

LNF - 69/17

13 Marzo 1969

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FOR THE VENEZIANO AMPLITUDE. -

Laboratori Nazionali di Frascati del CNEN  
Servizio Documentazione

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Nota Interna: n. 437  
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ABSTRACT. -

The Veneziano amplitude is projected on the irreducible representations of the Lorentz group. An infinite sequence of Toller poles, as well as an essential singularity as  $\lambda \rightarrow \infty$ , is shown to be present. Fixed poles at the non positive integers are also present. The residues of the Toller poles, with the exception of the first, do not factorize.

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(x) - Work supported in part by the Air Force Office of Scientific Research through the European Office of Aerospace Research, OAR, U.S. Air Force, under contract F 61052-67-C-0084.

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(1) In this note we study some properties of the Lorentz decomposition of the Veneziano(2) formula. For definiteness we shall discuss the Veneziano-Lovelace(3) model for  $\pi\pi$  scattering. However our method can be quite generally extended to study any scattering amplitude of the Veneziano type, and our results are of general validity.

Until now the Toller pole content of the Veneziano amplitude was not clear(4): it is quite apparent(2, 5) that such an amplitude does not correspond to a single Toller pole and it has been suggested(6) that, in fact, an infinite number of these poles is present: however no rigorous proof of this exists.

We perform the projection of the Veneziano amplitude, at fixed momentum transfer  $s$ , on the irreducible representations of the group  $O(3, 1)$ . We find that the projection of the terms  $A(s, t)$  and  $A(s, u)$  (see eq. 2) contains an infinite series of Toller poles at  $\lambda = \alpha(s) + 1 - m$  ( $m=0, 1, \dots$ ) and an essential singularity at infinity. The term  $A(t, u)$  gives fixed poles at the non positive integers: at  $s=0$  these fixed poles occur only at even values of  $\lambda$  and give rise, in the continued partial wave amplitudes, to the well known additive fixed poles at the wrong signature non-sense value of 1(4, 6, 7).

The residue of the poles is given in compact form.

We define the amplitude

$$(1) \quad B(s, t, u) = A_1(s, t) + A_2(s, u) + A_3(t, u)$$

with(3)

$$(2) \quad A_i(x, y) = -\gamma_i \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}$$

The amplitudes for the three  $\pi\pi$  isostates can be expressed in the form of Eq. (1), with a suitable choice of the coefficients  $\gamma_i$ (3). In Eq. (2)  $\alpha(x)$  is the degenerate  $\rho$ - $f^0$  trajectory, assumed to take the linear form

$$(3) \quad \alpha(x) = ax + b$$

We introduce the notation

$$(4) \quad D^\pm(s, t) = D_t^\pm(s, t) \pm D_u^\pm(s, t)$$

$D_t^\pm(s, t) [D_u^\pm(s, u)]$  being the  $t$  [ $u$ ] discontinuity of the amplitude  $B(s, t, u)$ .

The Lorentz projection can be defined as(8)

$$(5) \quad \alpha_{\lambda-1}^+(\lambda, s) = \frac{2}{\lambda\pi} \int_1^{\infty} d(\cosh\beta'_t) \sinh\beta'_t D_{\lambda-1}^{(1)}(\cosh\beta'_t) D^+(s, t')$$

where

$$(6) \quad \cosh\beta'_t = \frac{t' - 2m^2}{2m^2} = x$$

and  $D_{\lambda-1}^{(1)}$  is a Gegenbauer function of the second kind<sup>(x)</sup>.

Since

$$(7) \quad \sqrt{x^2-1} D_{\lambda-1}^{(1)}(x) = - \left[ x + \sqrt{x^2-1} \right]^{-\lambda}$$

We have

$$(8) \quad \alpha_{\lambda-1}^+(\lambda, s) = - \frac{2}{\pi\lambda} \int_{\frac{t_0}{2m^2}-1}^{\infty} D^+[s, 2m^2(1+x)] \left[ x + \sqrt{x^2-1} \right]^{-\lambda} dx$$

Since our discussion will treat the amplitudes linearly, we can study the single terms one by one and then superpose the results.

Let us start with the amplitude obtained by putting  $\gamma_3=0$  in (1). Introducing the expression for  $D^{\pm}$  calculated from (1) we easily obtain

$$(9) \quad \alpha_{\lambda-1}^+(\lambda, s) = - \frac{[\gamma_1^+ \gamma_2^-] \alpha(s)}{\lambda a m^2} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha(s)+n+1)}{\Gamma(\alpha(s)+1)n!} \left[ a_n + \sqrt{a_n^2-1} \right]^{-\lambda}$$

with

$$a_n = \frac{n+1-b}{2am^2} - 1$$

(x) - The Gegenbauer functions of the second kind are defined by:

$$D_{\lambda}^{(\alpha)}(x) = e^{i\pi\alpha} (2x)^{-\lambda-2\alpha} \Gamma(\lambda+2\alpha) \left[ \Gamma(\alpha) \Gamma(\lambda+\alpha+1) \right]^{-1} \cdot {}_2F_1\left(\frac{1}{2}\lambda+\alpha; \frac{1}{2}\lambda+\alpha+\frac{1}{2}; \lambda+\alpha+1; \frac{1}{x^2}\right)$$

4.

The expression (9) defines a holomorphic function of  $\lambda$  whenever it converges, i. e. for  $\text{Re } \lambda > \alpha(s)+1$ .

Using the expansion

$$(10) \quad \left[1 + \sqrt{1-z}\right]^{-\lambda} = 2^{-\lambda} \sum_{m=0}^{\infty} \frac{(-1)^m \lambda}{m!} \frac{\Gamma(-\lambda-m)}{(-\lambda-2m+1)} \left(\frac{z}{4}\right)^m$$

valid for  $|z| < 1$ , and the relation

$$(11) \quad z^{-\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} e^{-zx} x^{\lambda-1} dx$$

valid for  $\text{Re } \lambda > 0$ , we obtain, after an allowed interchange of the order of summation

$$(12) \quad a^+(\lambda, s) = \frac{[\gamma_1^+ \gamma_2^+]^{\alpha(s)} 2^{-\lambda}}{a m_{\pi}^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\Gamma(-\lambda-m)}{\Gamma(-\lambda-2m+1)} \left(\frac{1}{4}\right)^m \frac{1}{\Gamma(\lambda+2m)} \cdot \sum_{n=0}^{\infty} \frac{\Gamma(\alpha(s)+n+1)}{\Gamma(\alpha(s)+1)n!} \int_0^{\infty} e^{-a_n x} x^{\lambda+2m-1} dx$$

This expression has been obtained for  $\text{Re } \lambda > \max(0, \alpha(s)+1)$ : later on we shall analytically continue in  $\lambda$  the final answer. We can now perform both sums in Eq. (12) and the results is

$$(13) \quad a^+(\lambda, s) = \frac{[\gamma_1^+ \gamma_2^+]^{\alpha(s)}}{a m_{\pi}^2} \int_0^{\infty} dx x^{-\alpha-2} e^{-\left[\frac{1-b}{2am^2} - 1\right] x} \cdot \left[ \frac{-\frac{x}{2am_{\pi}^2}}{1 - e^{-\frac{x}{2am_{\pi}^2}}} \right]^{-\alpha(s)-1} I_{\lambda}(x)$$

Where  $I_{\lambda}(x)$  is the modified Bessel function of the first kind<sup>(9)</sup>, that behaves, for  $x \rightarrow 0$ , like  $x^{\lambda}$ .

The right hand side of Eq. (13) can be easily analytically continued in  $\lambda$ . To see explicitly the structure of the singularities of  $\mathcal{Q}^{\pm}(\bar{\lambda}, s)$ , let us define the function

$$(14) \quad f(z) = e^{-\left[\frac{1-b}{2am^2_\pi} - 2\right]z} \left[ \frac{1 - e^{-\frac{z}{2am^2_\pi}}}{z} \right]^{-\alpha(s)-1} \left(\frac{z}{2}\right)^{-\lambda} I_\lambda(z)$$

We expand  $f(z)$  according to the Taylor theorem, as

$$(15) \quad f(z) = \sum_{k=0}^N \frac{f^{(k)}(0)}{k!} z^k + R_{N+1}(z)$$

The non-negative integer  $N$  can be chosen arbitrarily large; however  $R_{N+1}(z)$  vanishes, for  $N \rightarrow \infty$ , only for  $|z| < 2\pi$ .

Using the expansion (14) we obtain

$$(16) \quad a^\pm(\lambda, s) = \frac{[\gamma_1 + \gamma_2]}{2^\lambda a m^2_\pi} \left\{ \sum_{k=0}^N \frac{f^{(k)}(0)}{k!} \Gamma(\lambda - \alpha(s) + k - 1) + \int_0^\infty x^{\lambda - 2 - \alpha(s)} R_{N+1}(x) e^{-x} dx \right\}$$

The only singularities in  $\lambda$  of the last term of (15) are due to the divergences of the integral at the lower limit of integration, where the behavior of the integrand is  $O(x^{\lambda+N-1-\alpha(s)})$ : therefore this term is regular for  $\text{Re } \lambda > \alpha(s) - N$ . By choosing  $N$  arbitrarily large, we can conclude that in any finite region of the  $\lambda$  plane, the only singularities are Toller poles with parallel "trajectories" spaced by one unit, starting at  $\lambda(s) = \alpha(s) + 1$ . However an essential singularity as  $\lambda \rightarrow \infty$  is also present. The situation in the  $\lambda$  plane is therefore similar to the one in the  $l$ -plane<sup>(6,7)</sup>. We stress that, due to the presence of the essential singularity, the Veneziano amplitude, after the analogue of the Mandelstam, Sommerfeld and Watson representation has been performed, cannot be expressed as a convergent series of Toller poles. It is therefore also unclear whether a single Toller pole can be reconstructed by superposing an infinite number of Veneziano amplitudes<sup>(5)</sup>; difficult problems of convergence, to be studied very carefully, arise.

The residue of the Toller pole at  $\lambda = \alpha(s) + 1 - m$  ( $m=0, 1, 2, \dots$ ) is given by:

$$(17) \quad \beta_m(s) = \frac{[\gamma_1 + \gamma_2] \alpha(s)}{a m^2_\pi 2^{\alpha(s)-m+1} m!} \left[ \frac{d^m}{dz^m} (f(z) e^{-z}) \right]_{z=0}$$

There is now an important observation to be made. It is usually accepted that at  $s=0$ ( $\mathbf{x}$ ) the residue of a Toller pole factorizes. It is easily seen, by considering two coupled channels, that only the residue of the pole at  $\lambda = \alpha(0)+1$  factorize: the residue of all the other poles, with  $m \geq 1$ , does not factorize.

We also remark that at  $s=0$ , if the condition  $\alpha(s)+\alpha(t)+\alpha(u)=1$  (i. e.  $4m_\pi^2 a+3b-1=0$ ) is satisfied, only Toller poles spaced by two units appear. This is the analogue of the condition imposed by Veneziano<sup>(2)</sup> in the  $\pi \pi \rightarrow \pi \omega$  case, where it turns out to be  $\alpha(s)+\alpha(t)+\alpha(u)=2$ . The  $\pi \pi$  case however is an example where the supplementary condition on the trajectory, besides being unnecessary, is also untenable, since it requires (for zero mass pions)  $\alpha(0)=1/3$ , while the positivity of the elastic widths requires  $\alpha(0) \geq 1/2$ <sup>(6, 10)</sup>.

We briefly discuss now the term  $A_3(t, u)$ . In evaluating  $D^\pm(s, t)$  we find that  $D^-(s, t) \equiv 0$ : this is due to the particular choice of our amplitude and has the effect of introducing fixed poles at the wrong signature non sense value of the angular momentum only in the even signature partial amplitude  $a^+(l, s)$ . For a general term of the form

$$\frac{\Gamma(1-\alpha(t))\Gamma(m-\alpha(u))}{\Gamma(n-\alpha(t)-\alpha(u))}$$

fixed poles at the wrong signature nonsense integer are present in both  $a^\pm(l, s)$ <sup>(7)</sup>. In the following we eliminate the  $\pm$  index. It turns out that the projection of the amplitude  $A_3(t, u)$  can be carried out in almost the same way as above. The final expression for the Lorentz projection turns out to be

$$(18) \quad A_3(\lambda, s) = - \frac{2 \delta_3 [\alpha(s) - \delta]}{a m_\pi^2} \int_0^\infty dx x^{\lambda-1} g(x)$$

with  $\delta = 4am_\pi^2 + 3b - 1$  and

$$(19) \quad g(x) = e^{-x \left[ \frac{1-b}{2am_\pi^2} - 1 \right]} \left[ 1 + e^{-\frac{x}{2am_\pi^2}} \right]^{-\alpha(s) + \delta - 1} \left( \frac{x}{2} \right)^{-\lambda} I_\lambda(x)$$

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( $\mathbf{x}$ ) - The Lorentz group is a symmetry group of the scattering amplitude only for pairwise equal mass scattering at  $s=0$ .

It is easily seen that Eq. (18) defines an analytic function of  $\lambda$  in the region  $\text{Re } \lambda > 0$ , and an analytic continuation for the whole  $\lambda$  plane. It gives simple poles at  $\lambda = -m (m=0, 1, 2, \dots)$  and an essential singularity as  $\lambda \rightarrow \infty$ . The residues of the poles are simply given by

$$(20) \quad \beta_m(s) = - \frac{2 \gamma_3 [\alpha(s) - \delta]}{a m^2 \pi m!} g^{(m)}(0)$$

At  $s=0$  (see the footnote at page 6) all the residues of the poles at the odd  $m$  vanish: this corresponds, in the angular momentum plane, to the fact that additive fixed poles are present only at the points of wrong signature.

#### REFERENCES. -

- (1) - M. Toller, Nuovo Cimento 53A, 671 (1968). References to previous works can be found in this paper.
- (2) - G. Veneziano, Nuovo Cimento 57A, 190 (1968).
- (3) - C. Lovelace, Phys. Letters 28B, 265 (1968).
- (4) - G. Veneziano, Invited paper at the Coral Gable Conference on Fundamental Interactions at High Energy (January 1969).
- (5) - M.L. Paciello, L. Sertorio and B. Taglienti, Rome preprint n. 200 (1968).
- (6) - F. Drago and S. Matsuda, California Institute of Technology preprint CALT-68/181 (revised version).
- (7) - V. Alessandrini and D. Amati, Orsay preprint 69/14.
- (8) - L. Durand, P.M. Fishbane and L.M. Simmons, Phys. Rev. Letters 21, 1654 (1968).
- (9) - Bateman Manuscript Project, Higher Transcendental Functions, Vol. II.
- (10) - J. Shapiro and J. Yellin, Berkeley preprint UCRL-18500.