigma_{jx} \sigma_{kx} e^{iq(r_j - r_k)} | i \rangle + \quad \text{(II.16)}$$

$$+ \frac{2}{M^2} \langle i | \sum_{j \neq k}^{A} e_j e_k p_{jx} p_{kx} e^{iq(r_j - r_k)} | i \rangle$$

The form (II.16) is more convenient for analysing T(q) for large q's as it exhibits explicitly the first leading (and model independent) term.

These sum rules should be corrected for the center of mass motion by means of the Gartenhaus-Schwartz transformation (II.2).

We obtain now after some simple manipulations[5]:

$$T_{GS}(q) = \frac{q^2}{2M^2} (Z\mu_p^2 + N\mu_n^2) + \frac{2Z}{M^2 A} \langle i | \sum_j^{A} p_{jx}^2 | i \rangle \quad \text{(II.17)}$$

$$+ \frac{q^2}{2M^2} \langle i | \sum_{j \neq k}^{A} \mu_j \mu_k \sigma_{jx} \sigma_{kx} e^{iq(r_j - r_k)} | i \rangle$$

$$+\frac{2}{M^2}\langle i|\sum_{j\neq k}^{A} e_j e_k p_j^x p_k^x\, e^{iq(r_j - r_k')}|i\rangle - \frac{2Z}{M^2A^2}\langle i|P_x^2|i\rangle +$$

(II.17)

$$+\frac{2}{M^2A}\langle i|\sum_{j\neq k}^{A} e_j e_k \left(\frac{P_x}{A} - p_j^x p_k^x\right) P_x \cdot e^{iq(r_j - r_k)}|i\rangle$$

where

$$\vec{P} = \sum_{j=1}^{A} \vec{p}_j$$

It is comparatively easy to evaluate (II. 17) in the shell model with oscillator potential. Using the antisymmetrized ground state wave function for nuclei with nucleons in the first s and p shells one obtains:

$$T_{GS}(q) = \frac{q^2}{2M^2} Z(\mu_p^2 + \mu_N^2) + \frac{5Z-4}{3}\frac{\alpha^2}{M^2} - \frac{Z}{2}\frac{q^2}{M^2}(\mu_p^2+\mu_N^2)\left(1+\frac{Z-2}{12Z}\frac{q^4}{\alpha^4}\right)$$

(II. 18)

$$\cdot \exp\left(-\frac{q^2}{2\alpha^2}\right) - \frac{2}{3}(Z-2)\frac{\alpha^2}{M^2}\left(1+\frac{q^2}{2\alpha^2}\right)\exp\left(-\frac{q^2}{2\alpha^2}\right) - \frac{Z}{A}\frac{\alpha^2}{M^2} -$$

$$-\frac{Z}{A}\frac{\alpha^2}{M^2}\left[Z-1-\frac{Z-2}{3}\frac{q^2}{\alpha^2}+\frac{Z-2}{12Z}\frac{q^4}{\alpha^4}\right]\exp\left(-\frac{q^2}{2\alpha^2}\right)$$

We have assumed here the same number of protons and neutrons in both spin states; α is the oscillator parameter. The first two terms are the diagonal "terms of (II. 17). The third and fourth are the σ- σ, and p-p correlation function of (II. 17). The fifth and sixth term are corrections due to subtraction of the c. m. motion. They are small for $q \gtrsim 1 fm^{-1}$ but for small q's they are very important.

There are some corrections to the formula (II. 18) one should consider before applying it confidently to the analysis of the experimentally measured sum rule.

As we evaluate sum rules which include summation over a host of complicated excitations, the details of the shell model we accept as our starting point (e. g. the spin-orbit coupling, the shape of the potential well) seem to be not important.

A more essential point is to estimate the effects of the nucleon--nucleon correlations. We would like to discuss the transverse sum rule at large momentum transfers and in this case the wave length of the virtual photon transferred to the nucleus is small enough to show some fluctuations of nucleons around their average shell model orbits.

We shall not go into details of the short range correlation calculations. We post-pone this problem to the chapter treating about the elastic electron scattering.

Let us note that the σ- σ correlation function $T_{\sigma\sigma}$ (the third term in (II.17) can be expressed through nucleon-nucleon spatial correlation function $\rho(1,2)$. Thus in order to estimate the short range effects in $T_{\sigma\sigma}$ one may employ $\rho(1,2)$ taken from other calculations (or experiments). The trouble is however that other model dependent terms in (II.17) cannot be expressed by means of $\rho(1,2)$. The hard-core effects on terms involving nucleon momenta can be evaluated only on the basis of a specific model.

We introduce the correlations into the wave function using the following procedure. Let us consider the wave function of a nucleon-nucleon pair. First, we carry out the so-called Talmi-Moshinsky transformation, i.e. we separate the relative and center-of-mass motions of the pair. Secondly, we modify the wave functions of the relative motion

(II.19)
$$|\widetilde{nlm}\rangle = \tilde{R}_{nl}(r)Y_{lm}(\theta,\phi), \quad \tilde{R}_{nl} = \frac{g(r)}{\sqrt{N_{nl}}} R_{nl}(r)$$
$$N_{nl} = \int_0^\infty dr\, r^2 R_{nl}^2 g^2(r)$$

where the function g(r) which modifies the standard radial oscillator function $R_{nl}(r)$ at short internuclear distances has the following properties

(II.20)
$$g(0) = 0, \quad g(r) \gtrsim 0 \quad \text{for} \quad r \lesssim r_c \qquad \text{and}$$
$$g(r) \lesssim 1 \quad \text{for} \quad r \geq r_h, \quad g(\infty) = 1$$

r_c is here the radius of the hard core, and r_h is the so-called "healing" distance.

In order to simulate the hard-core repulsion between nucleons one can use

(II.21)
$$g(r) = 1 - \exp(-\frac{1}{2}\gamma\alpha^2 r^2)$$

where the correlation parameter γ (α being the oscillator potential parame ter) may be somehow related to the hard-core radius. The form (II. 21) enables us to perform all the integrals analytically.

Fig. 3 shows the result of the calculations of T(q) for O^{16} nucleus[5] with an oscillator potential shell model wave function with and without two--particle repulsive correlations introduced.

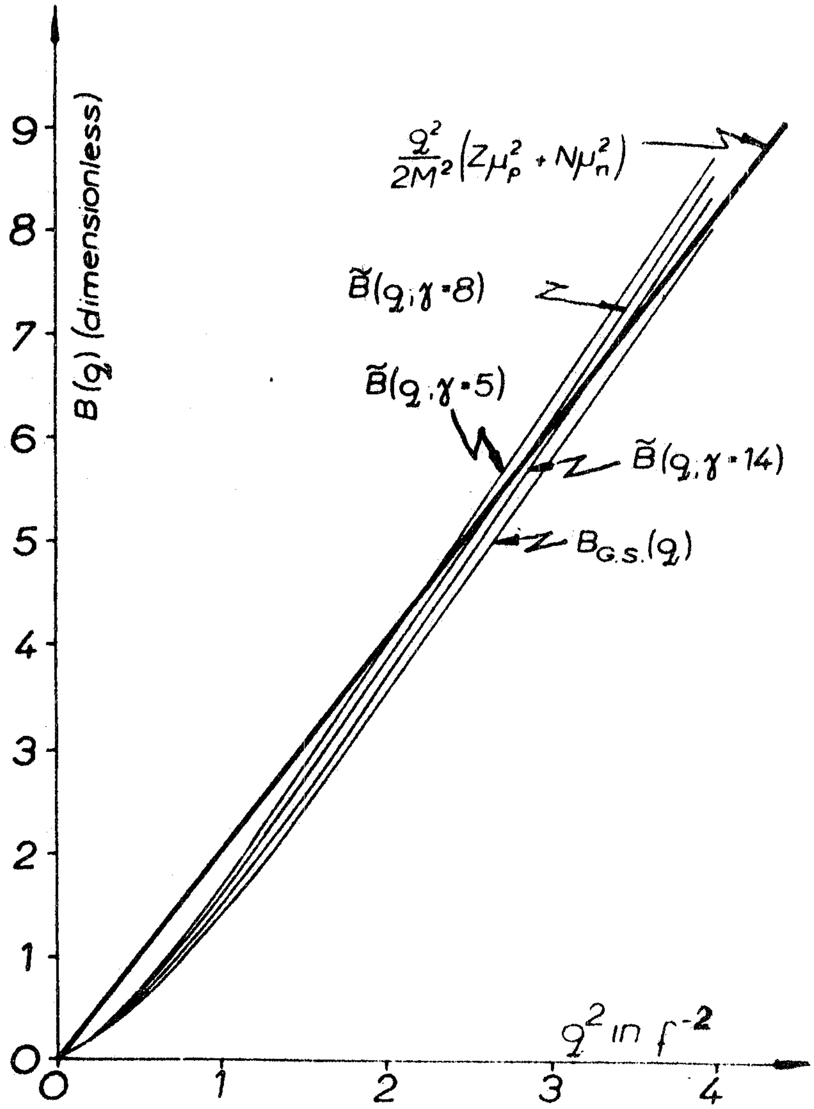


FIG. 3 - The transverse sum rule (denoted here[5] B(q)) for O^{16}. B_{GS} - statistical correlations and the c.m. motion correction included. $\tilde{B}$ (γ=5, 8, 14) repulsive short range correlations between nucleons included.

The striking feature of Fig. 3 is the dominance of the term coming from the magnetic moments: $q^2/2M^2\,(Z\mu_p^2+N\mu_N^2)$. The dominance is so strong that even drastic changes of the nucleon-nucleon correlations do not matter very much. For instance at $q=1.5\ fm^{-1}$ the leading term contributes almost 100% and at $q=2\ fm^{-1}$ about 95% to the sum rule. The transverse sum rule is, for large q's, very sensitive to the effective magnetic moment parameter in the current (I.1).

If one could measure T(q) with good accuracy one would have an important information about the magnetic properties of nucleons bound in nuclei. E.g. an effect of 5% or 10% "quenching" of magnetic moments would produce about 10% and 20% changes of the sum rule.

The short-range internucleon correlations do not influence very much T(q). The most important correlation correction comes from the $T_{\sigma\sigma}$ function. The hard-core effects may be evaluated here with a great confidence, as $T_{\sigma\sigma}$ can be expressed through nucleon pair correlation function $\rho(1,2)$, and one may employ $\rho(1,2)$ obtained from various calculations. Then there would be no need for calculating correlation correction on the basis of a specific model. The analysis would have virtually no free parameters except the parameters of the nuclear current. The precise measurements of the T(q) sum rule devised to obtain data on the magnetic properties of bound nucleons seem, therefore, to represent a very attractive problem.

High resolution transverse sum rule.-

In this sections we describe a way of analysing the transverse sum rule which is adopted to inelastic electron scattering experiments with good energy resolution, and devised for investigating groups of discrete nuclear levels like e.g. the Giant Dipole Resonance multiplet.

Let us remind the expression for the transverse form factor see Eq. (I.2), (I.3)

(II.22) $$T(\omega,q)=\bar{\sum_i}\sum_f\delta(\omega-E_f+E_i)(\vec{J}^{\,x}_{fi}\cdot\vec{J}_{fi})_\perp$$

Precisely speaking the nuclear current in (II.22) differs by a factor $f(q^2_\mu)$ from that in (I.2).

Employing the multipole expansion (I.4) one obtains the transverse sum rule in the form:

(II.23) $$T(q)=\frac{4\pi}{2J_i+1}\sum_f\sum_{j=1}^{\infty}\left[\left|\langle f,J_f\|T^{el}_J\|i,J_i\rangle\right|^2+\left|\langle f,J_f\|T^{mag}_J\|i,J_i\rangle\right|^2\right.$$

where the summation over f extends over all final states, J_i and J_f are the angular momenta of the initial and final nuclear states and the indices i and f specify the remaining quantum numbers of the initial and final states.

We shall consider only spin zero nuclear ground states. Then the Wigner-Echart theorem gives us

$$\langle f, J_f \| T_J^{\substack{el\\mag}} \| i, 0 \rangle = \sqrt{2J+1} \langle f, J_f | T_{J0}^{\substack{el\\mag}} | i, 0 \rangle \tag{II.24}$$

Suppose[6] that in an experiment one can identify the transitions of a chosen multipolarity, and that the energy avaiable in this experiment (notice that we have always $\omega \leq q$) covers all possible excitations of the multipolarity chosen. Then just one term of the sum over J's in (II.23) corresponds to the experimentally performed sum over all excitations of given multipolarity. We can, applying (II.24) and closure, write the following sum rule for it:

$$T^{\substack{el\\mag}}(J;q) = 4\pi (2J+1) \langle i, 0 | T_{J0(q)}^{+\substack{el\\mag}} \; J_{J0(q)}^{\substack{el\\mag}} | i, 0 \rangle \tag{II.25}$$

It is by no means obvious to what an extent the requirements specified above can be met in the present day experiments. Probably the clearest situation exists for the E1 excitations. It is known that E1 transition strength is concentrated around the so-called Giant Dipole Resonance at excitation energies around 15-25 MeV. This may suggest that one has in this case an effectively complete set of find states at disposal-even if one works at small, say $\gtrsim$ 50 MeV, momentum transfers.

It is much harder to tell whether the other multipoles have their transition trengths concentrated at low enough excitation energies to be tractable in the spirit of the sum rule at small q. We shall therefore concentrate on the E1 transition, although the analysis presented below can be easily extended to other multipole transitions.

Before going to concrete calculations let us say a few words about the Giant Dipole states[7]. The Giant Resonance was first seen in photonuclear reactions such as (γ, p), (γ, n) and in total photoabsorption. Increasing the photon energy one first observes the relatively weak and narrow magnetic levels (M1), and at about 20 MeV for light nuclei (15 MeV for heavy nuclei) the cross-section suddenly rises to tens or even hundreds of millibarns. The cross-section curve forms a broad bump (width of several MeV) which, depending on the nucleus, may show considerable fine structure. Above the Giant Resonance, the cross-section fulls off again and shows little structure.

Starting from 0^+ ground state such as in C^{12} or 0^{16} the Giant Resonance states must be 1^- since only the electric dipol (E1) can lead to as large a transition strength as is observed. Moreover one can show that the condition $\Delta T=0$ (T being the total isospin quantum number) should be satisfied. As the photoexcitation and electron scattering are the $\Delta T_3=0$ processes the Giant Resonance states are characterized by $1^-, T=1, T_3=0$, if the ground state is $0^+, T=0$. We see that the Giant Resonance represents an isotriplet whose $T_3=\pm 1$ components lie in neighbouring T=1 nuclei. These "analog" giant resonances may be excited by $\Delta T_3=1$ processes such as muon or pion capture or neutrino absorption.

The large cross-section of the Giant Resonance suggests that the excitation is a result of a collective motion of the nucleus, producing the large dipol moment. Goldhaber and Teller(8) described therefore the excited state as anharmonic vibration of the protons as a whole against the neutrons as a whole. This model can be generalized in order to account for the spin flip transitions which may be important for electron scattering with large q's. In the generalized Goldhaber-Teller(7) model collective vibrations of nuclear matter do not involve two fluids only, those of protons and neutrons, but involve four fluids, those of protons with spin up, protons with spin down, neutrons with spin up and neutrons with spin down. Possible modes of nuclear vibrations are given by the in - phase displacements of any two of these four fluids against the remaining two fluids.

Another description of the Giant Resonance states is based on the shell model. The shell model gives a much more detailed picture; it furnishes the energies and the widths of states. The independent-particle-model picture of the Giant Resonance states as originally used by Wilkinson(9) describes the $1^-, T=1$ states being created by removing a nucleon from a filled shell and raising it to a higher unfilled shell of opposite parity. Elliott and Flowers(10) improved the model by assuming a "residual interaction" between the raised particle in the higher shell, and the hole it left behind in the filled shell. This interaction is taken as the empirical nucleon-nucleon force.

Let us turn now to the discussion of our high resolution sum rule for the E1 excitations. There exist several measurements of the so-called form factor of the Giant Dipole State:

(II.26) $$\Phi(1^-, T=1;q) = \frac{1}{4\pi} T^{el}(1;q)$$

These measurments were succesfully compared with the particle-hole description of the Giant Dipole State. We shall see(6) that applying the sum rule approach the experimental data can be equally well explained without resorting to any detailed calculations of the particle-hole states and their interactions. As the particle-hole calculations become in practice quite involved we would like to stress the usefulness of the sum rule method which reduces the calculations of the form factor Φ to a few easy manipulations.

There are two ways of calculating the Giant Dipole State form factor

i) One calculates $\Phi(1^-, T=1;q)$ straight from the ground state expectation value expression with the electric dipole multipole.

As we consider nuclei with the $0^+, T=0$ states, and the final states are $1^-, T=1$ one obtains from (II.25) and (II.26)

$$\Phi(1^-, T=1;q) = 3\langle 0^+, T=0 \left| T^{el+}_{10,v}(q) T^{el}_{10,v}(q) \right| 0^+, T=0 \rangle \tag{II.27}$$

where the subscript v labels the isovector part of the multipole operator.

For the O^{16} nucleus employing the oscillator well shell model wave function one obtains[6]:

$$\Phi(1^-, T=1;q) = \frac{1}{\pi}\left[\frac{\omega_o}{M} F_1^2 + \frac{(\mu_p - \mu_n)^2}{8}\left(\frac{q}{M}\right)^3 \frac{q}{\omega_o} F_2^2\right] \tag{II.28}$$

where $\omega_o = \alpha^2/M$ (oscillator spacing) and

$$F_1^2 = 1 - 0.75\frac{q^2}{\alpha^2} + 0.3375\frac{q^4}{\alpha^4} - \ldots\ldots$$
$$F_2^2 = 1 - 0.75\frac{q^2}{\alpha^2} + 0.3125\frac{q^4}{\alpha^4} - \ldots\ldots \tag{II.29}$$

Thus for small q's F_1^2 and F_2^2 are virtually identical. Notice that for large momentum transfers (II.28) contains contributions from different than Giant Dipole States excitations, especially the quasi-elastic scattering contribution. Hence for large q's (II.28) cannot be interpreted as the form factor of the Giant Dipole Resonance.

ii) We are going now to describe the E1 excitations in terms of the collective dipole oscillations following thus the main idea of the Goldhaber-Teller model. We short with the isovector part of $T^{el}_{10}(q)$ for small q's

(II.30)
$$T^{el}_{10,v}(q) = \frac{1}{2\sqrt{6\pi}M}\left[\sum_{j=1}^{A}\tau_{3j}\,p_{jz} + (\mu_p - \mu_n)\frac{q^2}{2M}\sum_{j=1}^{A}\tau_{3j}\cdot (x_j\,\sigma_{yj} - y_j\,\sigma_{xj})\right]_{q^2\to 0}$$

In order to obtain collective oscillations of various groups of nucleons with respect to each other it is essential to have electromagnetic interaction, which induces them, linear in, r_j's. Then the collective coordinates can be introduced and only relative oscillations of the centers of mass of the whole groups of nucleons excited. In (II.30) we therefore neglected a contribution $\sim q^2 r_j^2$. The neglected term destroys the coherence of the collective oscillations and make them decay. The decay will be introduced into the picture in another way.

One can obtain the Φ form factor in the limit $q^2 \to 0$ employing the following sum rules:

(II.31)
$$\sum_{|f\rangle} (E_f - E_i) \left|\langle f|\sum_{j=1}^{A} \tau_{3j} z_j|i\rangle\right|^2 = \frac{A}{2M}$$

$$\sum_{f} (E_f - E_i) \left|\langle f|\sum_{j=1}^{A} \tau_{3j}(x_j\,\sigma_{yj} - y_1\,\sigma_{xj})|i\rangle\right|^2 = \frac{A}{M}$$

These sum rules one gets from the well known identity with the double commutator:

$$\langle i|\left[\theta^+, [H,\theta],\right]|i\rangle = 2\sum_{|f\rangle} (E_f - E_i) \left|\langle f|\theta|i\rangle\right|^2$$

which is correct for operators satisfying the relation

$$\left|\langle f|\theta^+|i\rangle\right| = \left|\langle f|\theta|i\rangle\right|,$$

and under the assumption that the interaction part of the Hamiltonian H is spin and isospin independent.

If we further assume that the transition strength is concentrated around certain energy ω_R we get from (II.30) and (II.31)[6]:

(II.32)
$$\Phi(1^-, T=1; q)_{q^2\to 0} = \frac{A}{16\pi}\left[\frac{\omega_R}{M} + \frac{(\mu_p - \mu_n)^2}{8}\left(\frac{q}{M}\right)^3 \frac{q}{\omega_R}\right]$$

where we have used the identity:

$$\langle f | p_{jz} | i \rangle = i M (E_f - E_i) \langle f | Z_j | i \rangle .$$

The formula (II.32) does not take into account the fact that the collective oscillations caused by the two terms of (II.30) (neutrons vs protons and neutrons with spin down + protons with spin up vs neutrons with spin up + protons with spin down) can be destroyed. The probability that the oscillating spheres of e.g. protons are not destroyed is given by the elastic form factor (normalized to unity at q=0).

So, we can easily improve (II.32) and write

$$\Phi(1^-, T=1; q) = \frac{A}{16\pi}\left[\frac{\omega_R}{M} + \frac{(\mu_p - \mu_N)^2}{8}\left(\frac{q}{M}\right)^3 \frac{q}{\omega_R}\right] F_{el}^2 \tag{II.33}$$

where we assumed that the two elastic form factors are the same and equal to the elastic electron scattering form factor F_{el}.

For 0^{16} the oscillator potential shell model wave function gives

$$F_{el}^2(q) = \left(1 - \frac{1}{8}\frac{q^2}{\alpha^2}\right)^2 \exp\left(-\frac{1}{2}\frac{q^2}{\alpha^2}\right) = 1 - 0.75\frac{q^2}{\alpha^2} + 0.266\frac{q^4}{\alpha^4} - \ldots . \tag{II.34}$$

Let us compose now for the 0^{16} nucleus, the results for the Giant Dipole State form factor obtained in i) and ii). See Eq. (II.28) and (II.33).

They are identical for small q's provided one put $\omega_R = \omega_o$. The calculated form factor is with $\omega_o \simeq 15$ MeV in good agreement with experimental data presented in Fig. 4.

On the other hand, if we used in (II.33) $\omega_R = 22$ MeV (the experimentally measured mean excitation energy of the Giant Dipole Resonance for 0^{16}) the agreement with experiment would become very poor.

We would like, however, to stress the point that in sum rules (II.31) which are based on the assumption of pure Wigner forces we should use $\omega_R = \omega_o$ (the oscillator spacing), and not the experimental ω_R. For the oscillator potential ground state and the operators

$$\sum_j^A \tau_{3j} Z_j \qquad \text{and} \qquad \sum_{j=1}^A \tau_{3j} (x_j \sigma_{yj} - y_j \sigma_{xj}) ,$$

the states $1^-, T=1$ form an effectively complete set of states only if they are the excited states of the same oscillator potential, hence are separated from the ground state by ω_o. Thus we obtain a consistent model of the form factor although the model does not reproduce the correct position of the Giant Dipole Resonance.

The following concluding remarks are in order here.

a) As the results obtained, for low q's, from the evaluation of the ground state expectation value of the complete multipole moment and from the application of the sum rules (II.31) are so much alike, we would like to stress the usefulness of the second method which gives the form factor Φ nearly without any evaluation.

b) The presented here analysis of the Giant Dipole Resonance Figs. 4 and 5, shows that the existing experimental data are in agreement with the sum rule approach which exhibits these features of the collective excitation which are independent on the details of the microspic theory of collective dipole vibrations. It would be very desirable to have more complete measurements of the Φ form factors, especially for large momentum transfers.

General sum rules. -

So far we have constructed the sum rules in such a way that in the theoretical prediction for the sum $d\omega\sigma(q, \theta, \omega)W(\omega)$ we had no terms depending on energy loss ω. This enabled us to use closure and eliminate thus complicated (or unknown) final state wave functions from the calculations.

However in more general cases ω-dependent terms will occur in the sum over final nuclear states and the closure relation cannot be used.

One can avoid this difficulty using the "closure approximation"

$$\text{(II.35)} \qquad \sum_{|f\rangle}^{\text{inel.}} g(\omega) \left| \langle f | \theta | i \rangle \right|^2 = \langle g \rangle \langle i \, \theta^+ \theta \, | i \rangle - g(0) \left| \langle i | \theta | i \rangle \right|^2$$

where $\langle g \rangle$ is a suitable average of an energy loss dependent function $g(\omega)$; θ being an operator.

Thus the powers of ω will be replaced by certain estimated averages e.g. $\omega \rightarrow \langle \omega \rangle = q^2/2M$ (from quasi-elastic scattering).

Another more complicated but exact method is to remove the -dependent terms using (n times) the identity[14]

$$\text{(II.36)} \qquad \langle f | \omega_f^n \theta | i \rangle = \langle f | \omega_f^{n-1} [H, \theta] | i \rangle ,$$

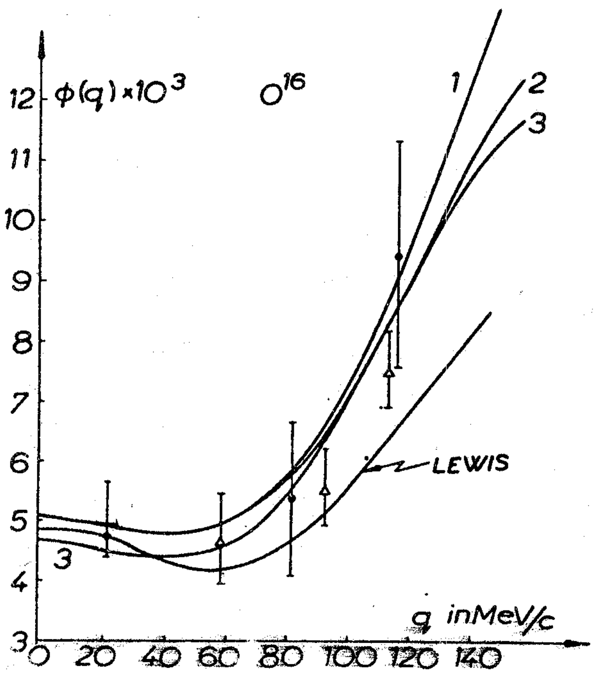


FIG. 4 - Electron scattering form factors of the 1^-, T=1 states for O^{16}. The curve 1 is given by E1 multipole expectation value (II. 28). The curves 2 and 3 have been obtained from (II. 33). The curve 1 and 2 are plotted for $\alpha^2 = 0.36\ fm^{-2}$. The same value was used by Lewis(11) in his particle-hole description of the Giant Dipole State. The curve 3 is plotted for $\alpha^2 = 0.334\ fm^{-2}$. The experimental data are those reported in Ref. (6).

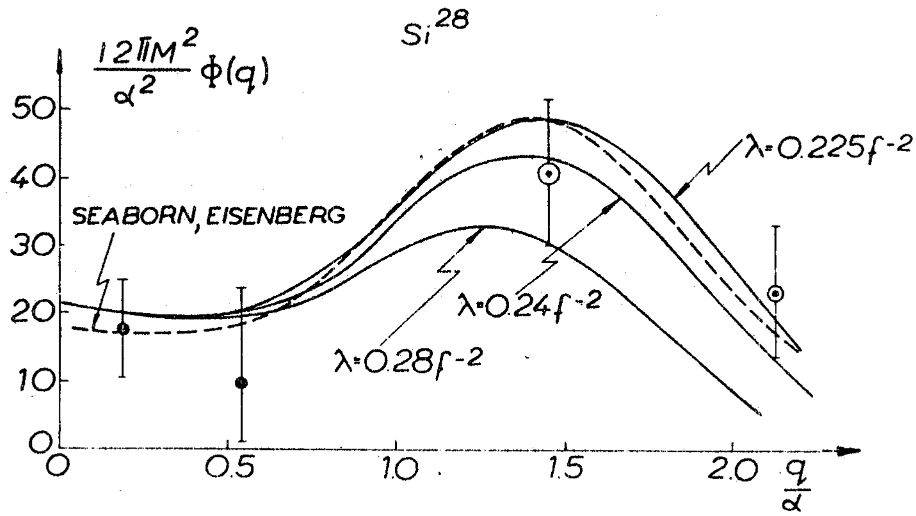


FIG. 5 - The Giant Dipole Resonance States form factors for Si(28). Form factors obtained from (II. 33) for three values of $\lambda = \alpha^2$ as indicated are compared with the result of Seaborn and Eisenberg(12) (particle-hole model) and the experimental data. The data are those reported in Ref. (6).

where H is the internal nuclear Hamiltonian

$$H = \sum_{j=1}^{A} \frac{p_j^2}{2M} + \sum_{i>j}^{A} V(i,j) ,$$

($\omega_f = \omega - \frac{q^2}{2AM}$ being the nuclear excitation energy);

and then apply closure.

In principle, using (II.36) one can construct various sum rules with different weighting factors $W(\omega)$. The price one pays for it is considerable complication of the formulas. Moreover one may worry about an uncertainty of the procedure(15) if the ground state wave function one uses to evaluate the expectation values is not completely consistent with the Hamiltonian.

The only case where one can confidently use this method is the Deuteron(15) where one calculates the ground state wave function from a given Hamiltonian. Therefore it seems to be especially interesting to study the sum rules for H^2 as carefully as possible both experimentally and theoretically.

In particular one may expect that the ω_f-weighted sum rule

$$\text{(II.37)} \qquad \tilde{\sigma}_\omega = \sum_{|f\rangle} \omega_f \left| \langle f | \theta | i \rangle \right|^2 = \langle i | \theta^+ H \theta | i \rangle$$

should be very sensitive to the "gauge" currents which arise from the presence of the charge exchange interaction in the Hamiltonian (in order to maintain the continuity equation). This was pointed out by Drell and Schwartz(14) who found the exchange effects in the energy-weighted cross--section to contribute up to 40% of the result.

III. - ELASTIC ELECTRON SCATTERING FROM NUCLEI. -

In elastic scattering both the initial and final nuclear states are the ground state: $|f\rangle = |i\rangle$. Therefore, electron scattering can tell us about the nuclear ground state charge distribution and other electromagnetic (quadrupole, magnetic dipole, magnetic octupole moments) properties of nuclei. We will also indicate that the elastic electron scattering can be a usefull tool for studying nucleon-nucleon correlations in the ground state.

The selection rules for the case of elastic scattering are more stringent than for the general case. We have $J_i = J_f$ and no change of parity. The parity eliminates (see Table I) the odd Coulomb, odd electric and even magnetic multipoles. Invariance of the theory under time reversal eliminates the even transverse electric multipoles. Thus in elastic scattering the only multipoles are the even Coulomb C0, C2, ... and odd magnetic multipoles M1, M3, ... Angular momentum conservation gives another restrictions: C2 if $J_i \geq 1$, M1 if $J_i \geq 1/2$, M3 if $J_i \geq 3/2$, etc.

We will confine ourselves to the discussion of electron elastic scattering from spin zero (J_i=0) nuclei. In this case only the monopole moment of the charge density (C0) can contribute.

We have for spin zero nuclei in the first Born approximation:

(III. 1)
$$\frac{d\sigma}{d\Omega}_{el} = \sigma_M Z^2 \left| F_o(q) \right|^2$$

where

$$\sigma_M = \frac{e^4 \cos^2\theta/2}{4 \varepsilon^2 \sin^4\theta/2} \left(1 + \frac{2\varepsilon}{AM} \sin^2\theta/2\right)^{-1}$$

and

(III. 2)
$$F_o = \frac{\sqrt{4\pi}}{Z} \langle i \| \hat{C}_o(q) \| i \rangle = \frac{1}{Z} \int d^3 r\, j_o(qr) \langle i | \hat{Q}(\vec{r}) | i \rangle$$

F_o is called the charge (monopole) elastic form factor of the target nucleus.

Let us define the nuclear charge density (spherically symmetric) per particle as follows:

(III. 3)
$$\rho_{ch}(r) = \frac{1}{Z} \langle i | \hat{Q}(\vec{r}) | i \rangle$$

Thus in the first Born approximation one has a direct relationship between the density distribution and the form factor (or cross-section).

The elastic scattering measurments can always be interpreted phenomenologically choosing ad hoc nuclear charge distribution $\rho_{ch}(r)$ which fits the data for the nucleus in question.

This is true also for heavy nuclei where the Born approximation is expected to break down. In that case a more sophisticated analysis is necessary. One can start with a particular density distribution $\rho_{ch}(r)$ and calculate the corresponding electrostatic potential V(r) given by Poisson's equation. Then one solves the Dirac equation for an electron moving in the potential V(r). This can be done exactly by means of a phase shift analysis[16] of each partial wave. The phase shifts δ_j determine the scattering amplitude $f(\theta)$.

In this case no simple relationship exists between the density distribution and the scattering cross-section. In general the only way to see what effect on the cross-section has a small change in the charge distribution is to go through the whole calculation:

$$\rho(r) \to V(r) \to \delta_j \to \frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

again for a slightly changed distribution. The simple connection between $\rho_{ch}(r)$ and $d\sigma/d\Omega$ given by the Born approximation can nevertheless be used as a guide for finding a suitable charge density distribution.

The main conclusion from many experiments on elastic scattering on spin zero nuclei is that the ground-state charge distribution is well fitted (for heavy enough nuclei) by the Fermi distribution

(II.4) $$\rho(r) = \frac{\rho_o}{1+\exp\left(\frac{r-R}{a}\right)}, \qquad R \gg a$$

The parameters of the distribution show[2,16] the following systematic behaviour in nuclei with $20 < A < 208$:

i) The radius to half the maximum ρ is given by $R=r_o A^{1/3}$, $r_o=$ $=1.07$ fm.

ii) A ρ_o, the central nuclear density is a constant $\approx 0.17\ \text{fm}^{-3}$

iii) The surface thickness s defined to be the distance over which the charge density falls from 90% to 10% of the central density is a constant: $s = 4\ a \ln 3 = 2.4$ fm. The two last features can be qualitatively understood as a consequence of the short range nature of nuclear forces[16]. Because of this a nucleon in the central region of a nucleus is unaffected by surface effects and thus the central nuclear density approaches that for infinite nuclear matter. On the other hand, nucleons at the surface are unaffected by the bulk of the nucleus, so that surface features are nearly independent of the size of the nucleus. These consideration naturally do not apply to very light nuclei.

The Fermi distribution (III.4) represents average properties of the nuclear density for many nuclei. The finer details of the ground state charge distribution have also been studied.

We mention the recent Stanford experiment[17] on the elastic scattering of 750 MeV electrons from calcium isotopes. It was found that a charge distribution $\rho_o(r)$ obtained by analyzing the scattering data at 250 MeV was quite inadequate at 750 MeV to explain the experimental results in the region of large momentum transfer $q \gtrsim 500$ MeV. An oscillating function, $\Delta\rho(r)$, had to added to the charge distribution $\rho_o(r)$ to obtain a good fit at 750 MeV. It was suggested that such a modulating correction corresponds at least qualitatively to filling shells in the shell model and thus reflects the shell structure of the nucleus.

So far we have interpreted the elastic electron scattering in terms of a phenomenological ground state charge density. A more ambitious approach should be based on the wave function of the ground state. Even if we use an approximate nuclear wave function given by a simple model (say, independent particle model) such a description represents a more interesting theoretical interpretation.

Let us discuss the elastic electron scattering from light, spin zero nuclei in this spirit. We can write the expression for the charge elastic form factor - see Eq. (III.2) as follows:

(III.5) $$F_{ch} = \frac{1}{Z} \langle i | \hat{Q}(\vec{q}) | i \rangle$$

In order to evaluate the form factor one has to know the nuclear charge density operator $\hat{Q}(q)$ and the nuclear ground state $|i\rangle$.

If we use the McVoy-Van Hove[1] charge operator - see Eq. (I.1) one gets

(III.6) $$F_{ch} = \Phi(q_\mu^2) \frac{1}{Z} \langle i | \sum_{j=1}^{A} e_j e^{i\vec{q}(\vec{r}_j - \vec{R})} | i \rangle , \quad \vec{R} = \frac{1}{A} \sum_{k=1}^{A} \vec{r}_k$$

where

(III.7) $$\Phi(q_\mu^2) = (G_{Ep} + G_{En})(1 + \frac{q_\mu^2}{8M^2})$$

In Eq. (III.7) we have introduced the electric form factors of the nucleons[18]; $\Phi(q_\mu^2)$ represents the correction due to finite sizes of the nucleon.

Let us note that in Eq. (III.6) the nuclear center-of-mass motion has been taken into account by means of the Gartenhaus-Schwartz transformation[4]. Thus, we can calculate the charge form factor by using ground state wave functions which are not translationally invariant.

It is well known that the elastic electron scattering from light nuclei (He^4, C^{12}, 0^{16}) can be well described in the harmonic oscillator potential shell model. In this model the individual nucleons are considered to move in a common oscillator well. The nuclear wave function is constructed from the single-particle oscillator states.

Let us consider a nucleus with two protons in the s-shell and Z-2 protons in the p-shell. Using the oscillator model ground state wave function one obtains the charge form factor:

$$F_{ch}(q) = \Phi(q_{\mu}^2) \exp\left(\frac{q^2}{4A\alpha^2}\right)\left(1 - \frac{Z-2}{6Z}\frac{q^2}{\alpha^2}\right)\exp\left(-\frac{q^2}{4\alpha^2}\right) \qquad \text{(III. 8)}$$

where α is the parameter of the Gaussian factor $\exp(-1/2\ \alpha^2 r^2)$ of the oscillator wave functions. Let us note that the c.m. motion correction reduces in the oscillator well model to a simple factor.

The elastic electron scattering data have been satisfactory interpreted(16) in terms of the oscillator well model for many light nuclei: He^4, Be^9, B^{11}, C^{12}, N^{14}, 0^{16}(x).

The only nucleus for which the harmonic oscillator potential shell model was not succesful is Li^6. It is likely that for this nuclid the p-nucleons are bound a good deal less firmly that in heavier nuclei in the p--shell. This can be simulated by allowing the s - and p - nucleons to move in different potential wells (different oscillator strenghts) as suggested by Elton(16). The model with two different oscillator wells provided a good fit to the experimental data for Li^6.

Recently, the elastic scattering measurements for He^4 and Li^6 have been(19, 20) extended to large momentum transfers. It turned out that the simple shell model with oscillator potential is no more able to explain the experimental results for these nuclei. In the case of the He^4 nucleus a well pronounced minimum of the form factor, at $q^2 \gtrsim 10\ fm^{-2}$, was found. This minimum is in a drastic disagreement with the prediction of the harmonic oscillator shell model.

It was suggested by Czyz and Lesniak(21) that such a minimum may arise from the hard-core repulsion between nucleons at short mutual distances. In fact, taking into account the two-nucleon correlations one can produce the minimum in the elastic form factor and, under suitable conditions (i.e. by choosing suitable parameters), one can well reproduce its position for He^4. Similar attempts to explain the large momentum transfer

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(x) - In the case of Be^9 and B^{11} the quadrupole scattering contribution had to be taken into account because of the large quadrupole moments of these nuclides.

behaviour of the form factor for Ca^{40}(22) and Li^{6}(23) have also been, at least qualitatively, succesful.

We will discuss now the influence of the short range nucleon-nucleon correlations on the elastic form factor in more detail. Let us stress that the short-range effects, e.g. a hard core repulsion between nucleons, may be visible if the momentum transferred to the nucleus is sufficiently high. The usual shell model does not account for possible dynamical nucleon-nucleon correlations. The wave function of the shell model has few high-momentum components. In experiments involving large momenta, high-momentum components of the wave function are important and the usual shell model is expected to break down. The high-momentum components can be introduced into the wave function by taking into account the strong short-range repulsion in the nucleon-nucleon interaction.

We introduce dynamical correlations between nucleons into the shell model wave function Ψ_{SM} employing a unitary(24) operator U:

(III. 9) $$|\tilde{\psi}\rangle = U|\psi_{SM}\rangle , \qquad U^{+} = U^{-}$$

Of course, it is possible to calculate ground-state expectation values (as the elastic form factor) using uncorrelated shell model state if one uses unitary transformed operators, $\tilde{\theta}$, related to the standard operators, θ, by:

(III. 10) $$\tilde{\theta} = U^{+}\,\theta\, U = U^{-}\theta\, U$$

As we will consider the short-range correlations arising from the hard core repulsion between nucleons we can take into account the two-particle correlations only. We neglect then the probability of simultaneous modification of the wave functions of more than two particles as the probability for three and more nucleons to come close together is expected to be small. Speaking the same language in the three-particle correlation approximation we will neglect four-nucleon (and higher) clusters in the system.

In order to account for the two-particle corrections we construct U as a Jastrow product(25) of unitary operators of two particles:

(III. 11) $$U = \prod_{j>k=1}^{A} \mu(j,k)$$

Let us consider the unitary transform of an one-body operator like that in Eq. (III.6)[x].

Because of unitarity one has

$$U^{+} \sum_{j=1}^{A} 0(j)\, U = \sum_{j=1}^{A} \prod_{k(\neq j)}^{A} \mu^{-1}(j,k)\; 0(j)\, \mu(j,k) \qquad \text{(III.12)}$$

as the operators for different particles are naturally supposed to commute.

Assuming now only the two-particle correlations we obtain:

$$U^{+} \sum_{j=1}^{A} 0(j)\, U \cong \sum_{j\neq k}^{A} \mu^{+}(j,k)\; 0(j)\, \mu(j,k) - (A-2) \sum_{j=1}^{A} 0(j) \qquad \text{(III.13)}$$

Where one has subtracted the contributions from these k's which are not "correlated" to a given j.

Proceeding in the same way we obtain in the three-particle correlation approximation

$$\begin{aligned} U^{+} \sum_{j=1}^{A} 0(j)\, U \cong \frac{1}{2} \sum_{j\neq k\neq i}^{A} \mu^{+}(j,k,i)\; 0(j)\, \mu(j,k,i) \\ -(A-3) \sum_{j\neq k}^{A} \mu^{+}(j,k)\, 0(j)\, \mu(j,k) + \frac{1}{2}(A-3)(A-2) \sum_{j=1}^{A} 0(j) \end{aligned} \qquad \text{(III.14)}$$

where the correlation operator was assumed in the form:

$$U = \prod_{j>k>i}^{A} \mu(j,k,i) = \prod_{j>k>i} \left[\mu(j,k)\, \mu(j,i)\, \mu(k,i) \right] \qquad \text{(III.15)}$$

(x) - We make here use of the fact that in the simple oscillator shell model the nuclear c.m. motion correction just factors out. Let us stress however, that in the model with two different oscillator wells, the c.m. correction becomes algebraically quite complicated[26].

Let us turn now to the elastic charge form factor. In the oscillator shell model we can write −see Eq. (III. 6)−the "correlated" form factor, as follows:

$$\tilde{F}_{ch} = \Phi(q_{\mu}^2)\exp\left(\frac{q^2}{4A\alpha^2}\right)\left[F_{SM} + \Delta F\right] \tag{III. 16}$$

where F_{SM} is the so-called shell model form factor

$$F_{SM} = \frac{1}{Z}\langle \Psi_{SM}|\sum_{j=1}^{A} e_j\, e^{i\vec{q}\cdot\vec{r}_j}|\Psi_{SM}\rangle \tag{III. 17}$$

and ΔF is the correction to F_{SM} which accounts for nucleon-nucleon corre lations.

Let us calculate the two-particle correlation correction to the ela stic form factor. Applying (III. 13) one obtains from (III. 17):

$$\Delta F = \frac{1}{Z}\sum_{\alpha B}\Big[\langle\widetilde{\alpha B}(1,2)|\, e_1\, e^{i\vec{q}\cdot\vec{r}_1}|\widetilde{\alpha B}(1,2) - \widetilde{B\alpha}(1,2)\rangle - \langle\alpha(1)B(2)|\, e_1\, e^{i\vec{q}\cdot\vec{r}_1}|\alpha(1)B(2) - B(1)\alpha(2)\rangle\Big] \tag{III. 18}$$

where the summation extends over all occupied single-particle states of the shell-model.

In (III. 18) we have introduced the correlated two-particle states:

$$|\widetilde{\alpha B}(1,2)\rangle = \mathcal{M}(1,2)\,|\alpha(1)\rangle\,|\,B(2)\rangle \tag{III. 19}$$

Performing in Eq. (III. 18) the summation over the spin and isospin quantum numbers one has:

$$\Delta F = \frac{2}{Z}\sum_{ab}\Big[4\,\Delta\langle ab|e^{i\vec{q}\cdot\vec{r}_1}|ab\rangle - \Delta\langle ab|e^{i\vec{q}\cdot\vec{r}_1}|ba\rangle\Big] \tag{III. 20}$$

where $\Delta(\ldots)$ denotes the difference between correlated and uncorrelated magnitudes. The single particle spatial quantum numbers we denote a, b...; the one-particle orbital state is $|a\rangle = |n_a l_a m_a\rangle$.

In the case of harmonic oscillator wave functions it is possible to define a transformation from motion of two particles about a common center to the relative and c.m. motion of the two particles. Following Moshinsky[27] this transformation may be written:

$$(III.21) \qquad |n_1 l_1, n_2 l_2, \lambda, \mu\rangle = \sum_{nlNL} \{nl, NL, \lambda | n_1 l_1, n_2 l_2 \lambda\} |nl, NL, \lambda\mu\rangle$$

where (nlm) are the quantum numbers of relative motion and (NLM) are the quantum numbers of the c.m. motion.

We introduce the two-particle correlations in (III.20) be modifying (in a unitary way thus preserving normalizations and orthogonalities[5]) the radial functions of the relative motion of a nucleon-nucleon pair:

$$(III.22) \qquad |\widetilde{nlm}\rangle = \tilde{R}_{nl}(r) Y_{lm}(\theta, \phi), \qquad \tilde{R}_{nl}' = \frac{g(r)}{\sqrt{N_{nl}}} R_{nl}(r)$$

$$N_{nl} = \int_0^\infty dr\, r^2 R_{nl}^2 g^2(r)$$

where g(r) is the "correlation" function which modifies the standard radial oscillator function $R_{nl}(r)$ at short internucleon distances.

Employing the Moshinsky technique[27] one obtains from (III.20) the following two-particle correlation correction to the elastic form factor:

$$(III.23) \qquad (\Delta F)_s = \frac{1}{Z} \exp(-t) \left\{ \left[6(Z-1)-(Z-2)(6-t)t\right] \cdot \Delta\langle 000| \exp(\frac{iqZ}{\sqrt{2}}) |000\rangle + \right.$$

$$\left. + \frac{1}{2}(Z-2)\Delta\langle 100| \exp(\frac{iqZ}{\sqrt{2}}) |100\rangle - (\frac{2}{3})^{1/2} t \cdot \Delta\langle 100| \exp(\frac{iqZ}{\sqrt{2}}) |000\rangle \right\}$$

where

$$t = q^2 / 8\alpha^2.$$

The formula (III.23) is valid for nuclei with two protons in the s-shell and Z-2 protons in the p-shell. It was assumed here that the short range correlations act on the relative s-states only[(x)].

One can derive a more exact formula by taking into account not only correlations in the s-state of relative motion, but those in all possible states[(28)]. The case of two different oscillator wells for s-and p-nucleons can also be included[(26)].

In order to evaluate the expressions in Eq. (III.23) we must assume a form for g(r). We write

$$g^2(s) = 1 - \exp(-\frac{1}{2}\Lambda^2 s^2) \tag{III.24}$$

This form enables us to perform all the integrations in (III.23) analytically. Such a correlation function represents the soft-core repulsion between nucleons at small relative distance s. The correlation parameter Λ may be somehow related to the hard-core radius, however, the relation is rather ambiguous.

In Figs. 6-8 we have presented, for some light nuclei, the elastic form factors corrected for the short-range nucleon-nucleon correlations. These are compared with the uncorrelated form factors and the experimental results.

In the case of He^4 (see Fig. 6) we have evaluated the effect of the two (curve 2) and three (curve 3) particle correlations; curve 0 represents the uncorrelated form factor obtained from the standard harmonic oscillator shell model. By introducing the correlations we are able to explain the existence of a diffraction minimum although the height of the second maximum is not well reproduced. Probably one should also consider a long-range correlation (change of the shape of the potential well) in order to get a good fit over the whole range of momentum transfer. Comparison of the curves 2 and 3 for He^4 shows that the effect of three-particle correlations in this small nucleus is particularly important.

The short-range nucleon-nucleon correlations seem to be very important also in the Li^6 nucleus[(+)]. We were able[(26)] to obtain a good fit to the experimental data for this nuclid in the model with two different oscil

(x) - Let us note that the oscillator function of the relative s-state does not vanish for r=0. This is in a drastic disagreement with the supposed hard-core repulsion between nucleons. The correlations correction coming from the modification of the relative s-state seems, therefore, to be the most important one.

(+) - Since the quadrupole moment of Li^6 is very small one can describe the elastic electron scattering from this nucleus in terms of the monopole form factor only.

lator wells provided one included the correlations between nucleons-see Fig. 7.

In Fig. 8 we have presented the correlated and uncorrelated charge form factor for C^{12}. One might infer from this figure that the short--range effects seem to decrease with the increase of the mass number. The existing experimental data for C^{12} can be sufficiently well explained in the oscillator shell model without the correlations. It could be qualitatively interpreted that with the mass number increasing, the effect of an average nuclear potential well becomes more and more important. In order to show some details of the fluctuations of nucleons around their average orbits one has to go then to very large momentum transfers.

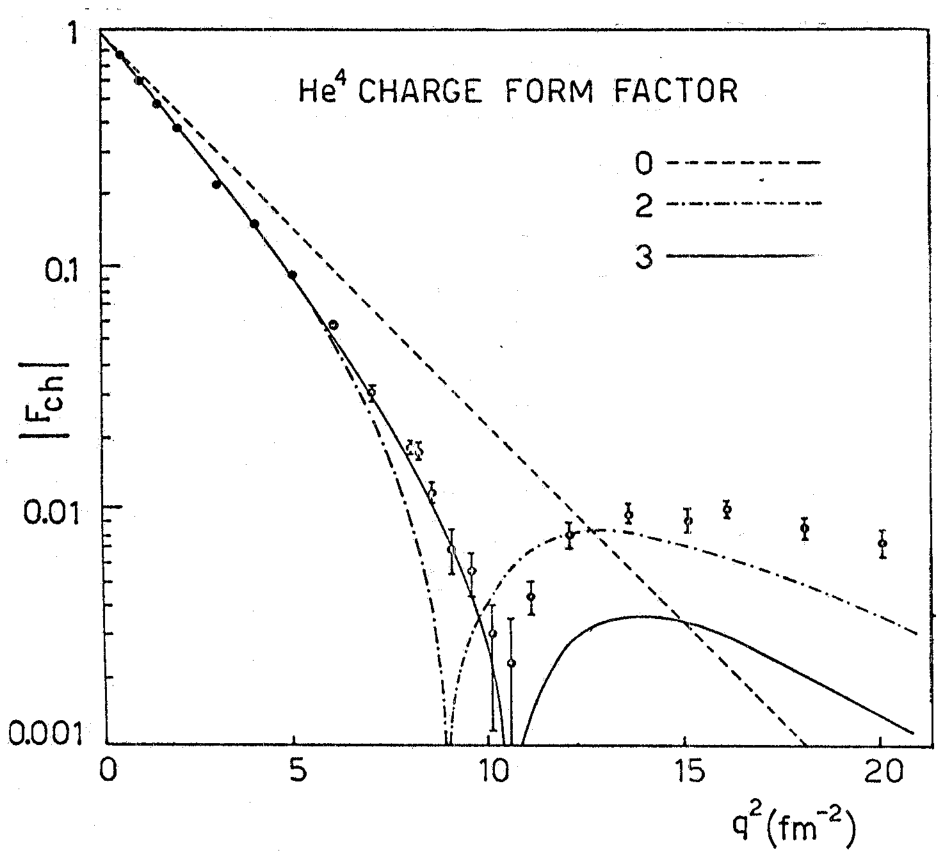

FIG. 6 - Charge form factor of He^4 versus momentum transfer square. Experimental points are from Ref. (19). 0- no correlation, 2- two-particle correlations, 3-three--particle correlations. The oscillator well parameter is α=155 MeV and the correlation parameter Eq. (III. 24) Λ=1.55 fm^{-1}.

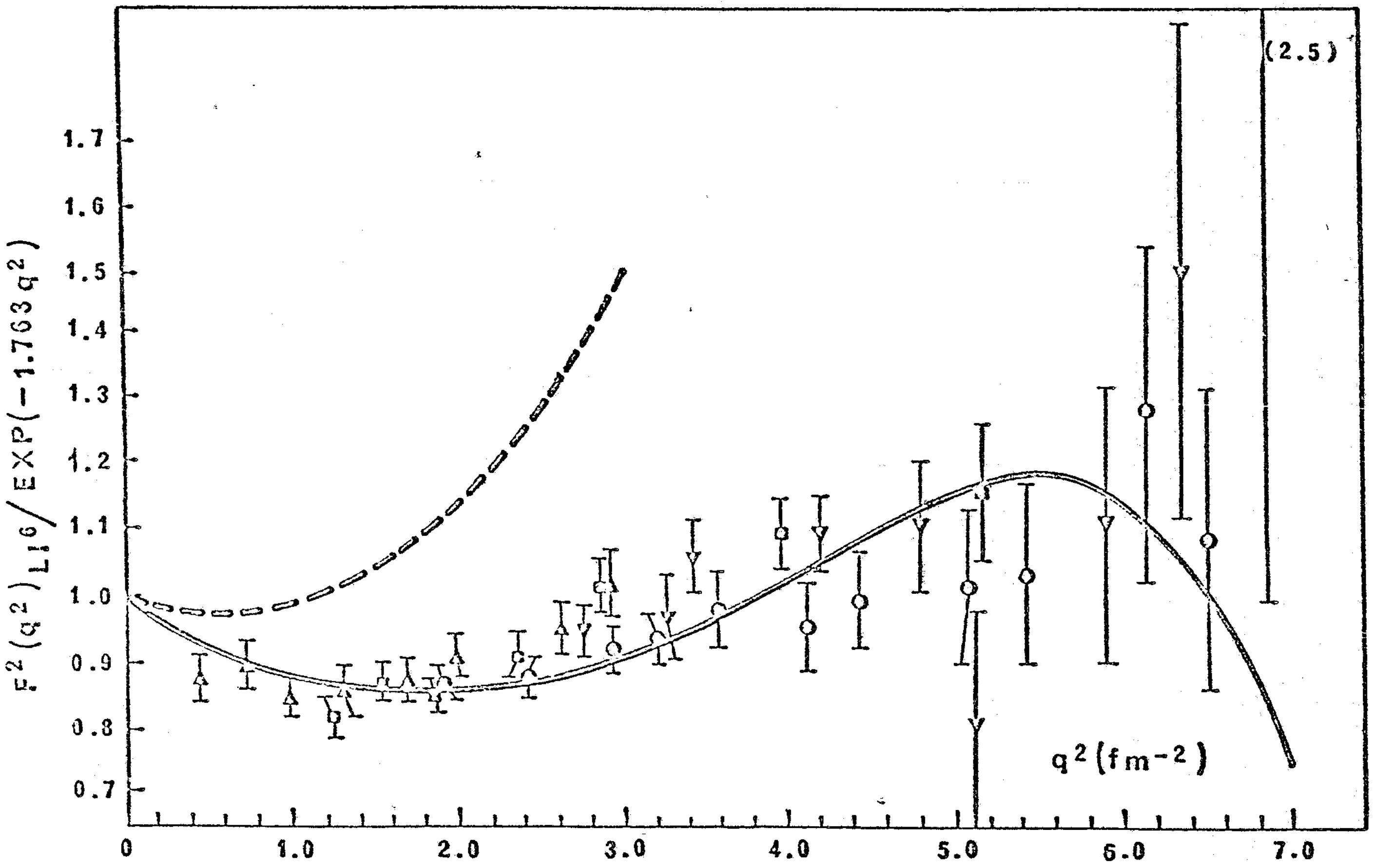

FIG. 7 - Charge form factor of Li^6. Experimental results are from Ref (20). Dashed line-no correlations. Full line-two-particle corre_ lations included. The oscillator parameters are $\alpha_s = 130.517$ MeV $\alpha_p = 98.588$ MeV. The correlation parameter $\Lambda = 1.678\ fm^{-1}$.

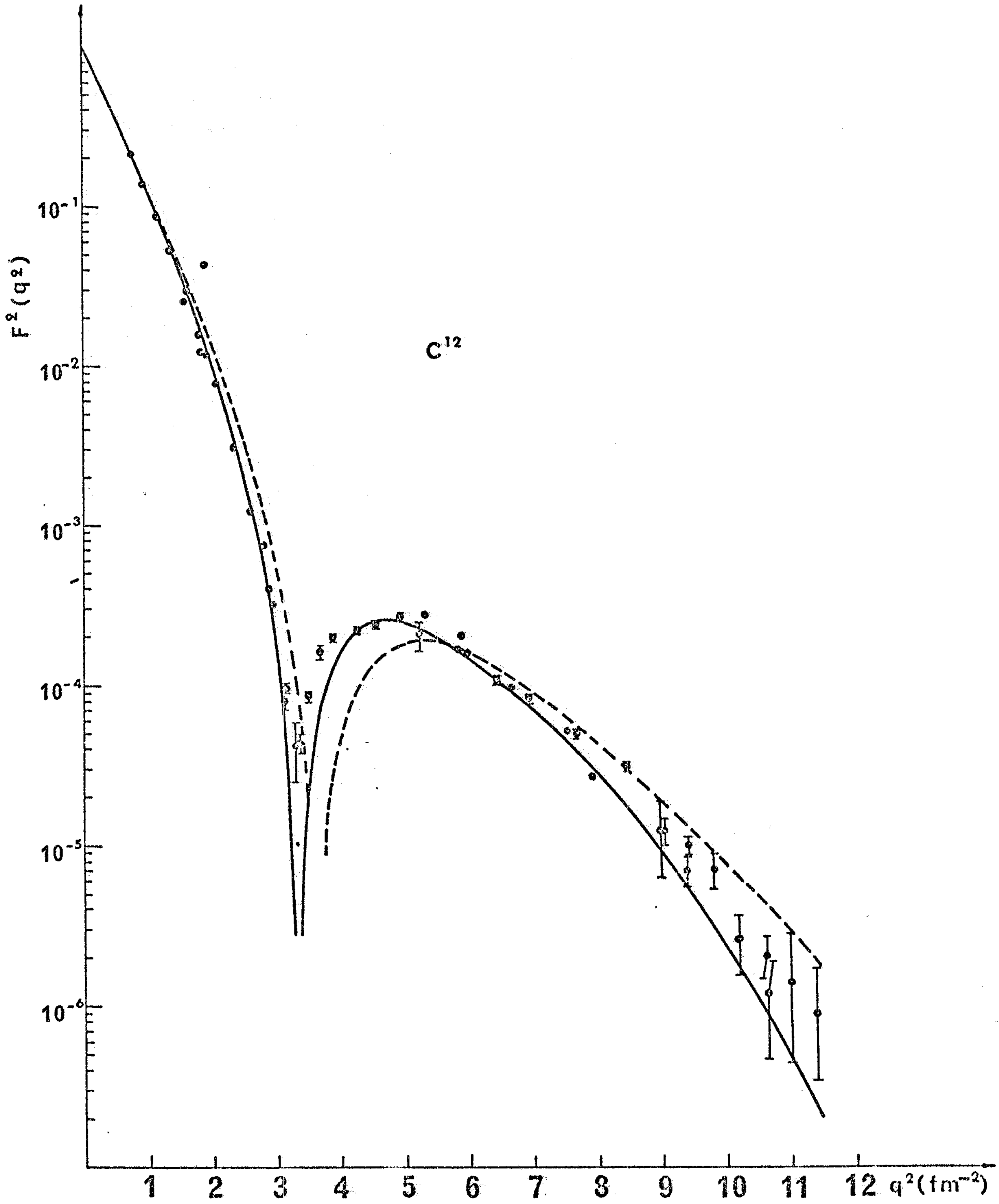

FIG. 8 - Charge form factor of C^{12}. Experimental points are from Ref. (30). Dashed line-no correlations. Full line-two-particle correlations included. The parameters are $\alpha_s = \alpha_p = 126.384$ MeV, $\Lambda = 2.1475$ fm^{-1}.

One should emphasize that if one reproduces the elastic charge form factor for He^4 or Li^6 by introducing short-range correlations one does not prove their existence. The electron scattering measurements can always be interpreted in terms of an effective nuclear-charge distribution - see Eqs. (III.23) - which may or may not come from correlations. In fact, it was possible to fit the He^4 results(19) by means of phenomenological charge densities with the inner part appreciably lower than that predicted by the oscillator potential model. Nevertheless the short-range correlation calculations described here may present a strong argument for the existence of the correlations. If the correlations were absent one should be able to derive the charge densities which fit the data from certain potential wells. However, the recent analysis(29) has shown that it is quite doubtful to find a central potential well which would correctly reproduce the diffraction minimum for He^4. In order to fit the form factor data up to large q an infinitely repulsive core had to be added to the central potential.

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