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B. Touschek: COVARIANT THERMODYNAMICS. -

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The relativistic transformation properties of thermodynamic quantities have recently received much attention. The work of this subject is reviewed by C. Møller[1] and it appears from this review, as well as from the perusal of the original papers, that the transformation properties of the temperature are still an open problem.

In this note we want to present an outline of a covariant formulation of statistical mechanics, from which a completely covariant form of thermodynamics can naturally be derived.

The controversy about the relativistic properties of thermodynamic quantities is rendered possible by the fact that no attempt has been made to either define the way in which the temperature of a moving medium can be measured or to discuss the scope of a measurement of temperature of a moving gas, which would supply an operative definition of the temperature. This feature of the controversy has been noted by F. Rohrlich[2]. An operational definition of the temperature is essential for a covariant formulation of the problem. In conventional thermodynamics a measurement of temperature allows one to create conditions in which two test bodies can be brought into contact in such a way that their inner equilibrium is not disturbed. For this it is necessary and sufficient that their temperatures are equal. In relativistic thermodynamics the equality of the temperature alone does not guarantee the equilibrium between two bodies: two jets of gas of equal temperature (measured with a thermometer in their respective rest frames) will only be in equilibrium if also their drift velocities coincide. It is therefore seen that the equilibrium of moving media must be defined in terms of at least 4 quantities: the temperature as well as the centre of mass velocity.

The principal tool for the formulation of covariant statistical mechanics is a generalization of Gibbs' ensemble. In the original version this ensemble allows only for the exchange of energy between its component systems and it is clear that in this form it cannot serve as a basis of relativistic theory since energy and momentum form a four vector. The generalized Gibbs ensemble will therefore have to allow an exchange of momentum as well as of energy.

The states of the component systems of the ensemble will be labelled by an index α. To every state of a component system there corresponds an energy momentum 4-vector $|p_\mu^\alpha|$. The state α also defines a mass $m_\alpha^2 = (p^\alpha, p^\alpha)$. The distribution of states of the systems which we assume to be macroscopic, can also be described in terms of a function $\rho(p)$, so that $\rho(p)\, d^4 p$ is the number of states in a four momentum interval $d^4 p$. If the systems composing the ensemble are not confined by the boxes which one usually associates with the conventional Gibbs ensemble or if the boxes are included in the definition of the systems and allowed to move freely the function ρ will be relativistically invariant and therefore only dependent on $p^2 = m^2$; we exclude gravitational forces.

The ensemble is composed of A systems and characterized by the occupation numbers a_α; A is assumed to be so large that also all the a_α can be considered large compared with unity. For a given distribution of occupation numbers a_α the number of states of the ensemble is $W = A!/\prod_\alpha a_\alpha!$.

Statistical equilibrium of the ensemble is determined by finding the maximum of logW subject to the conditions

$$(1) \qquad A = \sum_\alpha a_\alpha \qquad AP_\mu = \sum_\alpha a_\alpha p_\mu^\alpha$$

AP_μ is the total energy momentum fourvector of the ensemble and P_μ is the average momentum of a component system. Statistical equilibrium is therefore characterized by

$$(2) \qquad a_\alpha = \mathring{A} \exp(-\beta^\mu p_\mu^\alpha)/Z$$

with

(3) $$Z = \sum_{\alpha} e^{-(\beta, p^{\alpha})}$$

The sum over states Z will converge provided that β^{μ} is time like with $\beta^{o} > 0$ and that the level density of the component systems $\rho(p)$ increases less than exponentially, with $p^{o} > 0$. This follows from the fact that p^{o} is a timelike vector.

Introducing $f(x^{\alpha}) = \log Z$ $(x^{\alpha} = (\beta, p^{\alpha}))$ one can write for equation (2)

(4) $$a_{\alpha} = -A(\partial f/\partial x^{\alpha})$$

The second condition (1) can be written

(5) $$P_{\mu} = -(\partial f/\partial \beta^{\mu})$$

and β^{μ} has to be determined as a function of P_{μ} from this equation.

A variation of the four momentum of the ensemble is given by

(6) $$A\,\delta P_{\mu} = \sum_{\alpha} (\delta a^{\alpha}\, p^{\alpha}_{\mu} + a^{\alpha} \delta\, p^{\alpha}_{\mu})$$

The zeroth component of the first term corresponds to the heat added to the ensemble and the zeroth component of the second to the work done by the ensemble.

The present formalism therefore immediately suggests the introduction of a "four-heat" the existence of which has been demonstrated by Møller(1). This "four-heat" is given by

(7) $$A\,\delta Q_{\mu} = \sum_{\alpha} \delta a^{\alpha}\, p^{\alpha}_{\mu}$$

The integrating factor of (7) for reversible changes is β^{μ}, for in a reversible process one has, because of (4)

(8) $$\delta a^{\alpha} = -A \sum_{\beta} (\partial^{2} f/\partial x^{\alpha} \partial x^{\beta})\, \delta x^{\beta}$$

so that

$$\beta^{\mu} \delta Q_{\mu} = -\sum_{\alpha\beta} x^{\alpha} (\partial^{2} f/\partial x^{\alpha} \partial x^{\beta})\, \delta x^{\beta} = \sum_{\beta} (\partial G/\partial x^{\beta})\, \delta x^{\beta}$$

with

$$G = f - \sum_{\alpha} x^{\alpha} (\partial f/\partial x^{\alpha}) \quad .$$

Using (4) and the second condition (1) we can write

$$G = (\beta, P) + f \tag{9}$$

It is immediately seen, that if the space part of P is zero G becomes $\beta^o P_o + f$ and can therefore be identified with the entropy: $S = kG$, the temperature in this case being given by $T = 1/k\beta^o$.

The relativistic invariance of the formalism can now easily be verified. Writing for (3)

$$Z = \int d^4 p \rho (p) \, e^{-(\beta, p)} \tag{3'}$$

it is seen that with $\rho(p)$ a relativistically invariant function of p^2, Z can only depend on $\beta^2 = \beta_o^2 - \vec{\beta}^2$. It then follows immediately from equation (5) that β behaves like a four vector and that G defined by (9) is a scalar.

The obvious covariance of the formalism allows one to determine all quantities in the rest system, which is characterized by $\vec{P} = 0$ and therefore also by $\vec{\beta} = 0$ because of (5), for which one can write

$$P_\mu = -2\beta_\mu \, (\partial f / \partial \beta^2) \ . \tag{10}$$

It is seen that in this special frame of reference the theory developed here coincides completely with conventional statistical theory.

The behaviour of a system in equilibrium with a heat reservoir is regulated by the free energy. In the present case the heat reservoir can be represented by the ensemble of equal systems carrying momentum and energy. The free energy in this case is replaced by a four vector, precisely

$$F_\mu = P_\mu - \beta_\mu G / \beta^2 \tag{11}$$

In the rest system this coincides with the free energy $F = E - TS$. If the system is in equilibrium with the ensemble F its assumes minimum value. The condition of equilibrium is therefore $\delta F = 0$ in conventional thermodynamics. The present formalism allows us to generalize this consideration and to include heat reservoirs with a drift velocity $v_i = \beta_i / \beta_o$. The equilibrium condition is in this case

$$\delta F_\mu = 0 \ . \tag{12}$$

We finally consider black body radiation as a practical application: we want to write Planck's formula in a relativistically invariant fashion. To achieve this we introduce a flux density $d^4 j_\mu(k)$, the fourth component of which describes the density of photons in a four momentum interval d^4k. From the present formalism one obtains for this quantity:

$$d^4 j_\mu(k) = a\, d^4k\, k_\mu \delta(k^2)\, (e^{(\beta, k)} - 1)^{-1} \qquad (13)$$

in which a is a numerical constant. It is easily verified that this expression coincides with Planck's formula in the rest frame. In a moving frame of reference one verifies that (13) is identical with the expression for the radiation from a moving black body derived by F. Hasenöhrl(3) and others.

It is a pleasure to thank Dr. E. Etim for instructive discussions on this subject.

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(2) - F. Rohrlich, Syracuse University preprint, June 1966.

(3) - F. Hasenoehrl, Wien. Ber. 113, 1039 (1904); Ann. Phys. 15, 344 (1905); K. v. Mosengeil, Ann. Phys. 22, 867 (1907); Compare W.Pauli, Relativitätstheorie (Enz. Math. Wiss), p. 697.