A. Malecki : REAL PHOTON APPROXIMATION FOR ELECTRO MAGNETIC PROCESSES ON LIGHT NUCLEI. -

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A. Małecki[(x)]: REAL PHOTON APPROXIMATION FOR ELECTRO-
MAGNETIC PROCESSES ON LIGHT NUCLEI. -

ABSTRACT. -

   We are interested in the electrodynamic processes on light
nuclei and we try to relate them to the nuclear photoabsorption and
to the gamma emission processes. In particular, the Weizsäcker-
-Williams approximation is analyzed. The general inelastic sum
rule, correct through order $1/M^2$ (M - nucleon mass), with the
corrections due to subtraction of the c. m. motion, is derived.
High energy electron scattering and electron pair photoproduction
are discussed as examples.

(x) - On leave of absence from Instytut Fizyki Jądrowej, Cracow,
      Poland.

1.   Electrodynamic processes on nuclear targets are usually de-
scribed in the first Born approximation in $e^2 = 1/137$. In the framework of this approximation (which is expected to be valid for light
nuclei) such processes as e. g. electron scattering, electron pair
production and bremsstrahlung from a nucleus are represented by
diagrams with one photon line connected to a nuclear vertex (see
Fig. 1). It seems that it should be possible to relate these proces-
ses to the photoabsorption or gamma emission processes, indicated
in Fig. 2. Each of them is a photon - induced reaction on the same
nuclear target where the photon can either be real ($k^2 = 0$) as in
photoabsorption and gamma decay or virtual $(q_\mu^2 = \omega^2 - \vec{q}^2 \leq 0)$
as for the processes indicated in Fig. 1.

For an arbitrary photon - induced process the cross-section
is determined by the contraction of two second-rank tensors $M_{\mu\nu}$
$W_{\mu\nu}$, where $M_{\mu\nu}$ describes the electromagnetic and $W_{\mu\nu}$ the
nuclear part of the process. It is well known[1] that the nuclear part
of the interaction is summarized in terms of only two scalar form
factors $W_{1,2}(\omega, \vec{q}^2)$ as follows :

$$
W_{\mu\nu} = W_1 (g_{\mu\nu} - \frac{q_\mu q_\nu}{q_\alpha^2}) + W_2 \frac{1}{M_T^2} (P_\mu - \frac{P_\alpha q_\alpha}{q_\alpha^2} q_\mu) \cdot
$$

(1)

$$
\cdot (P_\nu - \frac{P_\alpha q_\alpha}{q_\alpha^2} q_\nu) ,
$$

where P_μ $(E, \vec{P})$ denotes the initial four momentum of the target
$(P_\mu^2 = M_T^2)$ and $q_\mu(\omega, \vec{q})$ is the four momentum of the photon
$(P_\mu' - P_\mu = q_\mu)$.

As any process connected to the nucleus through a single pho-
ton line depends on the same two form factors W_1 and W_2 it gives a
basis for relations among various electrodynamic processes. In
particular one can try to express them through the photoabsorption
cross-sections or the gamma emission transition rates.

2.   The photoabsorption cross-section and the transition rate for
gamma emission can be expressed through the nuclear tensor $W_{\mu\nu}$
as follows :

(2) $\sigma_\gamma (k) = (2\pi)^2 e^2 \frac{1}{k} \sum_{pol} \varepsilon_\mu^* \varepsilon_\nu W_{\mu\nu} \Big|_{q = \omega = k}$,

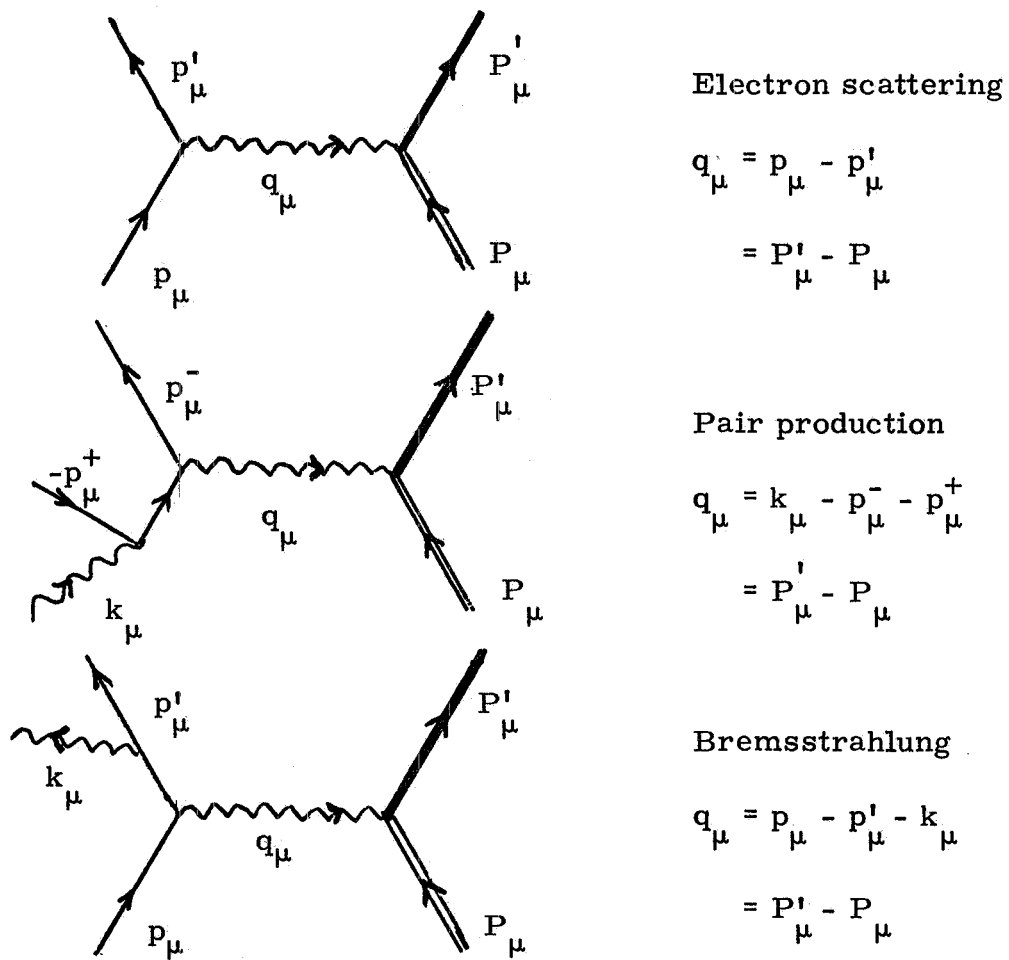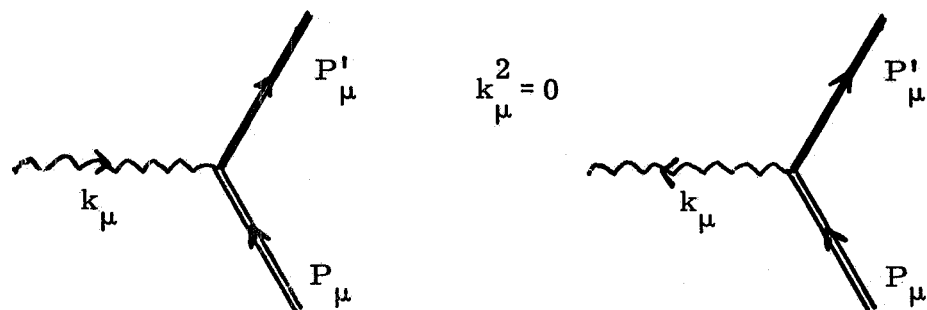FIG. 1 - Electrodynamic processes on a nucleus leading to arbitrary final nuclear states.

Electron scattering

$$q_\mu = p_\mu - p'_\mu$$

$$= P'_\mu - P_\mu$$

Pair production

$$q_\mu = k_\mu - p^-_\mu - p^+_\mu$$

$$= P'_\mu - P_\mu$$

Bremsstrahlung

$$q_\mu = p_\mu - p'_\mu - k_\mu$$

$$= P'_\mu - P_\mu$$

$$k^2_\mu = 0$$

FIG. 2 - Photoexcitation and gamma decay of a nucleus.

4.

(3)
$$\frac{d\,\rho_\gamma(k)}{dk} = 4\,e^2\,k\,\sum_{pol}\,\mathcal{E}^{\times}_\mu\,\mathcal{E}_\nu\,W_{\mu\nu}\Bigg|_{q\,=\,\omega\,=\,k}\,,$$

where $\mathcal{E}_\mu(\mathcal{E}_0,\vec{\mathcal{E}})$ is the unit polarization four vector. Three independent vectors of polarization, satisfying the transversality condition[2]

(4)
$$q^2\mathcal{E}_0 = \omega\,\vec{\mathcal{E}}\cdot\vec{q}\,,$$

can be chosen in the following way[3] ($\vec{q}$ along the x-axis) :

(5)
$$\mathcal{E}^{(1)}_\mu = (0,\ 0,\ 1,\ 0)$$
$$\mathcal{E}^{(2)}_\mu = (0,\ 0,\ 0,\ 1)$$
$$\mathcal{E}^{(3)}_\mu = \frac{1}{\sqrt{-q_\alpha^2}}\,(\omega,\ q,\ 0,\ 0)\ .$$

Using (1) and (5) we get :

(6)
$$\sum_{i=1}^{3}\,\mathcal{E}^{\times(i)}_\mu\,\mathcal{E}^{(i)}_\nu\,W_{\mu\nu} = \sum_{i=1}^{2}\,\mathcal{E}^{\times(i)}_\mu\,\mathcal{E}^{(i)}_\nu\,W_{\mu\nu} = -2\,W_1\,,$$

The photoabsorption cross-section[x] thus measures at the straight line $q = \omega = k$ the nuclear form factor $W_1(\omega,q)$ only.

3. To obtain the cross-section for an arbitrary electromagnetic process, as indicated in Fig. 3, we form the product of $W_{\mu\nu}$ with the electromagnetic tensor $M_{\mu\nu}$, observing that the terms in (1) proportional to q_μ or q_ν vanish by current conservation:

(7)
$$M_{\mu\nu}\,q_\mu = M_{\mu\nu}\,q_\nu = 0$$

(x) - We confine ourselves in the further considerations to the real photon approximation connected with the nuclear photoabsorption process. The approximation associated with gamma emission can be treated in the same way.

We get the following formula for the cross-section[(x)] :

$$\frac{d^{2F}\sigma}{\prod\limits_{j=1}^{F} d\mathcal{E}_f(j)\, d\Omega_f(j)} = \frac{2\, e^{2(F+1)}}{(16\pi^3)^{F-1}}\, M(\omega)\left[W_2 + \frac{M_{\mu\mu}}{M_{oo}}\, W_1\right]_{LAB}$$

(8)

$$M(\omega) = \frac{M_{oo}}{p_p\, q_\alpha^4}\, \prod\limits_{j=1}^{F} p_f(j) \ ,$$

where p_p is the projectile momentum in the laboratory frame. The product symbol refers to all particles in the final state with four mo-
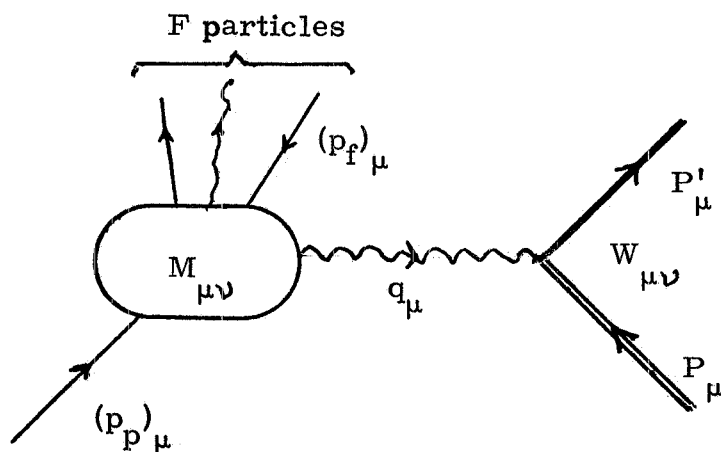
$(p_p)_\mu$ — projectile four momentum

$(p_f)_\mu$ — four momentum of outgoing particles

$q_\mu(\omega, \vec{q}) = P'_\mu - P_\mu$ — four momentum of the virtual photon

FIG. 3

menta $(p_f)_\mu = (\mathcal{E}_f, \vec{p}_f)$ which are produced in the electromagnetic part of the process. We have assumed that wave functions of particles in the electrodynamic part are normalized to twice their energies while the nuclear states are normalized to unity.

(x) — In the case of the heteroenergetic projectile beam with a spectrum $S(\mathcal{E}_p)$ this formula has to be changed slightly. The only change needed is to divide the L.H.S. of (8) and hence the integrand function in (18) and (19) by $d\mathcal{E}_p$, and to modify the definition of $M(\omega)$ multiplying it by $S(\mathcal{E}_p)$.

6.

4. The nuclear tensor $W_{\mu\nu}$ can be expressed through the three dimensional Fourier transform of the electromagnetic nuclear current as follows :

$$W_{\mu\nu} = \frac{1}{2} \sum_{|i\rangle} \sum_{|f\rangle} J^{x}_{\mu fi}(\vec{q}) \, J_{\nu fi}(\vec{q}) \, \delta(\omega - \omega_f)$$

(9)

$$J_{\mu fi}(x_\alpha) \equiv \langle f | J_\mu(x_\alpha) | i \rangle \equiv \left[Q_{fi}(x_\alpha), \, \vec{J}_{fi}(x_\alpha) \right] \, ,$$

where $Q(x_\alpha)$, $\vec{J}(x_\alpha)$ are the charge and current operators of the nucleus at the point $x_\alpha(t, \vec{x})$, and $|i\rangle, |f\rangle$ are initial and final nuclear states, respectively. We neglect the nuclear recoil energy[(x)] and thus $\omega_f = E_f - E_i$ is the excitation energy of the nucleus.

It can be shown, using (1) and the nuclear current conservation :

(10)
$$\vec{q} \cdot \vec{J}_{fi}(\vec{q}) = \omega \, Q_{fi}(\vec{q}) \, ,$$

that

$$W_1 = -\frac{1}{4} \sum_{|i\rangle}\sum_{|f\rangle} \delta(\omega - \omega_f) (\vec{J}^{x}_{fi} \cdot \vec{J}_{fi})_\perp \bigg|_{\text{LAB}}$$

(11)
$$W_2 = \frac{1}{2} \sum_{|i\rangle}\sum_{|f\rangle} \delta(\omega - \omega_f) \left[\frac{q^4_\alpha}{q^4} Q^{x}_{fi} Q_{fi} - \frac{q^2_\alpha}{q^2} (\vec{J}^{x}_{fi} \cdot \vec{J}_{fi})_\perp \right]_{\text{LAB}}$$

$$(\vec{J}^{x}_{fi} \cdot \vec{J}_{fi})_\perp \equiv \vec{J}^{x}_{fi} \cdot \vec{J}_{fi} - \frac{1}{q^2} (\vec{q} \cdot \vec{J}^{x}_{fi})(\vec{q} \cdot \vec{J}_{fi}) \, ,$$

where all the quantities are to be taken in the laboratory frame.

From (2), (6) and (11) we get :

(x) - The largest correction which comes from the nuclear recoil energy is the kinematical factor (see (17) - $(1 - \frac{QQ_1}{2M_T})^{-1}$.

This factor does not affect the ratios R_1 and R_2 we calculate in the last section.

$$W_1 = - \frac{\omega \gamma_1}{2(2\pi)^2 e^2} \mathbb{G}_\gamma (\omega \gamma_2) + O_1$$

(12)

$$W_2 = - \frac{\omega \gamma_1}{2(2\pi)^2 e^2} \frac{q_\alpha^2}{q^2} \mathbb{G}_\gamma (\omega \gamma_2) + \frac{q_\alpha^2}{q^2} O_1 + O_2 ,$$

where

$$O_1 = -\frac{1}{4} \sum_{|i\rangle} \sum_{|f\rangle} \delta(\omega - \omega_f) \left[\vec{J}_{fi}^*(\vec{q}) \cdot \vec{J}_{fi}(\vec{q}) \right]_\perp +$$

(13)

$$+ \frac{1}{4} \sum_{|i\rangle} \sum_{|f\rangle} \delta(\omega\gamma_2 - \omega_f) \frac{\gamma_1}{\gamma_2} \left[\vec{J}_{fi}^*(\frac{\omega\gamma_2}{q}\vec{q}) \cdot \vec{J}_{fi}(\frac{\omega\gamma_2}{q}\vec{q}) \right]_\perp ,$$

(14) $$\quad \theta_2 = \frac{1}{2} \sum_{|i\rangle} \sum_{|f\rangle} \delta(\omega - \omega_f) \frac{q_\alpha^4}{q^4} Q_{fi}^* Q_{fi} ,$$

and $\gamma_{1,2}(q,\omega)$ are some functions of three momentum transfer and energy loss.

One can hope that it would be possible to find such functions $\gamma_{1,2}(q,\omega)$ which would minimize $O_1(q,\omega)$ and thus obtain by putting in (12) : $O_1 = O_2 = 0$, a valuable (at least in certain region of the q, ω plane) approximation to the nuclear form factors.

We will discuss in the following only the approximation with $\gamma_1 = \gamma_2 = 1$. This is the so-called Weizsäcker-Williams approximation which is expected to be a good one above all for small $q_\mu^2 = \omega^2 - q^2$.

5.   We want to compare the W-W approximation with the formulas (11) in terms of the inelastic sum rule which one obtains when summing (8) over the whole nuclear spectrum. To do the calculations we accept the following nuclear model. We assume the charge and current operator of the nucleus to be the sum of the operators for the individual nucleons. Consistently we use the single-particle model (including Pauli correlations) for the description of the nuclear states. We choose the shell model potential to be the oscillator potential well, with the oscillator spacing $\omega_o \equiv \alpha^2/M$ (M - nucleon mass). Following McVoy and Van Hove we use the nonrelativistic form of the nuclear charge and current operators[4] including terms to the order $1/M^2$:

8.

$$Q(\vec{q}) = \sum_{j=1}^{A} \left\{ \left[e_j (1 - \frac{a}{2} \frac{q^2}{M^2} + \frac{q^2}{8M^2}) - \mu_j \frac{q^2}{4M^2} \right] e^{i\vec{q}\cdot\vec{r}_j} \right\}$$

(15)

$$\vec{J}(\vec{q}) = \sum_{j=1}^{A} \left\{ \frac{e_j}{2M} (\vec{p}_j e^{i\vec{q}\cdot\vec{r}_j} + e^{i\vec{q}\cdot\vec{r}_j}\vec{p}_j + \frac{\mu_j}{2M} i(\vec{\sigma}_j \times \vec{q}) e^{i\vec{q}\cdot\vec{r}_j} \right\},$$

where $e_j = \dfrac{1+\mathcal{T}_{3j}}{2}$, $\mu_j = \dfrac{\mu_p + \mu_n}{2} + \dfrac{\mu_p - \mu_n}{2} \mathcal{T}_{3j}$, and $\vec{r}_j$, $\vec{p}_j$,

$\frac{1}{2} \vec{\sigma}_j$ are the position, momentum and spin operators for the j-th nucleon. The nucleon form factor is assumed to be $f(q^2) = 1 - \frac{a}{2} \frac{q^2}{M^2}$. Ref. (5) gives the value of $a = 4.82$.

Consistently with (15) we will keep in our calculations only terms through order $1/M^2$.

Let us write :

(16)
$$M(\omega) = M(0) (1 + M_1 \frac{\omega}{Q} + M_2 \frac{\omega^2}{Q^2}) ,$$

where Q is the elastic three momentum transfer. When one performs the inelastic sum rule the momentum transfer q varies in general with the energy loss ω, as follows :

(17)
$$q^2 = Q^2 (1 + Q_1 \frac{\omega}{Q} + Q_2 \frac{\omega^2}{Q^2}) .$$

For nuclei with filled s-shell and Z-2 protons and neutrons in p-shell we get from (8), (11), (15), (16) and (17) - for more details see (6) and appendix - the inelastic sum rule :

$$\frac{(16\pi^3)^{F-1}}{e^{2(F+1)} M(0)} \int_{inel} d\omega \frac{d^{2F}\sigma}{\prod_{j=1}^{F} d\varepsilon_f(j) \, d\Omega_f(j)} =$$

(18)
$$= \left\{ 1 - 2aT \frac{\alpha^2}{M^2} - (2\mu_p - 1) T \frac{\alpha^2}{2M^2} + \left[1 - (\frac{M_{\mu\mu}}{M_{oo}})_o \right] \cdot \right.$$

$$\left. \cdot (\mu_p^2 + \mu_n^2) T \frac{\alpha^2}{2M^2} \right\} \left[Z - (Z + \frac{Z-2}{3} T^2) e^{-T} \right] + \quad \ldots \ldots$$

$$+ (-2 + M_2) \left[5Z - 4 + 3ZT - 2(Z-2) e^{-T} \right] \frac{\alpha^2}{6M^2} +$$

$$+ \left[1 - (\frac{M_{\mu\mu}}{M_{oo}})_o \right] \left[5Z - 4 - 2(Z-2)(1+T) e^{-T} \right] \frac{\alpha^2}{6M^2} +$$

(18)
$$+ \frac{Z\sqrt{T}}{\sqrt{2}} (Q_1 + M_1) \frac{\alpha}{M} + \frac{\alpha^2}{6M^2} \left\{ (Q_2 + Q_1 M_1) \left[5Z - 4 + 6ZT - 2(Z-2)(1-T) e^{-T} \right] + Q_1^2 T \left[3Z + (Z-2)(2-T) e^{-T} \right] \right\} -$$

$$- \frac{1}{3} (e^{T/A} - 1) e^{-T} \left\{ \left[1 - 2aT \frac{\alpha^2}{M^2} - (2\mu_p + 2\mu_n - 1) T \frac{\alpha^2}{2M^2} \right] \cdot \right.$$

$$\cdot \left[3Z^2 - 2Z(Z-2)T + 2(Z-2)T^2 \right] + \left[1 - (\frac{M_{\mu\mu}}{M_{oo}})_o \right] T \frac{\alpha^2}{4M^2} (8-Z)(Z-2) \right\} -$$

$$- \left[1 - (\frac{M_{\mu\mu}}{M_{oo}})_o \right] \frac{\alpha^2}{6AM^2} \left\{ 3Z + e^{-T} \left[3Z(Z-1) - 2Z(Z-2)T + (Z-2)T^2 \right] \right\}.$$

where $T = \dfrac{Q^2}{2\alpha^2}$ and $(\dfrac{M_{\mu\mu}}{M_{oo}})_o$ indicates the value of $\dfrac{M_{\mu\mu}}{M_{oo}}$ at $\omega = 0$.

Formula (18) was given in [6] except for the last two terms which are corrections due to the Gartenhaus-Schwartz[7] transformation. With the aid of this transformation we remove contributions from some spurious excitations of the centre of mass degrees of freedom. These spurious contributions to the sum rule are in general small, A times smaller than the other terms (to make things more graphic we retain A in (18) although A = 2Z), but for small three momentum transfer they become very important (as it was stressed in ref. (8)). The calculations devoted to subtraction of the c. m. motion are presented in more detail in the Appendix.

The Weizsäcker-Williams approximation yields a very simple results. Keeping terms through order $1/M^2$ only, we get from (8), (12), (2) and (15) the approximated inelastic sum rule :

(19)
$$\frac{(16\pi^3)^{F-1}}{e^{2(F+1)} M(0)} \int_{inel} d\omega \left. \frac{d^{2F}\sigma}{\prod\limits_{j=1}^{F} d\varepsilon_f(j) d\Omega_f(j)} \right|_{W-W} =$$

$$= \frac{Z^2}{A} \left[1 - (\frac{M_{\mu\mu}}{M_{oo}})_o \right] \frac{\alpha^2}{2M^2} .$$

6.   We end with a few numerical examples. We calculate the ratio R_1 of the inelastic sum rule in the Weizsäcker-Williams approximation (see eq. (19)) to the inelastic sum rule as given by (18). We are also interested in the ratio R_2 of the inelastic sum rule (18) to the elastic yield given by (A4). We chose carbon as nuclear target. In this case $Z=6$ and the oscillator potential parameter[5] is $\alpha = 121$ MeV.

a) High energy electron scattering at constant beam energy $\mathcal{E}$ and scattering angle θ.

In this case the inelastic sum rule is performed experimentally by varying the final electron energy. The required formulas are :

$$F = 1 \; ; \qquad \frac{M_{\mu\mu}}{M_{oo}}\bigg|_{LAB} = -2\tan^2\frac{\theta}{2}$$

(20)
$$M(0) = \frac{\cos^2\theta/2}{4\,\mathcal{E}^2\sin^4\theta/2} \; ; \qquad M_1 = M_2 = 0$$

$$Q = 2\,\mathcal{E}\sin\theta/2 \; ; \qquad Q_1 = -2\sin\theta/2 \; ; \qquad Q_2 = 1$$

For [9] $\mathcal{E} = 106$ MeV and $\theta = 30^\circ$ we have got $R_1 = 6.5\%$ and $R_2 = 1.4\%$. The first result shows that the W-W approximation is quite improper in this case while the second one suggests the inelastic contribution to be negligible (in disagreement with some experimental results[9]).

b) High energy electron pair photoproduction.

As usually one uses the heteroenergetic bremsstrahlung photon beam, the inelastic sume rule is performed automatically. We consider the symmetric pair case when the electron and positron are produced at the same angle θ with regard to the photon momentum, all the momenta are in the same plane, and both electrons have the same energy $\mathcal{E}$. To a good approximation the bremsstrahlung photon spectrum is $S(k) = f(\mathcal{E}, k_{max})/k$, where k_{max} is the end point energy of the spectrum.

The formulas for this case are :

$$F = 2 \; ; \qquad (\frac{M_{\mu\mu}}{M_{oo}})_o\bigg|_{LAB} = -2\,c\tan^2\theta$$

(21)
$$M(0) = \frac{\pi}{16}\frac{\cos^2\theta/2}{\mathcal{E}^4\sin^6\theta/2} f(\mathcal{E}, k_{max}) \; ; \quad M_1 = -2(1 + 4\sin^2\frac{\theta}{2}) \; ;$$

$$M_2 = 5 + 16 \sin^2 \frac{\theta}{2} + 40 \sin^4 \frac{\theta}{2} \; ;$$

(21)

$$Q = 4 \, \mathcal{E} \, \sin^2 \frac{\theta}{2} \; ; \qquad Q_1 = 2 \; ; \qquad Q_2 = 1$$

Calculations for carbon, at $\theta = 9^\circ 36'$ and $\mathcal{E} = 1973$ MeV (the kinematical conditions in a) and b) give the same value of $Q = 55.2$ MeV in both cases) give the results[x]: $R_1 = 66.0\%$, $R_2 = 8.7\%$. Also in this case the W-W approximation seems to be questionable. This approximation understimates considerably the inelastic sum rule. This may be the reason why the inelastic contribution quoted in [10] is smaller than ours.

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(x) - Ref. (6) in which the Gartenhaus-Schwartz corrections were not taken into account gives $R_2 = 14.7\%$. The difference between that and the present value shows how important are the effects of the c. m. motion at small momentum transfers. In addition, we should like to point out that in (6) there was a mistake in the formula for $\left. \dfrac{M_{\mu\mu}}{M_{oo}} \right|_{LAB}$. The result 14.7% however was based on the correct formulas.

APPENDIX. -

The shell model wave functions are functions of the 3A spatial coordinates of the nucleons. In fact there are only 3(A-1) independent internal coordinates because three of the nucleon coordinates merely give the position of the centre of mass. So, if the shell model functions are used, one gets contributions from some spurious states which correspond to excitation of the center of mass. As these states do not represent internal excitation they must be eliminated from the shell model calculations. To subtract the c. m. motion we employ the Gartenhaus-Schwartz transformation[7] which in our case reduces to the following transformation of the operators involved in calculations :

$$\vec{r}_j \to \vec{r}_j - \vec{R} \quad ; \qquad \vec{p}_j \to \vec{p}_j - \frac{1}{A}\vec{P} \, ,$$

(A1)   where

$$\vec{R} \equiv \frac{1}{A}\sum_{j=1}^{A}\vec{r}_j \qquad \text{and} \qquad \vec{P} \equiv \sum_{j=1}^{A}\vec{p}_j$$

are the c. m. vector of position and the c. m. momentum, respectively.

We are interested in the G-S correction to the inelastic sum rule (see eq. (18)). In order to calculate them we use the closure approximation[11]:

(A2) $$\sum_{|f\rangle}^{inel} g(\omega)\,|\langle f|O|i\rangle|^2 = \langle g\rangle\langle i|O^+O|i\rangle - g(0)\,|\langle i|O|i\rangle|^2$$

where $\langle g\rangle$ is a suitable average of an energy loss dependent function $g(\omega)$. From (A2) and (15) we can see that for the operators involved in our calculations the first term in the R. H. S. of (A2) is affected by the transformation (A1) of nucleon momenta only, while the second one is affected only by the G-S transformation of nucleon coordinates.

It is easy to evaluate the G-S correction to the second term in (A2) in the shell model with oscillator potential. In this case[12] the shell model wave function can be related easily to the c. m. wave function Φ_{CM} and to the actual wave function of internal coordinates.
Using the fact[13] that

(A3) $$\left|\langle\Phi_{CM}|e^{i\vec{Q}\cdot\vec{R}}|\Phi_{CM}\rangle\right|^2 = e^{-T/A}$$

we find the elastic yield to be (see [6] for more details) :

$$\frac{(16\pi^3)^{F-1}}{e^{2(F+1)}M(0)} \int_{el} d\omega \; \frac{d^2F_\sigma}{\prod\limits_{j=1}^{F} d\mathcal{E}_f(j)\, d\Omega_f(j)} =$$

$$(A4) \qquad = \frac{1}{3} e^{T/A} e^{-T} \left\{ \left[1 - 2aT \frac{\alpha^2}{M^2} - (2\mu_p + 2\mu_n - 1)T \frac{\alpha^2}{2M^2} \right] \cdot \right.$$

$$\cdot \left[3Z^2 - 2Z(Z-2)T + 2(Z-2)T^2 \right] + \left[1 - (\frac{M_{\mu\mu}}{M_{oo}})_o \right] \cdot$$

$$\left. \cdot (8-Z)(Z-2)T \frac{\alpha^2}{4M^2} \right\}$$

From (A2) and (A4) we get immediately the first Gartenhaus-Schwartz term in eq. (18).

The second correction in (18) comes from the transformation of nucleon momenta in the nuclear current interaction terms. Using (15) we get the G-S correction :

$$\delta \sum_{|f\rangle} \left[\vec{J}_{fi}^{*}(\vec{q}) \cdot \vec{J}_{fi}(\vec{q}) \right]_\perp =$$

$$= -\frac{2}{M^2} \frac{1}{A} (2 - \frac{1}{A}) \sum_{|f\rangle} \left| \langle f | \sum_{j=1}^{A} e_j \, p_{jz} \, e^{iqxj} | i \rangle \right|^2 \quad -$$

$$(A5) \qquad - \frac{4}{M^2} \frac{1}{A} (1 - \frac{1}{A}) \sum_{|f\rangle} \langle f | \sum_{j=1}^{A} e_j \, p_{jz} e^{iqxj} | i \rangle \cdot$$

$$\cdot \langle f | \sum_{j \neq k}^{A} e_j p_{kz} e^{iqxj} | i \rangle^x + \frac{2}{M^2 A^2} \sum_{|f\rangle} \left| \langle f | \sum_{j \neq k}^{A} e_j p_{kz} e^{iqxj} | i \rangle \right|^2$$

where we chose $\vec{q}$ along the x-axis.

The only final nuclear states which contribute in (A5) are one particle-one hole states $| f \rangle = | \alpha_p \alpha_h \rangle$ and two particle - two hole

14.

states $|f\rangle = |\alpha_p \beta_p \alpha_h \beta_h\rangle$ where $|\alpha_p\rangle$, $|\beta_p\rangle$ and $|\alpha_h\rangle$, $|\beta_h\rangle$ are single particle states, respectively above and below the Fermi level of the nucleus.

In the shell model with the oscillator potential we get, for nuclei with filled s-shell and $Z-2$ protons and neutrons in p-shell, the following results :

$$\sum_{\substack{|\alpha_p\alpha_n\rangle \\ (n\omega_0)}} \left| \langle \alpha_p\alpha_h| \sum_{j=1}^{A} e_j p_{jz} e^{iqxj}|i\rangle \right|^2 = \frac{\alpha^2}{6} e^{-t} \frac{t^{n-1}}{(n-1)!} (1 - \delta_{no}) \cdot$$

(A6)

$$\cdot \left[3Z+(Z-2)(n-2t+\frac{n+2}{n+1}\frac{t^2}{n} - \delta_{n1}) \right] + \frac{\alpha^2}{12} e^{-t} t \delta_{no}(8-Z)(Z-2) \; ,$$

$$\sum_{\substack{|\alpha_p\alpha_n\rangle \\ (n\omega_0)}} \left| \langle \alpha_p\alpha_h| \sum_{j=1}^{A} e_j p_{jz} e^{iqxj}|i\rangle \right.^* \langle \alpha_p\alpha_h| \sum_{j\neq k}^{A} e_j p_{kz} e^{iqxj}|i\rangle =$$

(A7)

$$= -\frac{Z-2}{6} \alpha^2 e^{-t} \frac{t^{n-1}}{(n-1)!} (1 - \delta_{no})(1 +\frac{1}{n}\frac{t^2}{n+1}) -\frac{\alpha^2}{12} e^{-t} t \delta_{no}(8-Z)(Z-2) +$$

$$+ \frac{\alpha^2}{6} e^{-t} \delta_{n1} \left[3Z^2-2Z-2-2(Z-1)(Z-2)t +(Z-2)t^2 \right] - \frac{Z-2}{6} \alpha^2 t e^{-t} \delta_{n2} \; ,$$

$$\sum_{\substack{|\alpha_p\alpha_n\rangle \\ (n\omega_0)}} \left| \langle \alpha_p\alpha_h| \sum_{j\neq k}^{A} e_j p_{kz} e^{iqxj}|i\rangle \right|^2 = \frac{\alpha^2}{6} e^{-t} \frac{t^{n-1}}{(n-1)!} (1 - \delta_{no})(Z-2) \cdot$$

(A8)

$$\cdot (1 +\frac{1}{n}\frac{t^2}{n+1}) + \frac{\alpha^2}{12} e^{-t} t \delta_{no}(8-Z)(Z-2) + \frac{\alpha^2}{6} e^{-t}\delta_{n1} \left[6Z^3-6Z^2+2Z+ \right.$$

$$\left. + 2-2(Z-2)(2Z^2-2Z+1)t +(Z-2)(4Z-3)t^2 \right] +\frac{\alpha^2}{12} e^{-t} t \delta_{n2}(Z-2)^2 \; ,$$

$$\sum_{\substack{|\alpha_p\beta_p\alpha_h\beta_h\rangle \\ (n\omega_0)}} \left| \langle \alpha_p\beta_p\alpha_h\beta_h| \sum_{j\neq k}^{A} e_j p_{kz} e^{iqxj}|i\rangle \right|^2 = \frac{\alpha^2}{6} e^{-t} \frac{t^{n-1}}{(n-1)!} (1- \delta_{no}-\delta_{n1}) \cdot$$

(A9)

$$\cdot (2Z-1) \left[2(Z+1)+(Z-2)n(1 - \frac{t}{n})^2 \right] - \frac{\alpha^2}{12} e^{-t} t \delta_{n2}(Z-2)(5Z-6) \; ,$$

where $t = q^2/2\alpha^2$ and all the final states one sums over have the same excitation energy $n\omega_0$. .

Performing for (A6) - (A9) the summation over n (from zero to infinity) we obtain on the basis of eq. (A5) the second Gartenhaus-Schwartz term in (18).

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