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P. Di Vecchia and F. Drago: MINIMAL SOLUTIONS TO THE
CONSPIRACY PROBLEM AND CLASSIFICATION OF REGGE
POLE FAMILIES, II. -

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ABSTRACT. -

A non group-theoretical approach to the conspiracy problem, based on analyticity, crossing symmetry and factorization, is presented. The solutions to all the physically interesting cases are given. A classification of Regge pole families is deduced.

2.

I. - INTRODUCTION. -

During the last few years the properties of the scattering amplitudes near $t=0$ have been extensively studied in the framework of the Regge pole model following two different ways: the group theoretical approach and the analytic approach.

The first one is based on the invariance of the scattering amplitude under the group $O(3,1)$ or $O(4)$ in the pairwise equal mass configuration. In fact Toller^(1,3) reggeized expansions of amplitudes in terms of the representations of the group $O(3,1)$; the simpler compact group $O(4)$ was used later by Freedman and Wang⁽⁴⁾ and by Domokos⁽⁵⁾. The Reggeization of these expansions shows that a Toller pole, that is a pole in the "four-dimensional angular momentum plane", leads to an infinite family of Regge poles, with definite relations between the trajectories and the residue functions at $t=0$. The families of Regge poles so generated are characterized, apart from the internal quantum numbers, by a Toller quantum number M , which, for boson trajectories, can take all the integer values.

Since this formalism applies rigorously only at the point $t=0$ and for equal mass scattering, more general formalisms have been developed in order to overcome these limitations^(6,7). The most general work in this direction has been done by Cosenza, Sciarrino and Toller^(8,9) by a generalization of the Lorentz group formalism and the introduction of quite strong assumptions. This formalism permitted them to construct a large class of families of conspiring Regge trajectories, which satisfy the factorization constraints for an arbitrary number of two-body reactions involving particles with arbitrary masses and spins. The contribution of one of these families of trajectories satisfies all the constraints which can be derived from the analytic properties of the amplitude.

The analytic approach on the other hand is based on the usual assumptions of the S-matrix theory, which, adapted to the Regge pole theory, permit us to understand the properties of the scattering amplitude near $t=0$ for any mass configuration. The assumptions made in the analytic approach are the following: a) analyticity; b) simplicity; c) crossing symmetry and; d) factorization. Some comment to these properties of the scattering amplitude adapted to the Regge pole model can be found in Ref. (10). The above assumptions require, for every parent trajectory exchanged, the exchange of an infinite family of Regge poles (daughter trajectories), with well defined quantum numbers with respect to the parent pole⁽¹¹⁾. The residues and the trajectories of the members of the family are strictly related near $t=0$ through some analyticity requirements⁽¹⁰⁾. In particular the assumptions a)...d) permitted to evaluate the contribution of a Regge pole family to the scattering amplitude at $t=0$

in the equal mass configuration and to show in this particular case the complete equivalence between the group theoretical and the analytic approach^(10, 12, 13, 14).

Other relations between the parameters of the various members of the family, like the "mass formulas", have been then evaluated^(12, 15, 16).

In this paper we consider the series of reactions:



where N is a nucleon and J(S) is a spin J(S) and mass $m_J(m_S)$ particle with $m_J \neq m_S \neq$ nucleon mass.

Then we evaluate the "minimal" behavior of the Regge pole residue functions satisfying the assumptions a)...d). In this way it is possible to define a quantum number M and to classify the Regge poles in families with well defined quantum numbers. This classification is equivalent to the group theoretical one; moreover the $t=0$ behavior of the factorized Regge pole residue functions, satisfying all the kinematical constraints, is the same as that derived in Refs. (8) and (9) by the group theoretical works.

In Section II we write down, in terms of the helicity amplitudes, the kinematical constraints at $t=0$ for the reactions of the type (I.1). The details of the calculation are given in Appendix A and B.

In Sect. III we write down the factorization requirements near $t=0$ for the Regge pole residue functions free from any kinematical singularity.

The "minimal" solutions of the factorization requirements and the constraint equations are given in Section IV, where our results are compared with those obtained by Capella, Contagouris and Tran Thauh Van⁽¹⁷⁾, and Cosenza, Sciarrino e Toller^(8, 9).

Finally in Section V we show how the introduction of the quantum number M permits to classify the Regge poles in families with well defined quantum numbers.

In Appendix C we determine the singularity structure near $t=0$ of the residues of the daughter trajectories in the various helicity amplitudes.

A short account of this work has been published elsewhere⁽¹⁸⁾.

II. - CONSPIRACY RELATIONS. -

We shall use the customary notation $f_{cd;ab}^t$ to denote a helicity amplitude⁽¹⁹⁾ for the t -channel reaction $a+b \rightarrow c+d$. Helicity amplitudes free from kinematical singularities in s and t must be used in the derivation of the constraints. The first step is to define amplitudes free from kinematical singularities in s ⁽²⁰⁾:

$$(II.1) \quad f_{cd;ab}^t = \left(\sqrt{2} \sin \frac{\theta_t}{2} \right)^{-|\lambda - \mu|} \left(\sqrt{2} \cos \frac{\theta_t}{2} \right)^{-|\lambda + \mu|} f_{cd;ab}^t$$

where $\lambda = a-b$, $\mu = c-d$ and θ_t is the scattering angle in the center of mass frame of the t -channel.

The works by Hara⁽²¹⁾ and Wang⁽²²⁾ show how one can then remove the t kinematical singularities from the parity conserving helicity amplitudes⁽²⁰⁾ $f_{cd;ab}^{t\pm}$ free from s kinematical singularities. The Wang result can be written:

$$(II.2) \quad \overline{f}_{cd;ab}^{t\pm} = K_{cd;ab}^{\pm}(t) \tilde{f}_{cd;ab}^{t\pm}$$

where $K_{cd;ab}^{\pm}(t)$ is a known factor containing the kinematical singularities at $t=0$.

However the analyticity requirements and the crossing symmetry will provide additional kinematical zeroes at $t=0$ in certain linear combinations of the parity conserving helicity amplitudes. We give now the kinematical constraints at $t=0$ for the various cases in our discussion:

1) EU case (i. e. $S+N \rightarrow J+N$). The constraints turn out to be:

$$(II.3) \quad i f_{cd; \frac{1}{2} - \frac{1}{2}}^{\sim(+)\ t} - f_{cd; \frac{1}{2} \frac{1}{2}}^{\sim(-)\ t} = 0(t)$$

for any c and d satisfactory the inequality $c \neq d$, and

$$(II.4) \quad i f_{cc; \frac{1}{2} - \frac{1}{2}}^{\sim(-)\ t} - f_{cc; \frac{1}{2} \frac{1}{2}}^{\sim(-)\ t} = 0(t)$$

for any $c^{(x)}$. The details of the derivation of these constraints are given in App. A. The number of the independent constraints (II. 3) is given by $J(2S+1)$, while the number of these (II. 4) is given by $(2S+u)/2$ with

$$(II. 5) \quad u \begin{cases} = 0 & \text{if } \sigma_J \sigma_S = 1 \\ = 1 & \text{if } \sigma_J \sigma_S = -1 \end{cases}$$

where $\sigma_J = \eta_J(-1)^J$ and η_J is the intrinsic parity of the particle J. We supposed above that $J \geq S$.

2) UU case (i. e. $S+S \rightarrow J+J$). The constraints are

$$(II. 6) \quad \tilde{f}_{cd;ab}^{(+t)} + \tilde{f}_{cd;ab}^{(-t)} = 0 (t^m)$$

if $|\lambda - \mu| < |\lambda + \mu|$, and

$$(II. 7) \quad \tilde{f}_{cd;ab}^{(+t)} - \tilde{f}_{cd;ab}^{(-t)} = 0 (t^m)$$

if $|\lambda - \mu| > |\lambda + \mu|$, where $m = \text{minimum}(|\lambda|, |\mu|)$. The derivation of these constraints is presented in App. B.

3) EE case (i. e. $N+N \rightarrow N+N$). In fact in the simplified treatment given here we consider only the nucleon-nucleon scattering in the equal mass configuration. In this case the constraint is well known⁽²³⁾:

$$(II. 8) \quad \tilde{f}_{\frac{1}{2} \frac{1}{2}; \frac{1}{2} \frac{1}{2}}^{(-t)} - z \tilde{f}_{\frac{1}{2} \frac{1}{2}; \frac{1}{2} \frac{1}{2}}^{(+t)} - \tilde{f}_{\frac{1}{2} \frac{1}{2}; \frac{1}{2} \frac{1}{2}}^{(-t)} = 0 (t)$$

The derivation can be found in Ref. (24).

If only Regge poles contribute to the scattering amplitudes, these

(x) - The factor $1/2$, which was present in the constraints (II. 4) in Ref.(18), has been eliminated because of the $\sqrt{2}$ factor in the definition (II. 1) of the helicity amplitudes free from s kinematical singularities. The constraints EU have been derived independently by J. D. Stack⁽²³⁾.

relations can be satisfied in either of two ways:

1) Each of the two amplitudes involved in an equation has an additional factor of t in the residue: in this case the relation is trivially satisfied (evasion).

2) The amplitudes \tilde{f} involved in the kinematical constraints go to a constant when $t \rightarrow 0$; this implies a relation between the intercepts $\alpha(0)$ and the residues $\beta(0)$ of the various trajectories which contribute to the \tilde{f} (conspiracy). In the following we will see that the mechanism chosen by the Regge pole families to satisfy the constraints will enable us to derive a classification of the Regge pole families.

III. - THE REGGE RESIDUE FACTORIZATION NEAR $t=0$. -

The partial wave expansion for the parity conserving helicity amplitudes, free from S kinematical singularities is given by⁽²⁰⁾:

$$(III.1) \quad \tilde{f}_{cd;ab}^{(\pm)t} = \sum_J (2J+1) \left[e_{\lambda\mu}^{J+} F_{cd;ab}^{J(\pm)} + e_{\lambda\mu}^{J-} F_{cd;ab}^{J(\mp)} \right]$$

where the $e_{\lambda\mu}^{J\pm}$ functions are defined in Ref. (20).

The contribution of a Regge trajectory α^{\pm} to the previous amplitudes is given by:

$$(III.2) \quad \tilde{f}_{cd;ab}^{(\pm)t} \approx \frac{2\alpha^{\pm}(t)+1}{\sin \alpha^{\pm}(t)} \beta_{cd;ab}^{\pm}(t) E_{\lambda\mu}^{+\alpha^{\pm}(t)}(-\cos \theta_t) + \\ + \frac{2\alpha^{\mp}(t)+1}{\sin \alpha^{\pm}(t)} \beta_{cd;ab}^{\mp}(t) E_{\lambda\mu}^{-\alpha^{\mp}(t)}(\cos \theta_t)$$

where the functions $E_{\lambda\mu}^{\pm}$ can be found in Ref. (20). Here and in the following we omit the signature factor which is not essential in our considerations.

Because of the conservation of the angular momentum, parity, G -parity and isotopic spin in strong interaction physics we can label the single Regge poles by the quantum numbers τ (signature), P (parity), $\mathcal{G} = P\tau$ (normality) and $\xi = G(-1)^T$ where T is the isospin of the Regge trajectory and G its G -parity. However the analyticity requires⁽¹¹⁾, for every parent Regge pole exchanged in unequal mass reactions the exchange of an infinite family of Regge trajectories with residues conveniently

singular near $t=0$ (daughter trajectories). If we label with n the quantum numbers of the n^{th} daughter trajectory, there exists the following relation between them and those of the parent:

$$(III. 3) \quad \sigma_n = \sigma \quad P_n = (-1)^n P \quad \tau_n = (-1)^n \tau \quad \xi_n = \xi$$

TABLE I

1	$\sigma = +$	$P = \xi$	n even
2	$\sigma = +$	$P = -\xi$	n odd
3	$\sigma = +$	$P = -\xi$	n even
4	$\sigma = +$	$P = \xi$	n odd
5	$\sigma = -$	$P = \xi$	n even
6	$\sigma = -$	$P = -\xi$	n odd
7	$\sigma = -$	$P = -\xi$	n even
8	$\sigma = -$	$P = \xi$	n odd

where the quantum numbers without subscript refer to the parent trajectory. Therefore a Regge pole can be labelled by a further quantum number n , which characterizes the behavior of its residue function near $t=0$ in the unequal mass scattering. Using the above considerations it is then possible to achieve the classification of Regge trajectories shown in Table I. The class 3 and 4 do not couple to the nucleon-antinucleon system because of the conservation of angular momentum, parity and G-parity. In the following we will determine the further zeroes of the residue functions which are necessary to satisfy the factorization requirements and the constraints and

we will get a more complete classification of the Regge trajectories in families with a well defined new quantum number M .

Now we impose the factorization conditions for the residue functions of the Regge poles.

The objects which factorize are the residues of the individual poles in $F_{cd;ab}^{J+}$; therefore in our case the residue is:

$$(III. 4) \quad \beta_{cd;ab}^{\pm}(t) = K_{cd;ab}^{\pm}(t) (P_{cd} P_{ab})^{\alpha^{\pm}-N} g_{cd;ab}^{\pm}(t) \gamma_{cd;ab}^{\pm}(t)$$

where $(P_{cd} P_{ab})^{\alpha^{\pm}-N}$ is the threshold factor, $N = \text{Maximum}(|\lambda|, |\mu|)$, $g_{cd;ab}^{\pm}(t)$ is a factor which takes into account of the further singularities present in the residues of the daughter trajectories near $t=0$ and $\gamma_{cd;ab}^{\pm}(t)$ is the reduced residue free from any kinematical singularity at $t=0$: it

8.

can contain at $t=0$ only eventual zeroes. Hence if reaction 1 is $a+b \rightarrow c+d$, reaction 2 is $c+d \rightarrow c+d$ and reaction 3 is $a+b \rightarrow a+b$ the factorization theorem can be stated as⁽²⁵⁾:

$$\begin{aligned}
 & \left[K_{1cd;ab}^{\pm}(t) \gamma_{1cd;ab}^{\pm}(P_{cd}; P_{ab})^{\alpha^{\pm} - N_1} g_{1cd;ab}(t) \right]^2 = \\
 \text{(III. 5)} \quad & = \left[K_{2cd;cd}^{\pm}(t) \gamma_{2cd;cd}^{\pm}(P_{cd})^{2(\alpha^{\pm} - N_2)} g_{2cd;cd}(t) \right] \times \\
 & \times \left[K_{3ab;ab}^{\pm}(t) \gamma_{3ab;ab}^{\pm}(P_{ab})^{2(\alpha^{\pm} - N_3)} g_{3ab;ab}(t) \right]
 \end{aligned}$$

In Tables II and III we give the behavior of the factor $K_{cd;ab}^{\pm}(t)$ and $g_{cd;ab}(t)$ for the reactions (I.1), while the behavior near $t=0$ of both sides of equation (III.5) is given in Table IVa for the reaction $S+S \rightarrow J+J$.

If an equal-mass vertex is present in the reaction considered, then the conservation laws imply the identical vanishing of the residue if the following conditions are not satisfied:

$$\begin{aligned}
 \text{(III. 6)} \quad & \sigma \sum_n \gamma_n (-1)^{S+1} = 1 \\
 & S=1 \quad \text{if} \quad \sigma = +1
 \end{aligned}$$

where the quantum numbers in equation (III.6) refer to the Regge pole exchanged and S is the total spin in the NN system. For the residues involving equal-mass vertices, which to not vanish, we give the factorization requirements in Tables IV b and IVc.

IV. - MINIMAL SOLUTIONS OF THE CONSPIRACY PROBLEM. -

In this section, under the assumption that only moving poles are present in the complex J plane, we give the behavior of the Regge pole residues near $t=0$ allowed by the kinematical constraints and the factorization theorem.

In general, from a given solution to all the factorization requirements, one can obtain solutions by increasing the number of t powers of some of the Regge pole residues in the original solution. If a given solution

TABLE IIa

	Amplitude $\bar{N}\bar{N} \rightarrow JS$	$ \lambda $	$ \mu $	Behaviour near $t=0$ $K_1(t)$
a	$\frac{-t^+}{f}$ $C'A'; \frac{1}{2} - \frac{1}{2}$	1	$\neq 0$	$t^{-\frac{1}{2}}$
b	$\frac{-t^-}{f}$ $C'A'; \frac{1}{2} \frac{1}{2}$	0	$\neq 0$	$t^{-\frac{1}{2}}$
c	$\frac{-t^-}{f}$ $C'A'; \frac{1}{2} - \frac{1}{2}$	1	0	$t^{-\frac{1}{2}}$
d	$\frac{-t^-}{f}$ $C'A'; \frac{1}{2} \frac{1}{2}$	0	0	$t^{-\frac{1}{2}}$
e	$\frac{-t^-}{f}$ $C'A'; \frac{1}{2} - \frac{1}{2}$	1	$\neq 0$	1
f	$\frac{-t^+}{f}$ $C'A'; \frac{1}{2} \frac{1}{2}$	0	$\neq 0$	1
g	$\frac{-t^+}{f}$ $C'A'; \frac{1}{2} - \frac{1}{2}$	1	0	1
h	$\frac{-t^+}{f}$ $C'A'; \frac{1}{2} \frac{1}{2}$	0	0	1

TABLE IIb

Amplitude $\bar{N}\bar{N}$	$\bar{N}\bar{N}$	$ \lambda - \mu $	Behaviour near $t=0$ $K_2(t)$
$\frac{-t^\pm}{f}$ $\mu; \lambda$	even		1
	odd		t

10.

TABLE IIc

Amplitude JS \rightarrow JS	Behaviour near $t=0$ $K_3(t)$
$\gamma_{\mu;\lambda}^{\pm}$	$t^{-\frac{1}{2}N}$ $N = \text{Max}(\lambda + \mu ; \lambda - \mu)$

TABLE III

Amplitude	Singularity of the residue of the n-th daughter
$(\gamma_{cd;ab}^{\pm})^{UU}$	$\frac{1}{t^n}$
$(\gamma_{cd;ab}^+)^{EU}$	$(\frac{1}{t})^{n/2}$ if $(-1)^n = 1$ $(\frac{1}{t})^{(n-1)/2}$ if $(-1)^n = -1$
$(\gamma_{cd;ab}^+)^{EU}$ Minimum $(\lambda , \mu) \neq 0$	$(\frac{1}{t})^{n/2}$ if $(-1)^n = 1$ $(\frac{1}{t})^{(n-1)/2}$ if $(-1)^n = -1$
$(\gamma_{cd;ab}^-)^{EU}$ Minimum $(\lambda , \mu) \neq 0$	$(\frac{1}{t})^{n/2}$ if $(-1)^n = 1$ $(\frac{1}{t})^{(n+1)/2}$ if $(-1)^n = -1$

TABLE IVa

Factorization conditions in JS \rightarrow JS
$(\gamma_{\mu;\lambda}^{\pm})^2 t^{ \mu - \lambda } = (\gamma_{\mu;\mu}^{\pm})(\gamma_{\lambda;\lambda}^{\pm})$

TABLE IVb

Factorization conditions in $\bar{N}N \rightarrow \bar{N}N$	$ \lambda - \mu $
$(\gamma_{\mu; \lambda}^{\pm})^2 = (\gamma_{\mu; \mu}^{\pm})(\gamma_{\lambda; \lambda}^{\pm})$	even
$(\gamma_{\mu; \lambda}^{\pm})^2 t = (\gamma_{\mu; \mu}^{\pm})(\gamma_{\lambda; \lambda}^{\pm})$	odd

TABLE IVc

	Factorization conditions in $\bar{N}N$ JS
a	$(\gamma_{cd; \frac{1}{2} - \frac{1}{2}}^+)^2 t^{ \mu - (1/2)} [1 + (-1)^n] = (\gamma_{cd; cd}^+) (\gamma_{\frac{1}{2} - \frac{1}{2}; \frac{1}{2} - \frac{1}{2}}^+)$
b	$(\gamma_{cd; \frac{1}{2} \frac{1}{2}}^-)^2 t^{ \mu - (1/2)} [1 + (-1)^n] = (\gamma_{cd; cd}^-) (\gamma_{\frac{1}{2} \frac{1}{2}; \frac{1}{2} \frac{1}{2}}^-)$
c	$(\gamma_{cd; \frac{1}{2} - \frac{1}{2}}^-)^2 t^{1/2} [1 - (-1)^n] = (\gamma_{cd; cd}^-) (\gamma_{\frac{1}{2} - \frac{1}{2}; \frac{1}{2} - \frac{1}{2}}^-)$
d	$(\gamma_{cd; \frac{1}{2} \frac{1}{2}}^-)^2 t^{-1/2} [1 + (-1)^n] = (\gamma_{cd; cd}^-) (\gamma_{\frac{1}{2} \frac{1}{2}; \frac{1}{2} \frac{1}{2}}^-)$
e	$(\gamma_{cd; \frac{1}{2} - \frac{1}{2}}^-)^2 t^{-1/2} [1 + (-1)^n] = (\gamma_{cd; cd}^-) (\gamma_{\frac{1}{2} - \frac{1}{2}; \frac{1}{2} - \frac{1}{2}}^-)$
f	$(\gamma_{cd; \frac{1}{2} \frac{1}{2}}^+)^2 t^{ \mu + (1/2)} [1 - (-1)^n] = (\gamma_{cd; cd}^+) (\gamma_{\frac{1}{2} \frac{1}{2}; \frac{1}{2} \frac{1}{2}}^+)$
g	$(\gamma_{cd; \frac{1}{2} - \frac{1}{2}}^+)^2 t^{1 + (1/2)} [1 - (-1)^n] = (\gamma_{cd; cd}^+) (\gamma_{\frac{1}{2} - \frac{1}{2}; \frac{1}{2} - \frac{1}{2}}^+)$
h	$(\gamma_{cd; \frac{1}{2} \frac{1}{2}}^+)^2 t^{1/2} [1 - (-1)^n] = (\gamma_{cd; cd}^+) (\gamma_{\frac{1}{2} \frac{1}{2}; \frac{1}{2} \frac{1}{2}}^+)$

cannot be obtained from another in this way, we will call it minimal. It is important to note that in general the minimality of a solution is strictly related to the set of reactions in which it is minimal. In fact if we consider two sets of reactions $\{A\}$ and $\{B\}$ a minimal solution in the reactions $\{B\}$ may be not minimal in the set of reactions $\{A\}$.

This means that even if some of the solutions given below are not minimal where looked at from the point of view of a particular reaction, they are minimal in the set of reactions (I.1).

As we said in Section III, on the basis of general analyticity properties the residues γ can contain only additional zeroes at $t=0$. Therefore if the factorization requires that $\gamma \sim t^a$, a is constrained to be a positive integer number greater than or equal to 0.

Taking into account this restriction it is easy to find the minimal solutions of the factorization requirements listed in Tables IVa, b, c. For the reactions with unequal masses, of the type $S+S \rightarrow J+J$ or $S+S' \rightarrow J+J'$, one finds:

$$(IV.1) \quad \gamma_{cd;cd}^{\sigma} \sim t^{|\mu| - M}$$

for any class of Regge poles listed in Table I^(*).

We can include into the set of reactions (I.1) also processes of the type of $S+S' \rightarrow J+J'$ because for them the factorization requirements are identical to those listed in Table IVa. The result (VI.1) is coincident with that obtained by Le Bellac using only UU scattering reactions⁽²⁶⁾.

When an equal mass vertex is involved, the selection rules due to parity and G-parity invariance must be taken into account. This implies the identical vanishing of the residue if the conditions (III.16) are not satisfied.

In the equal mass reaction $N+N \rightarrow N+N$, for the residues which do not vanish we have:

$$(IV.2) \quad \gamma_{ab;ab}^{\sigma} \sim \begin{cases} t^M & \text{if } (-1)^{\lambda+n} = \sigma \\ t^{|M-1|} & \text{if } (-1)^{\lambda+n} = -\sigma \end{cases}$$

(*) - The behaviour given above is different from that reported in Ref. (18) because there the singularity structure at $t=0$ of the residues of the daughter trajectories was included in γ .

For the families with $\sigma = +$ and $\tau = -\xi$, whose parent and even daughter trajectories do not couple to the NN system, we have for the odd daughters:

$$(IV.3) \quad \gamma_{ab;ab}^{(+)} \sim \begin{cases} t^{M+1} & \text{if } \lambda = 0 \\ t^{|M-1|+1} & \text{if } |\lambda| = 1 \end{cases}$$

Using the factorization conditions given in Table IVc it is then easy to obtain the behavior of the EU residues (Table V). The minimal solutions IV.1, IV.3, IV.4 and Table V of the factorization equations are consistent with the constraints given in Section II; such solutions are therefore the minimal solutions of the conspiracy problem.

M is a number that we introduced at this point in order to label the minimal solutions. It can assume all the integer values between zero and infinity. Every family of Regge poles is characterized by a value of M.

An interesting feature of our results is that, for all the values of the masses, a family with a given M contributes asymptotically only to the forward s-channel helicity amplitudes with helicity flip equal to $\pm M$. This property gives to M a clear physical meaning and permits to identify M with the Toller quantum number introduced in the group theoretical approach.

The previous results for the minimal solutions of the conspiracy problem are coincident with those found by Cosenza, Sciarrino and Toller in the general group theoretical approach.

However, besides the solutions given above, which are consistent with the group, theoretical one, we find two more solutions for the families with $\sigma = +1$, $\tau = -\xi$:

$$(V.4) \quad \begin{array}{l} \text{a) } \left\{ \begin{array}{l} \gamma_{ab;ab}^{(+)}(NN \rightarrow NN) \\ \gamma_{cd;cd}^{(+)}(JS \rightarrow J'S') \end{array} \right. \sim \begin{cases} \left\{ \begin{array}{l} t \\ \text{constant} \end{array} \right. & \begin{array}{l} \text{if } \lambda = 0 \\ \text{if } \lambda \neq 0 \end{array} \\ \left\{ \begin{array}{l} t^{|\mu|} \\ t^2 \end{array} \right. & \begin{array}{l} \text{if } \mu \neq 0 \\ \text{if } \mu = 0 \end{array} \end{cases} \\ \\ \text{b) } \left\{ \begin{array}{l} \gamma_{ab;ab}^{(+)}(NN \rightarrow NN) \\ \gamma_{cd;cd}^{(+)}(JS \rightarrow J'S') \end{array} \right. \sim \begin{cases} \left\{ \begin{array}{l} \text{constant} \\ t \end{array} \right. & \begin{array}{l} \text{if } \lambda = 0 \\ \text{if } \lambda \neq 0 \end{array} \\ \left\{ \begin{array}{l} t^{|\mu|+1} \\ t \end{array} \right. & \begin{array}{l} \text{if } \mu \neq 0 \\ \text{if } \mu = 0 \end{array} \end{cases} \end{array}$$

14.

TABLE V

Behavior of the EU residues near $t=0$	
a	$\gamma_{cd; \frac{1}{2} - \frac{1}{2}}^+ \sim t^{1/2} \{ \mu - M + 1 + M-1 - \mu \}$
b	$\gamma_{cd; \frac{1}{2} \frac{1}{2}}^- \sim t^{1/2} \{ \mu - M - \mu + M-(1/2)[1+(-1)^n] + (1/2)[1+(-1)^n] \}$
c	$\gamma_{cd; \frac{1}{2} - \frac{1}{2}}^- \sim t^{1/2} \{ M + M-(1/2)[1-(-1)^n] - (1/2)[1-(-1)^n] \}$
d	$\gamma_{cd; \frac{1}{2} \frac{1}{2}}^- \sim t^{1/2} \{ M + M-(1/2)[1+(-1)^n] + (1/2)[1+(-1)^n] \}$
e	$\gamma_{cd; \frac{1}{2} - \frac{1}{2}}^- \sim t^{1/2} \{ \mu - M - \mu + M-(1/2)[1-(-1)^n] + (1/2)[1-(-1)^n] \}$
f	$\gamma_{cd; \frac{1}{2} \frac{1}{2}}^+ \sim t^{1/2} \{ \mu - M + M - \mu \}$
g	$\gamma_{cd; \frac{1}{2} - \frac{1}{2}}^+ \sim t^{1/2} \{ M + M-1 - 1 \}$
h	$\gamma_{cd; \frac{1}{2} \frac{1}{2}}^+ \sim t^M$

We did not find any way to eliminate these solutions: their meaning and their relations to the group theoretical results are unclear.

The problem of the minimal solutions of the conspiracy problem in the set of reactions (I.1) has been also treated by Capella, Contagouris and Tran Thanh Van⁽¹⁷⁾ using the analytic approach.

They obtained

$$(IV.5) \quad \gamma_{ab;ab}^{\sigma} \sim t^{1/2} [1 - \sigma(-1)^{M+n+\lambda}]$$

for the EE mass configuration, while for the UU mass scattering their results are coincident with those given by (IV.1). The apparent disagreement between the expression (IV.2), (IV.3) and (IV.4) is due to the fact that their result is only valid for $M=0, 1$ because they did not impose the minimality in all the channels (I.1) at the same time. In so doing they obtained the solutions for $M=0, 1$, but lost other solutions which are minimal in the set of reactions (I.1). Finally the behavior of the EE residues for the family with $\sigma=+$ and $\nu=-\xi$ cannot be obtained from (IV.4) because the previous authors did not consider these families.

V. - CLASSIFICATION OF REGGE POLE FAMILIES. -

The introduction of the new quantum number M permits to classify the Regge poles in families according to the values of M, σ, ξ .

Class I: $M=0, \sigma=+1, \nu=\xi$

A parent trajectory requires the existence of a family of trajectories with "angular momentum":

$$\alpha_n(0) = \alpha(0) - n$$

where $\alpha(0)$ is the intercept of the parent trajectory. For n -even we have:

$$\nu_n = P_n = \xi = \nu$$

and for n odd

$$\nu_n = P_n = -\xi = -\nu$$

Only the trajectories with n even can couple to the NN system. This is the class I of Freedman and Wang⁽⁴⁾. Poles of this class never conspire, as can be seen from the behavior of the residues listed above.

16.

Class Ia: $M=0, \sigma=+1, \tau=-\xi$

Poles with n even have:

$$\tau_n = P_n = -\xi = \tau$$

and with n odd

$$\tau_n = P_n = \xi = -\tau$$

The parent and the even daughters of this class do not couple to the NN system: this explains why this class is not contained in the Freedman and Wang classification.

Also this poles never conspire.

Class II: $M=0, \sigma=-1, \tau=-\xi$

The poles of this class satisfy the constraints (II.4) and (II.8) by a daughter like conspiracy; that explains why such constraints are called "class II" constraints. All the others are satisfied by evasion. For n even we have:

$$\tau_n = -P_n = -\xi = \tau$$

and for n odd:

$$\tau_n = -P_n = \xi = -\tau$$

The parent trajectory contributes to the amplitude $f_{cd;1/2-1/2}^{(-)t}$ and conspires with the first daughter that contributes to the amplitude $f_{cd;1/2,1/2}^{(-)t}$.

The same mechanism of conspiracy applies between the amplitudes $f_{1/2-1/2;1/2-1/2}^{(-)t}$ and $f_{1/2,1/2;1/2,1/2}^{(-)t}$ in the nucleon-nucleon scattering.

Class IIa: $M=0, \sigma=-1; \tau=\xi$

Poles with n even have

$$\tau_n = -P_n = \xi = \tau$$

and with n odd

$$\tau_n = -P_n = -\xi = -\tau$$

The parent trajectory of this class, which is decoupled from the $N\bar{N}$ system at $t=0$, satisfies all the constraints by evasion. Also this class is therefore absent in the Freedman and Wang classification. Conspiracy between daughters is allowed in the EU configuration.

Class III: $M=1, \tau = \xi$

In this class we find the well known parity doubling phenomenon. The parity doublet structure not only allows to satisfy by conspiracy the constraints (II.3, II.6, II.7, II.8) but it is also imposed by the general analyticity requirements considered in detail in App. C and in Ref. (27). Therefore in some particular conditions the class III conspiracy is a necessity, not just a possibility, deriving from the analyticity and crossing symmetry requirements.

One has the following quantum numbers:

$$\sigma = +1 \begin{cases} n \text{ even} & \tau_n = P_n = \xi = \tau \\ n \text{ odd} & \tau_n = P_n = -\xi = -\tau \end{cases}$$

Only trajectories with n even can couple to $N\bar{N}$ system.

$$\sigma = -1 \begin{cases} n \text{ even} & \tau_n = -P_n = \xi = \tau \\ n \text{ odd} & \tau_n = -P_n = -\xi = -\tau \end{cases}$$

Class IIIa: $M=1, \tau = -\xi$

The poles of this class have the following quantum numbers:

$$\sigma = + \begin{cases} n \text{ even} & \tau_n = P_n = -\xi = \tau \\ n \text{ odd} & \tau_n = P_n = \xi = -\tau \end{cases}$$

Only trajectories with n odd can couple to the $N\bar{N}$ system.

$$\sigma = - \begin{cases} n \text{ even} & \tau_n = -P_n = -\xi = \tau \\ n \text{ odd} & \tau_n = -P_n = \xi = -\tau \end{cases}$$

The parent trajectories satisfy the constraints by evasion. Conspiracy between daughter trajectories of opposite value of σ is however allowed.

Poles with $M >$ are decoupled, at $t=0$, from the $N\bar{N}$ system, in agreement with the group theoretical approach.

18.

Such classification is coincident with that found in the general group theoretical approach by Cosenza, Sciarrino and Toller and results a generalization of the Freedman and Wang classification. In fact the class Ia, IIa, IIIa are absent in their classification because the parent Regge trajectories of these class are decoupled at $t=0$ in NN scattering.

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APPENDIX A. -

The method used for the derivation of the kinematical constraints in the channel $S+N \rightarrow J+N$ is due to Cohen-Tannoudji, Morel and Navelet⁽²⁸⁾.

Let us consider the reaction

$$(A.1) \quad S(a) + N(b) \rightarrow J(c) + N(d)$$

where the expressions between brackets refer to the helicity of the corresponding particle. The t-channel of the reaction (A.1) is:

$$(A.2) \quad \bar{N}(D') + N(b') \rightarrow J(c') + \bar{S}(A')$$

The helicity amplitudes of the processes (A.1) and (A.2) are related, through the crossing matrix, by

$$(A.3) \quad \bar{f}_{cd;ab}^S = (\sqrt{2} \operatorname{sen} \frac{\theta_s}{2})^{-|\lambda-\mu|} (\sqrt{2} \cos \frac{\theta_s}{2})^{-|\lambda+\mu|} \sum_{A'b'c'D'} d_{A'a}^S(\chi_a) d_{b'b}^{1/2}(\chi_b) d_{c'c}^J(\chi_c) d_{D'd}^{1/2}(\chi_d) (\sqrt{2} \operatorname{sen} \frac{\theta_t}{2})^{|\lambda'-\mu'|} (2 \cos \frac{\theta_t}{2})^{|\lambda'+\mu'|} f_{c'A';D'b'}^t$$

where $\cos \chi_a$, $\cos \chi_c$, $\cos(\theta_s/2)$, $\operatorname{sen}(\theta_s/2)$, $\cos(\theta_t/2)$ and $\operatorname{sen}(\theta_t/2)$ are regular functions near $t=0$, while $\cos \chi_b$ and $\cos \chi_d$ are singular⁽²²⁾.

Drawing out the singular behavior at $t=0$, the rotation matrices related to the two nucleons can be written near $t=0$ in the form:

$$(A.4) \quad d^{1/2}(\cos \chi_{b(d)}) \sim \frac{B(D)}{(-t)^{1/4}} \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$$

where

$$(A.5) \quad B(D) = \left[\frac{M(m_J^2 - m_S^2)}{2 [S - (m_{S(J)} - M)^2] [S - (m_{S(J)} + M)^2]} \right]^{1/2}$$

and M is the nucleon mass.

For sake of simplicity it is convenient to define, for any c' and A' , the following system of equations:

$$(A.6) \quad G_{bd}^{c'A'} = \sum_{b'D'} d_{b'b}^{1/2}(\chi_b) d_{D'd}^{1/2}(\chi_d) (\sqrt{2} \operatorname{sen} \frac{\theta_t}{2})^{|\lambda' - \mu'|} \cdot (\sqrt{2} \cos \frac{\theta_t}{2})^{|\lambda' + \mu'|} f_{c'A'; D'b'}^{-t}$$

Since the rank of the system (A.6) is one it is sufficient to consider only the equation which is obtained for $b=d=1/2$, so that we have near $t=0$:

$$(A.7) \quad G_{\frac{1}{2}\frac{1}{2}}^{c'A'} = -\frac{BD}{(-t)} \left[\begin{array}{c} \tilde{f}^{t(-)} \\ c'A'; \frac{1}{2}\frac{1}{2} \end{array} - i \begin{array}{c} \tilde{f}^{t(+)} \\ c'A'; \frac{1}{2} - \frac{1}{2} \end{array} \right]$$

for $c' \neq A'$ and

$$(A.8) \quad G_{\frac{1}{2}\frac{1}{2}}^{c'A'} = -\frac{BD}{(-t)} \left[\begin{array}{c} \tilde{f}^{t(-)} \\ c'A'; \frac{1}{2}\frac{1}{2} \end{array} - i \begin{array}{c} \tilde{f}^{t(-)} \\ c'A'; \frac{1}{2} - \frac{1}{2} \end{array} \right]$$

for $c'=A'$.

In the previous expressions we extracted from the parity conserving helicity amplitudes the same $t^{-1/2}$ singularity near $t=0$. (Table IIa). If now we put the relations (A.7) and (A.8) into (A.3) we see that we must have:

$$(A.9) \quad \sum_{\substack{c'A' \\ c' \neq A'}} d_{c'c}^J(\chi_c) d_{A'a}^S(\chi_a) \left[\begin{array}{c} \tilde{f}^{t(-)} \\ c'A'; \frac{1}{2}\frac{1}{2} \end{array} - i \begin{array}{c} \tilde{f}^{t(+)} \\ c'A'; \frac{1}{2} - \frac{1}{2} \end{array} \right] + \sum_{\substack{c'A' \\ c'=A'}} d_{c'c}^J(\chi_c) \cdot d_{A'a}^S(\chi_a) \left[\begin{array}{c} \tilde{f}^{t(-)} \\ c'A'; \frac{1}{2}\frac{1}{2} \end{array} - i \begin{array}{c} \tilde{f}^{t(-)} \\ c'A'; \frac{1}{2} - \frac{1}{2} \end{array} \right] = 0(t)$$

in order to not have a kinematical pole at $t=0$, which is forbidden because the left hand side of the equation (A.3) contains quantities free from kinematical singularities in t .

Because of the presence of the $d_{\lambda\mu}$ functions the determinant of the system (A.9) is equal to a finite value; therefore the only allowed solution is:

$$(A.10) \quad \tilde{f}_{c'A'; \frac{1}{2} \frac{1}{2}}^{t(-)} - i \tilde{f}_{c'A'; \frac{1}{2} - \frac{1}{2}}^{t(+)} = 0(t)$$

for any $c' \neq A'$ and

$$(A.11) \quad \tilde{f}_{c'c'; \frac{1}{2} \frac{1}{2}}^{t(-)} - i \tilde{f}_{c'c'; \frac{1}{2} - \frac{1}{2}}^{t(-)} = 0(t)$$

for any c' .

APPENDIX B. -

The kinematical constraints between the t-channel, helicity amplitudes of the reactions:

$$(B.1) \quad S(a) + S'(b) \rightarrow J(c) + J'(d)$$

can be obtained following the method of Cohen-Tannondji et al. (28). However we will follow here a more direct approach suggested by Frautschi and Jones (29).

The relation between the amplitudes $\bar{f}_{\mu; \lambda}^t$ and $f_{\mu; \lambda}^{\pm t}$ is:

$$(B.2) \quad \bar{f}_{\mu'; \lambda'}^t = \frac{1}{2} \left[\bar{f}_{\mu'; \lambda'}^{t+} + \bar{f}_{\mu'; \lambda'}^{t-} \right]$$

$$\bar{f}_{-\mu'; \lambda'}^t = \frac{\sigma_J \sigma_S (-)^{\lambda - N}}{2} \left[\bar{f}_{\mu'; \lambda'}^{t+} - \bar{f}_{\mu'; \lambda'}^{t-} \right]$$

where $\lambda' = D' - b'$; $\mu' = c' - A'$, $N = \text{Max}(|\lambda'|; |\mu'|)$ and $\sigma_J(\sigma_S)$ is the natural parity of the particle J(S). The t-kinematical singularities of the amplitudes $\bar{f}_{\mu'; \lambda'}^t$ and $\bar{f}_{\mu'; \lambda'}^{t\pm}$ are:

$$(B.3) \quad \bar{f}_{\mu'; \lambda'}^t \sim t^{-\frac{1}{2} |\lambda' - \mu'|} \quad \bar{f}_{\mu'; \lambda'}^{t\pm} \sim t^{-\frac{1}{2} N}$$

The kinematical constraints follow there immediately from the requirement that the two sides of the identities (B.2) have the same behavior near $t=0$.

APPENDIX C. -

The daughter trajectories have been introduced by Freedman and Wang⁽¹¹⁾ in order to eliminate any kind of singularity at $t=0$ in the full amplitude. They treated in detail the spin-less case and evaluated in this case the singularity of the residue of the n -th daughter at $t=0$ in the unequal-unequal mass scattering:

$$(C.1) \quad (\gamma)^{UU} \sim \frac{1}{t^n}$$

and in equal-unequal mass configuration:

$$(C.2) \quad \gamma^{EU} \sim \begin{cases} \left(\frac{1}{t}\right)^{n/2} & \text{if } (-1)^n = 1 \\ \left(\frac{1}{t}\right)^{(n-1)/2} & \text{if } (-1)^n = -1 \end{cases}$$

Those are the most singular behaviors required for the residue of the daughter trajectories in order to have an amplitude analytic at $t=0$; of course if the residue of the parent Regge pole vanishes at $t=0$ the residue of the daughter trajectories will be less singular than in (C.1) and (C.2).

We will extend the Freedman-Wang approach to the spin case, where the singularity structure of the daughter residues will show some differences with respect to the spin-less case.

We will see furthermore that in some case the introduction of under Regge pole family with opposite value of σ is necessary because the daughter trajectories alone are not sufficient to restore the analyticity of the full amplitude.

We start from the equation (III.2), which can be written:

$$(C.3) \quad \begin{aligned} \tilde{f}_{cd;ab}^{(\pm)t} \approx & \frac{2\alpha^{\pm}+1}{\sin \alpha^{\pm}} g_{cd;ab}^{\pm}(t) \gamma_{cd;ab}^{\pm} (P_{cd} P_{ab})^{\alpha^{\pm}-N} E_{\lambda\mu}^{+\alpha^{\pm}}(-\cos\theta_t) + \\ & + \frac{2\alpha^{\mp}+1}{\sin \alpha^{\mp}} g_{cd;ab}^{\mp}(t) \frac{K_{cd;ab}^{\mp}}{K_{cd;ab}^{\pm}} (P_{cd} P_{ab})^{\alpha^{\mp}-N} \gamma_{cd;ab}^{\mp} E_{\lambda\mu}^{-\alpha^{\mp}}(-\cos\theta_t) \end{aligned}$$

if we use the expression (III.4) for the residues and we eliminate the t kinematical singularities. For sake of clearness it is convenient to consider separately the UU and EU mass configurations.

In the UU case (C.3) becomes

$$(C.4) \quad \begin{aligned} \tilde{f}_{cd;ab}^{(\pm)t} &\sim g_{cd;ab}^{\pm} \gamma_{cd;ab}^{\pm} (P_{cd} P_{ab})^{\alpha^{\pm}-N} E_{\lambda\mu}^{+\alpha^{\pm}}(-\cos\theta_t) + g_{cd;ab}^{\mp} \times \\ &\times P_{cd} P_{ab} (P_{cd} P_{ab})^{\alpha^{\mp}-N-1} E_{\lambda\mu}^{-\alpha^{\mp}}(-\cos\theta_t) \gamma_{cd;ab}^{\mp} \end{aligned}$$

where we inglobed the factor $(2\alpha+1)/(\sin\pi\alpha)$ in the residue $\gamma_{cd;ab}$ and we used $K_{cd;ab}^{\pm}(t) = K_{cd;ab}^{\mp}(t)$.

We note that the factors $(P_{cd} P_{ab})^{\alpha^{\pm}-N} E_{\lambda\mu}^{+\alpha^{\pm}}(-\cos\theta_t)$ and $(P_{cd} P_{ab})^{\alpha^{\mp}-N-1} E_{\lambda\mu}^{-\alpha^{\mp}}(-\cos\theta_t)$ have the same singularity structure at $t=0$ of $P_{\alpha^{\pm}-N}(-\cos\theta_t) (P_{cd} P_{ab})^{\alpha^{\pm}-N}$ involved in the spin-less case.

The only difference therefore with the spin-less case arises from the presence of two terms in (C.4) and from the fact that the second term contains a factor that behaves like $1/t$ near $t=0$. This singular behavior of $P_{cd} P_{ab}$ complicates in the spin case the analysis of the singularity structure of the the residues of the daughter trajectories.

Obviously in the amplitudes with Minimum $(|\lambda|, |\mu|) = 0$ the singularity structure of the daughter poles is the same than in the spin-less case because of the identical vanishing of the $E_{\lambda\mu}^{-}$ function. For the other amplitudes because of the presence of the factor $P_{cd} P_{ab}$ in the second term of equation (C.2) the introduction of daughter trajectories is necessary to eliminate some singularities, but is not in general sufficient to have an amplitude analytic at $t=0$.

In order to restore the analyticity of the full amplitude we need the contribution of another Regge pole family with opposite value of σ with respect to the family primarily introduced in the amplitude. One can then study the singularity structure of the daughter trajectories of these two families and it is easy to check that such structure is not different from that found by Freedman and Wang in the spin less case. Finally we can conclude that in the spin case for the UU mass configuration the singularity structure of the residues of the daughter trajectories is the same of that given in (C.1). Furthermore such analysis permitted us to get that the analyticity requires the existence of the

parity doubling phenomenon⁽²⁷⁾.

Obviously the possibility of the parity doublet can be avoided if $\gamma_{cd;ab}^{\sigma} \sim t$ at least when $t \rightarrow 0$ for the pole primarily exchanged, but we are interested at this stage in the most singular behavior of the residues of the daughter trajectories and we expect that such further zeroes will come out from the factorization conditions.

Let us then consider the EU mass configuration. The expression (C.3) in this case becomes

$$(C.5) \quad \begin{aligned} \tilde{f}_{cd;ab}^{(+)} t &\sim g_{cd;ab}^{\pm} \gamma_{cd;ab}^{\pm} (P_{cd} P_{ab})^{\alpha^{\pm-N}} E_{\lambda\mu}^{+\alpha^{\pm}}(-\cos\theta_t) + \\ &+ \left[\frac{K_{cd;ab}^{+}(t)}{K_{cd;ab}^{\pm}} P_{cd} P_{ab} \right] g_{cd;ab}^{\mp} (P_{cd} P_{ab})^{\alpha^{\mp-N-1}} E_{\lambda\mu}^{-\alpha^{\mp}}(-\cos\theta_t) \gamma_{cd;ab}^{\mp} \end{aligned}$$

Also in this mass configuration the only difference with the spin-less case arises from the presence of two terms in (C.5) and from the expression

$$\frac{K_{cd;ab}^{+}(t)}{K_{cd;ab}^{\pm}} P_{cd} P_{ab} ,$$

which in some amplitudes behaves like $1/t$ for $t \rightarrow 0$. Because of the identical vanishing of the E^{-} for λ and μ not both different from zero, the analysis can therefore be restricted to the amplitudes with $\mu \neq 0$ and $\lambda = 1$; for the other amplitudes the singularities near $t=0$ of the residues of the daughter trajectories is evidently the same than in the spin-less case.

If $\lambda=1$ and $\mu \neq 0$ the expression (C.5) becomes:

$$(C.6) \quad \begin{aligned} \tilde{f}_{cd; \frac{1}{2} - \frac{1}{2}}^{(+)} t &\sim g_{cd; \frac{1}{2} - \frac{1}{2}}^{+} \gamma_{cd; \frac{1}{2} - \frac{1}{2}}^{+} (P_{cd} P_{ab})^{\alpha^{\pm-N}} E_{1\mu}^{+\alpha^{\pm}}(-\cos\theta_t) + \\ &+ g_{cd; \frac{1}{2} - \frac{1}{2}}^{-} (P_{cd} P_{ab})^{\alpha^{\mp-N-1}} E_{1\mu}^{-\alpha^{\mp}}(-\cos\theta_t) \gamma_{cd; \frac{1}{2} - \frac{1}{2}}^{-} \end{aligned}$$

$$\begin{aligned}
& \tilde{f}_{cd; \frac{1}{2} - \frac{1}{2}}^{(-)t} \tilde{g}_{cd; \frac{1}{2} - \frac{1}{2}}^{-} \gamma_{cd; \frac{1}{2} - \frac{1}{2}}^{-} (P_{cd} P_{ab})^{\alpha^{-} - N} E_{1\mu}^{+\alpha^{-}}(-\cos\theta_t) + \\
(C.7) \quad & + \frac{f(t)}{t} (P_{cd} P_{ab})^{\alpha^{\pm} - N - 1} E_{1\mu}^{-\alpha^{+}}(-\cos\theta_t) \gamma_{cd; \frac{1}{2} - \frac{1}{2}}^{+}
\end{aligned}$$

Where $f(t)$ is a function regular at $t=0$, which inglobes the non singular part of $(P_{cd} P_{ab})$. Because of the factor $1/t$ in the second term of (C.7), also in this mass configuration the parity doubling phenomenon is in general necessary to restore the analyticity of the amplitude at $t=0$: the singularities can be canceled only if in the first term of the right hand side of equation (C.7) contribute the odd daughters of a Regge pole family with $\sigma = -1$, whose residues have the singularity structure

$$(C.8) \quad \gamma_{cd; \frac{1}{2} - \frac{1}{2}}^{-} \sim \left(\frac{1}{t}\right)^{n+1} \quad \text{with} \quad (-1)^n = -1$$

which is different from the expression (C.2) valid in the spin-less case^(x).

In conclusion the analyticity properties require the singularity structure (C.8) for the residues of the odd daughters belonging to a family with $\sigma = -1$, while for all the other amplitudes and Regge pole families the singularities, are given by (C.2).

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REFERENCES. -

- (1) - M. Toller, *Nuovo Cimento* 37, 631 (1965); University of Roma reports n. 76 (1965) and n. 84 (1966) (unpublished).
- (2) - A. Sciarrino and M. Toller, *J. Math. Phys.* 7, 1670 (1967).
- (3) - M. Toller, *Nuovo Cimento* 53 A, 671 (1968) and 54 A, 295 (1968).
- (4) - D.Z. Freedman and J.M. Wang, *Phys. Rev.* 160, 1560 (1967).
- (5) - G. Domokos, *Phys. Letters* 24 B, 293 (1967) and 19, 137 (1967).
- (6) - R. Delbourgo, A. Salam and J. Strathdee, *Phys. Letters* 25 B, 230 (1967); and *Phys. Rev.* 164, 1981 (1967).
- (7) - G. Domokos and G.L. Tindle, *Phys. Rev.* 165, 1906 (1968).
- (8) - G. Cosenza, A. Sciarrino and M. Toller, *Phys. Letters* 27 B, 398 (1968).
- (9) - G. Cosenza, A. Sciarrino and M. Toller, University of Rome report n. 158 (1968); CERN preprint TH. 906 (1968).
- (10) - P. Di Vecchia and F. Drago, Cal-Tech preprint, CALT 68-172 (1968).
- (11) - D.Z. Freedman and J.M. Wang, *Phys. Rev.* 160, 1560 (1967).
- (12) - P. Di Vecchia and F. Drago, *Phys. Letters* 27 B, 387 (1968).
- (13) - J.B. Bronzan and C.E. Jones, *Phys. Rev. Letters* 21, 564 (1968).
- (14) - J.B. Bronzan, "Daughter sequences in unequal mass vector meson-scalar meson scattering, Massachusetts Institute of Technology preprint, November (1968).
- (15) - P. Di Vecchia and F. Drago, Cal-Tech preprint, CALT-68-176 (1968).
- (16) - J.B. Bronzan, C.E. Jones and P.K. Kuo, *Phys. Rev.* 175, 2200 (1968).
- (17) - A. Capella, A.P. Contagouris and J. Tran Thanh Van, *Phys. Rev.* 175, 1892 (1968).
- (18) - P. Di Vecchia, F. Drago and M.L. Paciello, *Nuovo Cimento* 56, 1185 (1968).
- (19) - M. Jacob and G.C. Wick, *Ann. Phys.* 7, 404 (1959).
- (20) - M. Gell-Mann, M.L. Goldberger, F.E. Low, E. Marx and F. Zachariasen, *Phys. Rev.* 133 B, 145 (1964).
- (21) - Y. Hara, *Phys. Rev.* 136 B, 507 (1964).
- (22) - L.L. Wang, *Phys. Rev.* 142, 1187 (1966); 153, 1664 (1967).
- (23) - J.D. Stack, *Phys. Rev.* 171, 1666 (1968).
- (24) - M.L. Goldberger, M.T. Grisaru, S.W. MacDowell and D.Y. Wong, *Phys. Rev.* 120, 2250 (1960).
- (25) - F. Arbab and J.D. Jackson, *Phys. Rev.* 176, 1796 (1968).
- (26) - M. Le Bellac, *Nuovo Cimento* 55 A, 318 (1968).
- (27) - M.A. Jacobs and M.H. Vanghn, *Phys. Rev.* 172, 1677 (1968).
- (28) - G. Cohen-Tannoudji, A. Morel and H. Navelet, *Ann. Phys.* 46, 239 (1968).
- (29) - S. Frantschi and L. Jones, *Phys. Rev.* 167, 1335 (1968).