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M. Bassetti: FINITE DIFFERENCE EQUATIONS CALCULATION  
OF BEAM-CAVITY COUPLING INSTABILITY.

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The problem of the influence of the beam loading on the stability of the coherent synchrotron oscillations is resumed and discussed in this work by means of finite difference equations i. e. in a way that differs from the ones employed by preceding authors (1, 2, 3).

#### INTRODUCTION -

##### Model of the phenomenon -

- 1) It is assumed to have only one beam and a single radiofrequency cavity on the first harmonic.
- 2) The cavity can be represented by an equivalent resonant circuit in parallel, with its three R, L, C, parameters.
- 3) The cavity is powered by an amplifier which, at intervals of time T equal to the synchronous period of the particles, brings about a constant perturbation of voltage and current.
- 4) The beam is described by a rigid distribution of charges determining, at every turn, a constant perturbation of voltage and current.
- 5) The structure of the machine is characterized by the "momentum compaction",  $\alpha_c$ , the period T of rotation of synchronous particles, the energy  $W_0$  irradiated by the beam in each turn and its variations which will be later specified.

Furthermore, the machine is assumed to work at constant energy  $U_0$ .

2.

1 - EQUATIONS AND MAGNITUDES CONCERNING THE ISOLATED CAVITY -

In our schematization of a resonant cavity as an R, L, C. oscillating circuit in parallel, the evolution in time of the electromagnetic field is represented by the homogeneous differential equation.

1. 1. 
$$\frac{d^2 x}{dt^2} + \frac{1}{RC} \frac{dx}{dt} + \frac{1}{LC} x = 0$$

where x can be a current flowing through any one branch of the circuit shown in fig. 1 or the voltage difference between two points of the same circuit. Let us define some parameters connected to the magnitudes of R, L, and C.

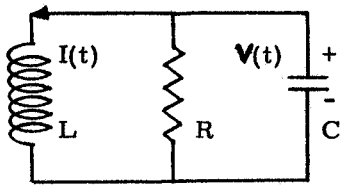


FIG. 1

1. 2. 
$$\begin{cases} \alpha = 1/2RC & \beta = \beta_0 \cos \psi & Q = R/Z_0 \\ \beta_0^2 = 1/LC & Z_0 = 1/\beta_0 C \\ \beta^2 = \beta_0^2 - \alpha^2 & Z_0 = \beta_0 L \\ \alpha = \beta_0 \sin \psi & Z_0 = (L/C)^{1/2} \end{cases}$$

From fig. 1, one obtains

1. 3. 
$$V(t) = L \frac{dI(t)}{dt}$$

If I(t) is taken as the variable x of equation 1. 1 it follows that

1. 4. 
$$\begin{cases} x(t) = I(t) \\ L \dot{x}(t) = V(t) \end{cases}$$

and by means of formula A1. 4 obtained in Appendix 1, the solution of equation 1. 1 is found to be

1. 5. 
$$\begin{cases} V(t) = e^{-\alpha t} \left[ (\cos \beta t - \frac{\alpha}{\beta} \sin \beta t) V(0) - \frac{\sin \beta t}{\beta C} I(0) \right] \\ I(t) = e^{-\alpha t} \left[ \frac{\sin \beta t}{L} V(0) + (\cos \beta t + \frac{\alpha}{\beta} \sin \beta t) I(0) \right] \end{cases}$$

or else

1. 6. 
$$\begin{cases} V(t) = e^{-\alpha t} \left[ \frac{\cos(\beta t + \psi)}{\cos \psi} V(0) - \frac{\sin \beta t}{\cos \psi} Z_0 I(0) \right] \\ I(t) = e^{-\alpha t} \left[ \frac{\sin \beta t}{\cos \psi} \frac{V(0)}{Z_0} + \frac{\cos(\beta t - \psi)}{\cos \psi} I(0) \right] \end{cases}$$

If one writes

1. 7. 
$$\vec{v}(t) = \begin{vmatrix} V(t) \\ I(t) \end{vmatrix} \quad \text{and} \quad M(\beta t) = \begin{vmatrix} \frac{\cos(\beta t + \psi)}{\cos \psi} & -Z_0 \frac{\sin \beta t}{\cos \psi} \\ \frac{\sin \beta t}{Z_0 \cos \psi} & \frac{\cos(\beta t - \psi)}{\cos \psi} \end{vmatrix}$$

the expression 1.6. can be written more concisely as:

$$1.8. \quad \vec{v}(t) = e^{-\alpha t} M(\beta t) \vec{v}(0)$$

Since  $M(\beta t)$  has a determinant equal to 1 and a trace  $< 2$ , in absolute value the Twiss formulae can be applied.

$$1.9. \quad M(\beta t) = I \cos \beta t + J \sin \beta t = e^{J \beta t}$$

with

$$1.10. \quad I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$1.11. \quad J = \begin{vmatrix} -\operatorname{tg} \psi & -\frac{Z_0}{\cos \psi} \\ \frac{1}{Z_0 \cos \psi} & \operatorname{tg} \psi \end{vmatrix}$$

$$1.12. \quad J^2 = -I$$

1.8. then becomes

$$1.13. \quad \vec{v}(t) = e^{(-\alpha I + \beta J)t} \vec{v}(0)$$

In all of the following calculation, the only matrices involved are I and J which commute with each other. This property makes the order of the matrices non-essential. In the intermediate calculations I can be neglected and J treated as an imaginary unit.

## 2 - EVOLUTION OF THE CAVITY IN THE PRESENCE OF EXTERNAL PERTURBATION.

Let  $\vec{p}(t) dt$  be the perturbation applied to the cavity at time  $t'$ . By the application of the superposition principle, equation 1.13 can be generalized as follows:

$$2.1. \quad \vec{v}(t) = e^{(-\alpha + \beta J)t} \vec{v}(0) + \int_0^t e^{(-\alpha + \beta J)(t-t')} \vec{p}(t') dt'$$

In the more restrictive hypotheses that perturbation be different from 0 only in the interval of time  $t_1 - t_2$  and that  $t$  be external to this interval, and indicating by  $t_s$  a general time within the interval  $t_1 - t_2$ , the equation 2.1. can be written:

$$2.2. \quad \vec{v}(t) = e^{(-\alpha + \beta J)t} \vec{v}(0) + e^{(-\alpha + \beta J)(t-t_s)} \left\{ \int_{t_1}^{t_2} e^{(-\alpha + \beta J)(t_s-t')} \vec{p}(t') dt' \right\} u(t-t_s)$$

where

4.

$$u(x) = 0 \quad \text{for} \quad x < 0$$

$$u(x) = 1 \quad \text{for} \quad x > 0$$

Finally assuming that

$$2.3. \quad \vec{v}_p(t_s) = \int_{t_1}^{t_2} e^{(-\alpha + \beta J)(t_s - t')} \vec{p}(t') dt'$$

equation 2.2. becomes

$$2.4. \quad \vec{v}(t) = e^{(-\alpha + \beta J)t} \vec{v}(0) + e^{(-\alpha + \beta J)(t - t_s)} \vec{v}_p(t_s) \quad u(t - t_s)$$

If one considers 2.4. to be valid at any moment, only local errors are made which have no effect before or after the perturbation.

Formula 2.3. can be used to evaluate the perturbation induced on the cavity by the supply and by the beam (going through). The two perturbations will later be assumed not to be both present at the same time. For the first one,  $\vec{v}_A$ , it is not necessary to specify the components, it is enough to assume that it is constant and periodic.

The perturbation due to the beam will be called  $\vec{v}_B$ . Let us consider the case of a rigid beam structure: the perturbation is then constant, however, because of coherent synchrotron oscillations it is not strictly periodical.

To calculate  $\vec{v}_B$ , the following hypotheses will be made:

- a) the rigid beam structure can be described by the charge density function  $\rho(t)$  which it is assumed to be positive.
- b) an infinitesimal charge  $\rho(t) dt$  crossing the cavity induces a perturbation.

$$2.5. \quad d\vec{v} = - \frac{\rho(t)}{C} \vec{p}_0 dt$$

where in the case of purely capacitive coupling

$$2.6. \quad \vec{p}_0 = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$

- c) Because of energy conservation, an infinitesimal charge  $\rho(t) dt$  crossing the cavity receives an energy.

$$2.7. \quad dE = \rho(t) V(t) dt$$

Through equations 2.3. and 2.5., at a general time  $t_s$  one obtains

$$2.8. \quad \vec{v}_B(t_s) = - \int_{t_1}^{t_2} e^{(-\alpha + \beta J)(t_s - t')} \frac{\rho(t')}{C} \vec{p}_0 dt'$$

The energy  $E_B$  that the beam takes from the cavity can be thought of as consisting of two terms. The largest is the energy which the beam would take if the voltage in the ca

vity were not affected by the beam.

Using formula 2.7. the following equation can be written for this part:

$$2.9. \quad E_B [\vec{v}^-(t_s)] = \vec{p}_0 \left[ \int_{t_1}^{t_2} e^{(-\alpha + \beta J)(t' - t_s)} \rho(t') \vec{v}^-(t_s) dt' \right] = M_{11} V^-(t_s) + M_{12} I^-(t_s)$$

where  $\vec{v}^-(t_s)$  is the vector associated to the cavity prior to the perturbation  $\vec{v}_B$  and  $M_{11}$  and  $M_{12}$  are two time independent coefficients.

$\vec{p}_0$  is to be multiplied (lines by columns) with the vector on the right.

The second term takes into account the beam-induced voltage variation. As the energy delivered to the beam depends linearly on the voltage this second term is independent of the voltage in the cavity prior to the beam crossing. By 2.5 and 2.7. one finds:

$$2.10. \quad \Delta E_B = - \vec{p}_0 \left[ \int_{t_1}^{t_2} \rho(t) dt \int_{t_1}^t e^{(-\alpha + \beta J)(t - t')} \frac{\rho(t')}{C} dt' \right] \vec{p}_0$$

### 3 - DEFINITION OF THE VECTORS $\vec{v}_c$ AND $\vec{v}_g$ -

In this paragraph it is assumed that the beam revolution frequency be identical to that,  $1/T$ , of the RF power supply pulses and delayed with respect to it by an amount  $\tau$ . If at the initial time, the cavity is characterized by the vector  $\vec{v}(0)$  generalizing the expression 2.4., at the general time  $t$ , we have:

$$3.1. \quad \vec{v}(t) = e^{(-\alpha + \beta J)t} \vec{v}(0) + \sum_{h=0}^n e^{(-\alpha + \beta J)(t - hT)} \vec{v}_A + \sum_{h=0}^m e^{(-\alpha + \beta J) [t - (hT + \tau)]} \vec{v}_B$$

where

$$3.2. \quad n = [t/T] \quad m = [(t - \tau)/T]$$

with  $[a]$  integral part of  $a$ .

In A2 it is shown that, assuming

$$3.3. \quad \vec{v}_s(t) = e^{(-\alpha + \beta J)t} \left\{ \vec{v}(0) - \frac{(e^{\beta JT} - e^{-\alpha T})(\vec{v}_A + e^{(\alpha - \beta J)\tau} \vec{v}_B)}{2 [\cosh \alpha T - \cos \beta T]} \right\}$$

$\vec{v}(t)$  can be expressed as a sum of two terms:  $\vec{v}_s(t)$  which tends to zero due to the factor  $e^{-\alpha t}$ , and the asymptotic expression 3.4. function of  $\tau$  and  $t$ .

$$3.4. \quad \vec{v}_c(\tau, t) = \frac{e^{\alpha T} - e^{-\beta JT}}{2 [\cosh \alpha T - \cos \beta T]} \left\{ e^{(-\alpha + \beta J)(t - nT)} \vec{v}_A + e^{(-\alpha + \beta J) [t - (mT + \tau)]} \vec{v}_B \right\}$$

From 3.4. and 3.2., it can be easily drawn that  $\vec{v}_c(\tau, t)$  is periodical with a pe-

6.

riod T. The following equation is obtained:

$$3.5. \quad \vec{v}_c(\tau, t+q \cdot T) = \vec{v}_c(\tau, t)$$

where q is an integer.

This result directly depends on the periodical character of the cavity perturbation as also on the damping of the electro-magnetic field oscillations.

From 3.4. it is possible to see that if one indicates with  $\vec{v}_c^-(\tau, nT)$ , and  $\vec{v}_c^+(\tau, nT)$ , the values of  $\vec{v}_c$  before and after time nT then

$$3.6. \quad \vec{v}_c^+(\tau, nT) - \vec{v}_c^-(\tau, nT) = \vec{v}_A$$

In the same way:

$$3.7. \quad \vec{v}_c^+(\tau, nT + \tau) - \vec{v}_c^-(\tau, nT + \tau) = \vec{v}_B$$

Other than from eq. s 3.2. and 3.4., one can define  $\vec{v}_c^+(\tau, t)$  in the following way:  $\vec{v}_c^+(\tau, t)$  is a vector which at times  $(nT)_+$  has the value given by eq. 3.8.

$$3.8. \quad \vec{v}_c^+(\tau, nT) = \frac{e^{\alpha T} - e^{-\beta J T}}{2[\cosh \alpha T - \cos \beta T]} \left[ \vec{v}_A + e^{(-\alpha + \beta J)(T - \tau)} \vec{v}_B \right]$$

at times  $(nT + \tau)_+$  has the value given by eq. 3.9.

$$3.9. \quad \vec{v}_c^+(\tau, nT + \tau) = \frac{e^{\alpha T} - e^{-\beta J T}}{2[\cosh \alpha T - \cos \beta T]} \left[ e^{(-\alpha + \beta J)\tau} \vec{v}_A + \vec{v}_B \right]$$

$$3.10. \quad \left\{ \begin{array}{l} \text{and during the time intervals } [nT - nT + \tau] \text{ and } [nT + \tau - (n+1)T] \\ \text{behaves as a homogeneous solution of eq.11 according to eq.} \\ \text{1.13.} \end{array} \right.$$

Let's point out the fact, that, in the hypothesis of a periodic perturbation,  $\vec{v}_c(\tau, t)$  can be a physical vector.

Later on, we will consider the case where the beam crossings are not synchronous with the RF.

It is therefore convenient to define a vector which has a definition similar to that of  $\vec{v}_c(\tau, t)$  and accounts for the non periodicity of the beam crossings.

Let us suppose that the crossings occur at times  $nT + \tau(n)$  (indicating the dependence of  $\tau$  from n explicitly). We define a new vector  $\vec{v}_g(t)$  in the following way:

$$3.11. \quad \vec{v}_g^+[nT + \tau(n)] = \vec{v}_c^+[\tau(n), nT + \tau(n)]$$

$$3.12. \quad \vec{v}_g^+[(n+1)T] = \vec{v}_c^+[\tau(n), (n+1)T]$$

$$3.13. \quad \left\{ \begin{array}{l} \text{During the time intervals } [nT + \tau(n) - (n+1)T] \text{ and } [(n+1)T - \\ - (n+1)T + \tau(n+1)] , \vec{v}_g(t) \text{ behaves as a homogeneous solution} \\ \text{of eq. 1.1. consistent with 1.13.} \end{array} \right.$$

We would like to underline the fact that  $\vec{v}_g(t)$  can be a physical vector only during

an interval  $[nT + \tau(n) - (n+1)T + \tau(n+1)]$  whenever  $\tau(n)$  is a function of  $n$ , but it is coincident with  $\vec{v}_c(\tau, t)$  if  $\tau(n)$  does not vary with  $n$ .

#### 4 - DEFINITION OF VECTOR $\vec{v}_r$ -

Let us suppose that the beam crossings are not synchronous and occur at times  $nT + \tau(n)$ . Eq. 3.1. has to be replaced by:

$$4.1. \quad \vec{v}(t) = e^{(-\alpha + \beta J)t} \vec{v}(0) + \sum_{h=0}^n e^{(-\alpha + \beta J)(t-hT)} \vec{v}_A + \sum_{h=0}^m e^{(-\alpha + \beta J)[t-(hT + \tau(h))]} \vec{v}_B$$

where:

$$4.2. \quad n = \lfloor t/T \rfloor \quad m = \lfloor (t - \tau(n))/T \rfloor$$

Due to  $\tau(n)$  varying from one crossing to the other  $\vec{v}(t)$  will not have an asymptotically periodic expression of type 3.4., but a different one that depends on  $\tau(n)$ .

To go further we have to introduce a new vector  $\vec{v}_r(t)$ , which is a generalized form of  $\vec{v}_s(t)$ , already defined in eq. 3.3.:

$$4.3. \quad \vec{v}_r(t) = \vec{v}(t) - \vec{v}_g(t)$$

where  $\vec{v}_g(t)$  is defined by eq. s 3.11, 3.12., 3.13.

The new vector  $\vec{v}_r(t)$  is not influenced by the discontinuity  $\vec{v}_A$  at times  $(n+1)T$  and follows eq. 1.13. during the entire time interval  $[nT + \tau(n) - (n+1)T + \tau(n+1)]$ . In fact, considering eq. 4.3.,  $\vec{v}(t)$  and  $\vec{v}_g(t)$  are both physically possible during such time interval.

Taking for simplicity:

$$4.4. \quad \vec{v}_r^+(n) = \vec{v}_r^+(nT + \tau(n))$$

$$4.5. \quad \vec{v}_r^-(n) = \vec{v}_r^-(nT + \tau(n))$$

we obtain:

$$4.6. \quad \vec{v}_r^-(n+1) = e^{(-\alpha + \beta J)[T + \tau(n+1) - \tau(n)]} \vec{v}_r^+(n)$$

At the  $(n+1)^{\text{th}}$  crossing  $\vec{v}_r(t)$  varies from  $\vec{v}_r^-(n+1)$  to  $\vec{v}_r^+(n+1)$ . To deduce the relation existing between these two values, let us apply eq. 4.3. to the right and to the left of time  $(n+1)T + \tau(n+1)$

$$4.7. \quad \vec{v}^- [(n+1)T + \tau(n+1)] = \vec{v}_r^-(n+1) + \vec{v}_g^- [(n+1)T + \tau(n+1)]$$

$$4.8. \quad \vec{v}^+ [(n+1)T + \tau(n+1)] = \vec{v}_r^+(n+1) + \vec{v}_g^+ [(n+1)T + \tau(n+1)]$$

According to eq. s 3.13, 3.12 and 3.5

$$4.9. \quad \vec{v}_g^- [(n+1)T + \tau(n+1)] = e^{(-\alpha + \beta J)\tau(n+1)} \vec{v}_c^+ [\tau(n), T]$$

According to eq. s 3.11., 3.7., 3.10. and 3.5.



8.

$$4.10. \quad \begin{cases} \vec{v}_g^+ [(n+1)T + \tau(n+1)] = \vec{v}_c^+ [\tau(n+1), (n+1)T + \tau(n+1)] \\ \vec{v}_g^+ [(n+1)T + \tau(n+1)] = \vec{v}_c^- [\tau(n+1), T + \tau(n+1)] + \vec{v}_B \\ \vec{v}_g^+ [(n+1)T + \tau(n+1)] = e^{(-\alpha + \beta J) \tau(n+1)} \vec{v}_c^+ [\tau(n+1), T] + \vec{v}_B \end{cases}$$

According to eq. s 3.1. and 3.2.

$$4.11. \quad \vec{v}^+ [(n+1) \cdot T + \tau(n+1)] = \vec{v}^- [(n+1)T + \tau(n+1)] + \vec{v}_B$$

Subtracting eq. 4.7. from 4.8. and using eq. s 4.9., 4.10., 4.11. and 4.6. we finally obtain:

$$4.12. \quad \vec{v}_r^+(n+1) = e^{(-\alpha + \beta J) [T + \tau(n+1) - \tau(n)]} \vec{v}_r^+(n) + e^{(-\alpha + \beta J) \tau(n+1)} \cdot \left\{ \vec{v}_c^+ [\tau(n)T] - \vec{v}_c^+ [\tau(n+1), T] \right\}$$

## 5 - BEAM CAVITY COUPLING -

In this paragraph, we want to establish one more relation between  $\tau(n)$  and vector  $\vec{v}_r^+(n)$  which, together with eq. 4.12., will completely define  $\tau(n)$  and  $\vec{v}_r^+(n)$ .

Eq. 4.12. includes the cavity parameters, the relation that we shall deduce now will essentially include the parameters defining the machine structure and the radiation mode.

Let us first consider the relation between two successive beam crossings and the beam energy.

$$5.1. \quad \tau(n+1) - \tau(n) = h [E(n) - (W_s + 0.5 W_o)]$$

where

$$5.2 \quad h = \frac{\alpha_c T}{W_s}$$

- $W_s$  synchronous energy of the beam (not of the particle)  $\simeq NeU_o$
- $N^s$  number of particles;
- $e$  elementary charge;
- $U_o$  energy of machine;
- $W_o$  energy irradiated per turn at equilibrium conditions;
- $\alpha_c$  momentum compaction;
- $E(n)$  beam energy after the nth passage through the cavity;
- $T$  period of the synchronous particle and of the power supply.
- $\tau(n)$  delay of the nth crossing with respect to the nth power supply pulse.

At equilibrium,  $\tau(n)$  does not vary and we obtain:

$$5.3. \quad E(n) = W_s + 0.5 W_o$$

In fact, under those conditions, the beam loses an energy  $W_o$  in one turn, comes back to the cavity with energy  $W_s - 0.5 W_o$ , gains an energy  $W_o$  and returns to  $W_s + 0.5 W_o$ .

Considering the beam energy variation from one crossing to the next, we may write, recalling eq. s 2.9 and 2.10.

$$5.4. \quad E(n+1) = E(n) - W_o \left[ 1 + K \frac{E(n) - (W_s + 0.5 W_o)}{W_s} \right] + E_B E_B \left[ \vec{v}^- \left[ (n+1)T + \tau(n+1) \right] \right] + \Delta E_B$$

where  $W_o$  represents the total radiation energy losses per turn both coherent and incoherent, and  $K$  accounts for variations in such losses.

The following terms in eq. 5.4 give the cavity contribution (2.9., 2.10).

As  $E_B$  defined by eq. 2.9 is a linear function of  $\vec{v}(t_s)$ , we may write (eqs. 3.1, 4.3, 4.4, 3.11, 3.8).

$$5.5. \quad \left\{ \begin{array}{l} E_B \left[ \vec{v}^- \left[ (n+1)T + \tau(n+1) \right] \right] = E_B \left[ \vec{v}^+ \left[ (n+1)T + \tau(n+1) \right] \right] - E_B(\vec{v}_B) \\ E_B \left[ \vec{v}^- \left[ (n+1)T + \tau(n+1) \right] \right] = E_B \left[ \vec{v}_g^+ \left[ (n+1)T + \tau(n+1) \right] \right] + E_B \left[ \vec{v}_r^+(n+1) \right] - E_B(\vec{v}_B) \\ E_B \left[ \vec{v}^- \left[ (n+1)T + \tau(n+1) \right] \right] = E_B \left[ \vec{v}_c^+ \left[ \tau(n+1), \tau(n+1) \right] \right] + E_B \left[ \vec{v}_r^+(n+1) \right] - E_B(\vec{v}_B) \end{array} \right.$$

Substituting eq. 5.5. into 5.4., we deduce

$$5.6. \quad E(n+1) = E(n) - W_o \left[ 1 + K \frac{E(n) - (W_s + 0.5 W_o)}{W_s} \right] + E_B \left[ \vec{v}_c^+ \left( \tau(n+1), \tau(n+1) \right) \right] + E_B \left[ \vec{v}_r^+(n+1) \right] - E_B(\vec{v}_B) + \Delta E_B.$$

At equilibrium conditions for both beam and cavity we also have:

$$5.7. \quad \vec{v}_r(n) = 0$$

and eq. 5.6 determines the value of  $\tau_s$  synchronous delay

$$5.8. \quad E_B \left[ \vec{v}_c^+ \left( \tau_s, \tau_s \right) \right] = W_o + E_B(\vec{v}_B) - \Delta E_B$$

Eq. 5.8. determines the beam position relative to the power supply pulses.

It might also be written:

$$5.9. \quad E_B \left[ \vec{v}_c^-(\tau_s, \tau_s) \right] = W_o - \Delta E_B$$

Multiplying eq. 5.6 times  $h$ , using eq. 5.1 and 5.8 and defining

$$5.10. \quad K_w = K \frac{W_o}{W_s}$$

10.

$$5.11. \quad \chi(n) = \tau(n) - \tau_s$$

eq. 5.6 becomes:

$$5.12. \quad \chi(n+2) - \chi(n+1)(2-Kw) + \chi(n)(1-Kw) = h \left\{ E_B \left[ \vec{v}_c^+(\tau(n+1), \tau(n+1)) \right] - E_B \left[ \vec{v}_c^+(\tau_s, \tau_s) \right] + E_B \left[ \vec{v}_r^+(n+1) \right] \right\} .$$

## 6 - LINEARIZATION AND PROJECTION OF EQ. S 4.12 AND 5.12 -

Let us consider simultaneously eq. s 4.12 and 5.12, that we rewrite as:

$$6.1. \quad \vec{v}_r^+(n+1) = e^{(-\alpha + \beta J) [T + \tau(n+1) - \tau(n)]} \vec{v}_r^+(n) + e^{(-\alpha + \beta J) \tau(n+1)} \cdot \left\{ \vec{v}_c^+[\tau(n), T] - \vec{v}_c^+[\tau(n+1), T] \right\}$$

$$6.2. \quad \chi(n+2) - \chi(n+1)(2-Kw) + \chi(n)(1-Kw) = h \left\{ E_B \left[ \vec{v}_c^+(\tau(n+1), \tau(n+1)) \right] - E_B \left[ \vec{v}_c^+(\tau_s, \tau_s) \right] + E_B \left[ \vec{v}_r^+(n+1) \right] \right\}$$

According to the definition 5.11, we assume in eq. 6.1. and 6.2:

$$6.3. \quad \tau(n) = \tau_s + \chi(n)$$

Therefore we may consider  $\chi(n)$  and  $\vec{v}_r^+(n)$  as independent variables.

To progress with the discussion, it is convenient to linearize eq. s 6.1 and 6.2.

There is only one way to do it and namely to expand each term of these equations in series of  $\chi(n)$  and  $\vec{v}_r^+(n)$  to first order.

There are no zero order terms because  $\chi(n)$  and  $\vec{v}_r^+(n)$  are zero at equilibrium.

Considering the two equations and remembering (2.9) that  $E_B$  is a linear operator, we note that the variable  $\vec{v}_r^+(n)$  is only present at the first order. Therefore  $\vec{v}_r^+(n)$  is linear already.

As for  $\chi(n)$ , we have first of all to make the substitution:

$$6.4. \quad e^{(-\alpha + \beta J) [T + \tau(n+1) - \tau(n)]} \longrightarrow e^{(-\alpha + \beta J) T}$$

As far as the other terms are concerned, we have shown in App. III that, assuming

$$6.5. \quad P_B = (\alpha - \beta J) \frac{e^{-\alpha T} - e^{\beta J T}}{2 [\cosh \alpha T - \cos \beta T]}$$

and

$$6.6. \quad P_A = (-\alpha + \beta J) \frac{(e^{\alpha T} - e^{-\beta J T}) e^{(-\alpha + \beta J) \tau_s}}{2 [\cosh \alpha T - \cos \beta T]}$$

there are two more substitutions to be made:

$$6.7. \quad e^{(-\alpha + \beta J) \tau(n+1)} \left\{ \vec{v}_c^+ [\tau(n), T] - \vec{v}_c^+ [\tau(n+1), T] \right\} \longrightarrow \\ \longrightarrow [\chi(n+1) - \chi(n)] P_B \vec{v}_B$$

$$6.8. \quad \vec{v}_c^+ [\tau(n+1), \tau(n+1)] - \vec{v}_c^+ [\tau(s), \tau(s)] = \chi(n+1) P_A \vec{v}_A$$

Using eq. s 6.4, 6.7 and 6.8, eq. s 6.1 and 6.2 become:

$$6.9. \quad \vec{v}_r^+(n+1) - e^{(-\alpha + \beta J) T} \vec{v}_r^+(n) - [\chi(n+1) - \chi(n)] P_B \vec{v}_B = 0$$

$$6.10. \quad \chi(n+2) - \chi(n+1) [2 - Kw - hE_B (P_A \vec{v}_A)] + \chi(n) (1 - Kw) - hE_B [\vec{v}_r^+(n+1)] = 0$$

To continue the discussion of eq. s 6.9 and 6.10 we must now obtain the two scalar equations corresponding to 6.9.

By defining an angle  $\theta$  as:

$$6.11. \quad \beta T = 2\pi + \theta$$

We obtain from eq. s 1.9, 1.7 and 6.11:

$$6.12. \quad e^{(-\alpha + \beta J) T} \vec{v}_r^+(n) = \frac{e^{-\alpha T}}{\cos \psi} \begin{pmatrix} \cos(\theta + \psi) V_r^+(n) - Z_o I_r^+(n) \sin \theta \\ \frac{\sin \theta}{Z_o} V_r^+(n) + \cos(\theta + \psi) I_r^+(n) \end{pmatrix}$$

In App. IV we have derived the product:

$$6.13. \quad P_B \vec{v}_B = \frac{\beta_0}{2[\cosh \alpha T - \cos \theta]} \left\{ \begin{array}{l} [e^{-\alpha T} \sin 2\varphi - \sin(2\varphi + \theta)] V_B + [e^{-\alpha T} \cos \varphi - \cos(\theta + \varphi)] Z_0 I_B \\ \\ [\cos(\theta + \varphi) - e^{-\alpha T} \cos \varphi] \frac{V_B}{Z_0} - \sin \theta I_B \end{array} \right.$$

In a similar way, in the same App. IV we have obtained for  $P_A \vec{v}_A$ :

$$6.14. \quad P_A \vec{v}_A = \frac{\beta_0}{2 \cos \varphi [\cosh \alpha T - \cos \theta]} \left\{ \begin{array}{l} [-e^{-\alpha T} \sin(\phi_s + 2\varphi) + \sin(\phi_s + \varphi - \theta)] V_A + [\cos(\phi_s + \varphi - \theta) - \\ - e^{-\alpha T} \cos(\phi_s + \varphi)] Z_0 I_A \\ \\ [e^{-\alpha T} \cos(\phi_s + \varphi) - \cos(\phi_s + \varphi - \theta)] \frac{V_A}{Z_0} + [\sin(\phi_s - \theta) - e^{-\alpha T} \sin \phi_s] I_A \end{array} \right.$$

6.14 shows that vector  $P_A \vec{v}_A$  depends on  $\vec{v}_B$  also, through the phase ( $\phi_s = \beta \tau_s$ ) which is in turn determined from eq. 5.9.

We will assume, in the following,  $\vec{v}_A$  to vary with  $\vec{v}_B$  in such a way that  $E_B(P_A \vec{v}_A)$  stays constant and positive. This corresponds, with good approximation, to the hypothesis that the amplitude of the RF voltage does not vary with load.

If one defines

$$6.15. \quad 2 e^{-\gamma} \cos \delta = 2 - Kw - h \cdot E_B(P_A \vec{v}_A)$$

$$6.16. \quad 1 - Kw = e^{-2\gamma}$$

$$6.17. \quad d = \frac{\beta_0}{2 \cos \varphi [\cosh \alpha T - \cos \theta]}$$

eq. s 6.9 and 6.10 may be written using eq. s 6.12, 6.13, 6.14, 6.15, 6.16, 6.17 and 2.9:

$$6.18. \quad \left\{ \begin{array}{l} V_r^+(n+1) - \frac{e^{-\alpha T}}{\cos \varphi} \cos(\theta + \varphi) V_r^+(n) + \frac{e^{-\alpha T}}{\cos \varphi} \sin \theta Z_0 I_r^+(n) + d \left\{ [\sin(2\varphi + \theta) - e^{-\alpha T} \sin 2\varphi] V_B + \right. \\ \left. + [\cos(\theta + \varphi) - e^{-\alpha T} \cos \varphi] Z_0 I_B \right\} [\chi(n+1) - \chi(n)] = 0 \\ \\ I_r^+(n+1) - \frac{e^{-\alpha T}}{\cos \varphi} \sin \theta \frac{V_r^+(n)}{Z_0} - \frac{e^{-\alpha T}}{\cos \varphi} \cos(\theta - \varphi) I_r^+(n) + d \left\{ [e^{-\alpha T} \cos \varphi - \cos(\theta + \varphi)] \frac{V_B}{Z_0} + \right. \\ \left. + \sin \theta I_B \right\} [\chi(n+1) - \chi(n)] = 0 \\ \vdots \end{array} \right.$$

$$6.18. \quad \begin{cases} \chi(n+2) - 2e^{-\gamma} \cos \delta \chi(n+1) + e^{-2\gamma} \chi(n) - hM_{11} V_r^+(n+1) - hM_{12} I_r^+(n+1) = 0 \end{cases}$$

7 - 4<sup>th</sup> ORDER POLYNOMIAL ASSOCIATED WITH THE EQUATIONS SYSTEM 6.18. -

The solutions of the homogeneous equations system 6.18 are of the type:

$$7.1. \quad \begin{cases} \chi(n) = a_\chi X^n \\ V_r^+(n) = a_v X^n \\ I_r^+(n) = a_I X^n \end{cases}$$

Introducing these solutions in eq. 6.18 and simplifying one obtains a system of linear equations, homogeneous in  $a_\chi$ ,  $a_v$ ,  $a_I$ . For the system to yield solutions, the coefficient determinant  $P(x)$

$$7.2. \quad P(x) = \begin{vmatrix} d \left\{ \left[ \sin(2\psi + \theta) - e^{-\alpha T} \sin 2\psi \right] V_B + \left[ \cos(\psi + \theta) - e^{-\alpha T} \cos \psi \right] Z_0 I_B \right\} (x-1) & x - \frac{e^{-\alpha T}}{\cos \psi} \cos(\theta + \psi) & \frac{e^{-\alpha T}}{\cos \psi} Z_0 \sin \theta \\ d \left\{ \left[ e^{-\alpha T} \cos \psi - \cos(\theta + \psi) \right] \frac{V_B}{Z_0} + \sin \theta I_B \right\} (x-1) & -\frac{e^{-\alpha T} \sin \theta}{Z_0 \cos \psi} & x - \frac{e^{-\alpha T}}{\cos \psi} \cos(\theta - \psi) \\ x^2 - 2xe^{-\gamma} \cos \delta + e^{-2\gamma} & -hM_{11} x & -hM_{12} x \end{vmatrix}$$

must vanish.

Eq. 7.2 shows that  $P(x)$  is a fourth order polynomial in the variable  $x$ .  $P(x)$  is sum of a first part, which is independent of the load and a perturbation term, which depends from the load through  $V_B$  and  $I_B$ .

Assuming:

$$7.3. \quad P(x) = P_0(x) + \Delta P(x) = \sum_{m=0}^4 (a_m + \Delta a_m) x^{4-m}$$

one quite easily deduces

$$7.4. \quad P_0(x) = (x^2 - 2xe^{-\gamma} \cos \delta + e^{-2\gamma})(x^2 - 2xe^{-\alpha T} \cos \theta + e^{-2\alpha T})$$

and:

$$7.5. \quad \begin{cases} a_0 = 1 \\ a_1 = -2(e^{-\gamma} \cos \delta + e^{-\alpha T} \cos \theta) \\ a_2 = e^{-2\gamma} + e^{-2\alpha T} + 4e^{-(\gamma + \alpha T)} \cos \theta \cos \delta \\ a_3 = -2e^{-(\gamma + \alpha T)} [e^{-\gamma} \cos \theta + e^{-\alpha T} \cos \delta] \\ a_4 = e^{-2(\gamma + \alpha T)} \end{cases}$$

Through a long series of algebraical and trigonometrical relationships one obtains:

$$7.6. \left\{ \begin{array}{l} \Delta a_0 = 0 \\ \Delta a_4 = 0 \\ \Delta a_1 = h \cdot d \left\{ M_{11} V_B \left[ \text{sen}(2\psi + \theta) - e^{-\alpha T} \text{sen} 2\psi \right] + \left( M_{12} \frac{V_B}{Z_0} - M_{11} Z_0 I_B \right) \cdot \right. \\ \quad \left. \cdot \left[ e^{-\alpha T} \cos \psi - \cos(\theta + \psi) \right] + M_{12} I_B \text{sen} \theta \right\} \\ \Delta a_2 = h d \left\{ M_{11} V_B \left[ e^{-2\alpha T} \text{sen}(2\psi - \theta) - \text{sen}(2\psi + \theta) \right] + \left( M_{12} \frac{V_B}{Z_0} - M_{11} Z_0 I_B \right) \cdot \right. \\ \quad \left. \cdot \left[ \cos(\theta + \psi) - e^{-2\alpha T} \cos(\theta - \psi) \right] - M_{12} I_B \text{sen} \theta (1 + e^{-2\alpha T}) \right\} \\ \Delta a_3 = h d \left\{ M_{11} V_B \left[ e^{-\alpha T} \text{sen} 2\psi - e^{-2\alpha T} \text{sen}(2\psi - \theta) \right] + \left( M_{12} \frac{V_B}{Z_0} - M_{11} Z_0 I_B \right) \cdot \right. \\ \quad \left. \cdot \left[ e^{-2\alpha T} \cos(\theta - \psi) - e^{-\alpha T} \cos \psi \right] + M_{12} I_B \text{sen} \theta e^{-2\alpha T} \right\}. \end{array} \right.$$

## 8 - BEAM STRUCTURE AND PERTURBATIONS -

From eq. s 2.8 and 2.9, one deduces that the four quantities  $V_B$ ,  $I_B$ ,  $M_{11}$ ,  $M_{12}$  which appear in eq. s 7.6, depend on the choice of  $t_s$ ,  $t_s$  being the time where we consider the beam perturbation.

On the other side from par. 2, it appears that, within wide limits, the choice of  $t_s$  is arbitrary and does not influence the physical results.

Mathematically this means that  $\Delta a_1$ ,  $\Delta a_2$ ,  $\Delta a_3$  as obtained from eq. s 7.6, must not depend explicitly on  $t_s$ .

It should be possible therefore to express the perturbations as functions of quantities independent from  $t_s$ .

Infact let us consider eq. s 2.8 and 2.9 again. We can write using eq. s 1.2 and 1.9.

$$8.1. \quad \vec{v}_B = -\beta_0 Z_0 \int_{t_1}^{t_2} \left[ \cosh \alpha(t-t_s) + \sinh \alpha(t-t_s) \right] \left[ I \cos \beta(t-t_s) - J \text{sen} \beta(t-t_s) \right] \rho(t) p_0 dt$$

$$8.2. \quad E_B = \vec{p}_0 \int_{t_1}^{t_2} \left[ \cosh \alpha(t-t_s) - \sinh \alpha(t-t_s) \right] \left[ I \cos \beta(t-t_s) + J \text{sen} \beta(t-t_s) \right] \rho(t) \vec{v}(t_s) dt$$

Assuming

$$8.3. \quad A_{11} = \int_{t_1}^{t_2} \rho(t) \left\{ \begin{array}{l} \cosh \alpha(t-t_s) \\ \sinh \alpha(t-t_s) \end{array} \right\} \left\{ \begin{array}{l} \cos \beta(t-t_s) \\ \text{sen} \beta(t-t_s) \end{array} \right\} dt$$

and using 1.11 for J, one obtains for  $V_B$  and  $I_B$

$$8.4. \quad \begin{cases} V_B = -\frac{\beta_o Z_o}{\cos \psi} [(A_{11} + A_{21})\cos \psi + (A_{12} + A_{22})\sin \psi] \\ I_B = \frac{\beta_o}{\cos \psi} (A_{12} + A_{22}) \end{cases}$$

and for the terms  $M_{11}$  and  $M_{12}$  defined by eq. 2.9:

$$8.5. \quad \begin{cases} M_{11} = \frac{1}{\cos \psi} [(A_{11} - A_{21})\cos \psi + (A_{22} - A_{12})\sin \psi] \\ M_{12} = \frac{Z_o}{\cos \psi} (A_{22} - A_{12}) \end{cases}$$

Replacing the coefficients of eq. 7.6 with eq. s 8.4 and 8.5:

$$8.6. \quad \begin{cases} M_{11} V_B = -\frac{\beta_o Z_o}{\cos^2 \psi} \left\{ [A_{11}^2 + A_{12}^2] - [A_{22}^2 + A_{21}^2] \right\} \cos^2 \psi + (A_{11} A_{22} - A_{12} A_{21}) \cdot \\ \quad \cdot \sin \psi + (A_{22}^2 - A_{12}^2) \left. \right\} \\ M_{12} \frac{V_B}{Z_o} - M_{11} Z_o I_B = -\frac{2\beta_o Z_o}{\cos^2 \psi} \left\{ (A_{11} A_{22} - A_{12} A_{21}) \cos \psi + \right. \\ \quad \left. + (A_{22}^2 - A_{12}^2) \sin \psi \right\} \\ M_{12} I_B = \frac{\beta_o Z_o}{\cos^2 \psi} (A_{22}^2 - A_{12}^2) \end{cases}$$

and writing

$$8.7. \quad \begin{cases} Q_1 = (A_{11}^2 + A_{12}^2) - (A_{22}^2 + A_{21}^2) \\ Q_2 = 2(A_{11} A_{22} - A_{12} A_{21}) \end{cases}$$

eq. s 7.6 become:

$$8.8. \quad \begin{cases} \Delta a_1 = -dh \beta_o Z_o \left\{ Q_1 [\sin(2\psi + \theta) - e^{-\alpha T} \sin 2\psi] - Q_2 [\cos(2\psi + \theta) - \cos 2\psi e^{-\alpha T}] \right\} \\ \Delta a_2 = -dh \beta_o Z_o \left\{ Q_1 [e^{-2\alpha T} \sin(2\psi - \theta) - \sin(2\psi + \theta)] + Q_2 [\cos(2\psi + \theta) - e^{-2\alpha T} \cos(2\psi - \theta)] \right\} \\ \Delta a_3 = -dh \beta_o Z_o \left\{ Q_1 [e^{-\alpha T} \sin 2\psi - e^{-2\alpha T} \sin(2\psi - \theta)] - Q_2 [e^{-\alpha T} \cos 2\psi - e^{-2\alpha T} \cos(2\psi - \theta)] \right\} \end{cases}$$



Defining now:

$$8.9. \quad \begin{cases} Q_0 = (Q_1^2 + Q_2^2)^{1/2} \\ Q_1 = Q_0 \cos \psi \\ Q_2 = Q_0 \sin \psi \end{cases}$$

and using definitions 5.2 and 6.17

$$8.10. \quad g = \frac{\alpha_c T \beta_0^2 Q_0 Z_0}{2W_s \cos \psi [\cosh \alpha T - \cos \theta]}$$

we may rewrite eq. s 8.8

$$8.11. \quad \begin{cases} \Delta a_1 = g \left\{ e^{-\alpha T} \sin(2\psi - \psi) - \sin[(2\psi - \psi) + \theta] \right\} \\ \Delta a_2 = g \left\{ \sin[(2\psi - \psi) + \theta] - e^{-2\alpha T} \sin[(2\psi - \psi) - \theta] \right\} \\ \Delta a_3 = g \left\{ e^{-2\alpha T} \sin[(2\psi - \psi) - \theta] - e^{-\alpha T} \sin(2\psi - \psi) \right\} \end{cases}$$

We will show in App. V that the terms  $Q_1$  and  $Q_2$ , defined by 8.7, and therefore  $Q_0$  and  $\psi$ , are independent of  $t_s$ . These are the quantities we mentioned at the beginning of this paragraph.

Through eq. s 8.11, we can easily evaluate the modifications brought about by a finite length beam structure as compared to a point-like beam.

Eq. s 8.11 include the three terms  $W_s$ ,  $Q_0$ ,  $\psi$  that we want to consider now in detail. For a point source distribution,  $W_s$ , the synchronous bunch energy, is  $q_B U_0$ .  $Q_0$  is coincident with  $q_B^2$  (choosing  $t_s$  coincident with the point charge crossing time and using eq. s 8.3, 8.7, 8.9) and  $\psi$  is zero.

For the finite source distribution for  $W_s$  we have to consider also the energy distribution around the synchronous energy and as far as  $Q_0$  is concerned, the general definition 8.9.

Finally  $\psi$  is always positive (see App. VI) and given by

$$8.12. \quad \psi \approx \frac{\psi}{2\sqrt{5}} \frac{(\rho^2)^{1/2}}{\rho} \delta_B^2$$

In eq. 8.11  $\delta_B$  is the interval of phase angles that contains the beam.

As  $\delta_B$  is normally very small, we have that:

$$8.13. \quad \psi \ll \psi.$$

Eq. 8.12 is right only to an order of magnitude; the exact definition of  $\psi$  still comes from eq. s 8.9.

Resuming the comparison of the point source with the finite source structures, the new definitions of  $W_s$  and  $Q_0$  are equivalent to a slight difference of total charge and do not affect the perturbations mathematical expression.

Eq. 8.12 and the fact that  $\psi$  is a contribution to a second order effect (eq. 8.11) allow us to neglect the correction due to  $\psi$ .

All this is with regard to the effect of the beam structure on the polynomial perturbation terms.

Let us add now that every cause of coherent losses, strongly dependent on the beam structure, generally causes variations of the parameter  $\gamma$  (6.16) associated with coherent synchrotron oscillations damping.

## 9 - QUALITATIVE DISCUSSION OF THE SOLUTIONS -

Let us first consider the unperturbed polynomial. The solutions of eq. 7.4 are:

$$9.1. \quad e^{-\gamma \pm i \delta}$$

and

$$9.2. \quad e^{-\alpha T \pm i \theta}$$

Eq. 9.1, represents the synchrotron oscillations of the bunch when the RF voltage is not affected by the beam oscillations.

From eq. s 6.15 and 6.16 obtains:

$$9.3. \quad \gamma \approx \frac{K_w}{2}$$

$$9.4. \quad \delta \approx h E_B (P_A \vec{v}_A)$$

$\gamma$  represents the damping of the synchrotron oscillation due to coherent and incoherent radiation losses.

$\delta$  represents the zero-load phase variation of the synchrotron oscillations over one turn.

Eq. 9.2 shows the time dependence of the cavity overexcitation  $\vec{v}_r$ , defined exactly by eq. 4.3.

From eq. s 1.2. and 6.11 obtains

$$9.5. \quad \alpha T = \frac{\alpha}{\beta} \beta T = \text{tg } \varphi \cdot (2\pi + \theta) \approx 2\pi \text{tg } \varphi \approx \frac{\pi}{Q}$$

$$9.6. \quad \theta = \beta T - 2\pi.$$

When the cavity is loaded, there is coupling. In such a situation to every polynomial root there corresponds a normal mode with both  $\vec{v}_r(n)$  and  $\mathcal{X}(n)$  simultaneously different from zero.

From eq. s 7.1 one derives that, in order to have stability, the four perturbed solutions must be smaller than one in absolute value.

As generally  $\alpha T$  is greater than  $\gamma$ , the most stable pair of solutions when no perturbation is present, is the one corresponding to oscillations inside the cavity.

One can deduce from eq. s 7.6 that the product of the four roots is constant because  $\Delta a_4$  is zero. Therefore in order for the perturbation, not to destroy the stability, the damping must be transferred from the highly damped couple to the least damped one. If it is the least damped couple to reduce its damping, the stability range is very small.

Let's show now that the sign of parameter  $\theta$  determines the way the damping is

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transferred.

From eq. s 7.5 we obtain that, for the case of complex conjugate damped roots (no load condition) one can write, considering  $\epsilon_i$  positive

$$9.7. \quad \left\{ \begin{array}{l} a_1 = -4 + \epsilon_1 \\ a_2 = 6 - \epsilon_2 \\ a_3 = -4 + \epsilon_3 \end{array} \right.$$

while from eq. s 8.11, neglecting second order terms one has:

$$9.8. \quad \left\{ \begin{array}{l} \Delta a_1 = -g \cdot \theta \\ \Delta a_2 = 2g \cdot \theta \\ \Delta a_3 = -g \cdot \theta \end{array} \right.$$

where  $g$  is positive (see 8.10) varying with the load,  $\theta$  must be negative if one wants the perturbed coefficients to stay of the type given by 9.7. We then qualitatively find the same criterion already deduced by other authors<sup>(1, 2, 3)</sup>.

In the next paragraph we will quantitatively analyse the influence of the load and of parameter  $\theta$  on the stability of the solutions.

## 10 - QUANTITATIVE ANALYSIS OF THE FOURTH ORDER POLYNOMIAL -

One way to obtain quantitative information from the fourth order polynomial, is to calculate the four solutions for every series of values of the parameters  $\gamma, \delta, \alpha, T, \theta$  and load (the influence of beam shape represented by the parameter  $\psi$  is negligible) and to check for stability.

This way is certainly the most likely to give results.

Nevertheless we do not need to know both polar coordinates of the solutions. As far as stability is concerned it is enough to know the modulus. Starting from these considerations we will present a way of translating the stability conditions into an inequality.

Let us consider the factor expansion of a fourth order polynomial.

$$10.1. \quad P(x) = \sum_{m=0}^4 b_m x^m = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

and assume:

$$10.2. \quad \left\{ \begin{array}{l} x_1 = y_1 y_2 \\ x_2 = y_1 / y_2 \\ x_3 = y_3 y_4 \\ x_4 = y_3 / y_4 \end{array} \right.$$

For the time being, we will consider only the case of complex roots. Under such an hypo

thesis, the new variables  $y_1$  and  $y_3$  represent the moduli and  $y_2$  and  $y_4$  the phase components of the roots.

Using 10.1 and 10.2 we obtain

$$10.3. \quad \left\{ \begin{array}{l} b_0 = 1 \\ b_1 = - \left[ y_1 \left( y_2 + \frac{1}{y_2} \right) + y_3 \left( y_4 + \frac{1}{y_4} \right) \right] \\ b_2 = y_1^2 + y_3^2 + y_1 y_3 \left( y_2 + \frac{1}{y_2} \right) \left( y_4 + \frac{1}{y_4} \right) \\ b_3 = - \left[ y_1 y_3^2 \left( y_2 + \frac{1}{y_2} \right) + y_1^2 y_3 \left( y_4 + \frac{1}{y_4} \right) \right] \\ b_4 = y_1^2 y_3^2 \end{array} \right.$$

We might eliminate  $y_2$  and  $y_4$  from eq. s 10.3 and obtain a symmetrical relation between  $y_1$  and  $y_3$ .

We will show in App. VII that, assuming:

$$10.4. \quad \left\{ \begin{array}{l} z = y_1^2 + y_3^2 = y_1^2 + b_4/y_1^2 = y_3^2 + b_4/y_3^2 \\ G(z, b_1, b_2, b_3, b_4) = z^3 - b_2 z^2 + (b_1 b_3 - 4b_4) z + [4b_2 b_4 - (b_3^2 + b_1^2 b_4)] \end{array} \right.$$

the solution of polynomial 10.1 is reduced to solving a third order equation in  $z$ :

$$10.5. \quad G(z, b_1, b_2, b_3, b_4) = 0$$

where remembering eq. s 7.3:

$$10.6. \quad \left\{ \begin{array}{l} b_0 = a_0 \\ b_1 = a_1 + \Delta a_1 \\ b_2 = a_2 + \Delta a_2 \\ b_3 = a_3 + \Delta a_3 \\ b_4 = a_4 \end{array} \right.$$

In no-load conditions  $y_1$  and  $y_3$  assume the values  $e^{-\gamma}$  and  $e^{-\alpha T}$ . If at higher load values one of them should cross the value one instability would result.

Correspondingly one of the three solutions of eq. 10.5, must assume the value  $1 + a_4$ , as from eq. s 10.4. If we define:

$$10.7. \quad F(b_1, b_2, b_3) = G(1+a_4, b_1, b_2, b_3, a_4)$$

initially, under stable conditions, one has:

$$10.8. \quad F(a_1, a_2, a_3) \neq 0$$

and stability is maintained until  $F(b_1, b_2, b_3)$  does not cross a zero at some load value.

In the case of complex conjugate roots, one can conclude that the sufficient as well as necessary condition for the stability of the four solutions is that function

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$F(b_1, b_2, b_3)$  maintains the initial sign or reacquires it. That is:

$$10.9. \quad \frac{F(b_1, b_2, b_3)}{F(a_1, a_2, a_3)} > 0$$

Let us now consider the possibility of real roots. In this case eq. 10.9 is still necessary in order to have stability, but is not sufficient any more because  $y_2$  and  $y_4$ , as defined from eq. s 10.2, may now make the moduli greater than 1.

Anyway we will show in App. VIII that the particular perturbation structure considered brings us to the conclusion that condition 10.9 is sufficient, from a practical point of view, also in the case of real roots.

Let us now consider eq. 10.9 again in order to show the load influence.

In (10.6) the load affects the  $\Delta Q_i$  values. In fact recalling definition 8.10

$$10.10. \quad g = \frac{\alpha_c T \beta_o^2 Q_o Z_o}{2 W_s \cos \varphi [\cosh \alpha T - \cos \theta]}$$

in the limiting of point load we have

$$10.11. \quad Q_o = q_B^2$$

$$10.12. \quad W_s = q_B U_o$$

and taking into account (1, 2)

$$10.13. \quad q_B \beta_o Z_o = \frac{q_B}{C} = -V_B$$

we obtain

$$10.14. \quad g = \frac{\alpha_c (\beta T) (q_B \beta_o Z_o)}{2 U_o \cos^2 \varphi [\cosh \alpha T - \cos \theta]} = \frac{(2\pi + \theta)}{2 \cos^2 \varphi (\cosh \alpha T - \cos \theta)} r$$

where

$$10.15. \quad r = \frac{\alpha_c q_B}{C U_o}$$

For physical values of  $\theta$  (i. e. for small ones) there is a one to one correspondence between the values of  $g$  and  $r$ .

Through 8.11 and 10.14 we can write 10.6 as

$$10.16. \quad \left\{ \begin{array}{l} b_1 = a_1 + c_1 \cdot r \\ b_2 = a_2 + c_2 \cdot r \\ b_3 = a_3 + c_3 \cdot r \end{array} \right.$$

where

$$10.17. \quad \begin{cases} c_1 = \left\{ e^{-\alpha T} \operatorname{sen} \xi - \operatorname{sen}(\xi + \theta) \right\} \cdot f \\ c_2 = \left\{ \operatorname{sen}(\xi + \theta) - e^{-2\alpha T} \operatorname{sen}(\xi - \theta) \right\} \cdot f \\ c_3 = \left\{ e^{-2\alpha T} \operatorname{sen}(\xi - \theta) - e^{-\alpha T} \operatorname{sen} \xi \right\} \cdot f \end{cases}$$

and

$$10.18. \quad \begin{cases} \xi = 2\varphi - \gamma \\ f = \frac{2\pi + \theta}{2\cos^2 \varphi (\cosh \alpha T - \cos \theta)} \end{cases}$$

Substituting 10.16 into 10.9 from 10.7 and 10.4 we have for stability

$$10.19. \quad \left\{ \frac{Ar^2 + Br + D}{D} > 0 \right.$$

where

$$10.20. \quad \begin{cases} D = (1 + a_4)^3 - a_2(1 + a_4)^2 + (a_1 a_3 - 4a_4)(1 + a_4) + [4a_2 a_4 - (a_3^2 + a_1^2 a_4)] \\ B = (1 - a_4)^2 (c_1 + c_3) + (a_1 c_3 + a_3 c_1)(1 + a_4) - 2(a_3 c_3 + a_1 c_1 a_4) \\ A = (1 + a_4) c_1 c_3 - c_3^2 - c_1^2 a_4 = (c_1 - c_3)(c_3 - c_1 a_4) \end{cases}$$

It is by no means easy to formulate 10.19 explicitly in a general way. It is, therefore useful to illustrate this formula by some observations and graphs.

In App. IX we will prove that coefficient A of 10.20 usually has the following representation

$$10.21. \quad A = - \frac{4e^{-(3\alpha T + \gamma)} f^2}{(1 + \operatorname{tg}^2 \frac{\theta}{2})^2} \left\{ \operatorname{tg}^2 \frac{\theta}{2} \operatorname{sen}(\xi) [1 + \cosh \alpha T] - 2 \operatorname{tg} \frac{\theta}{2} \cos(\xi) \operatorname{senh} \alpha T + \operatorname{sen}(\xi) \cdot \right. \\ \left. \cdot (1 - \cosh \alpha T) \right\} \left\{ \operatorname{tg}^2 \frac{\theta}{2} \operatorname{sen} \xi [\cosh \gamma + \cosh(\gamma + \alpha T)] + 2 \operatorname{tg} \frac{\theta}{2} \cos(\xi) \operatorname{senh} \gamma + \right. \\ \left. + \operatorname{sen}(\xi) [\cosh(\gamma + \alpha T) - \cos \gamma] \right\}$$

The first factor vanishes for two values of  $\operatorname{tg} \theta/2$ ; the first

$$\operatorname{tg} \frac{\theta}{2} = \operatorname{tgh} \frac{\alpha T}{2} \cot \gamma$$

has no importance; the second corresponds to

$$10.22. \quad \theta_{A1} \approx - \frac{\pi}{2Q^2}$$

The second factor (see App. IX) vanishes for the following values of  $\theta_A$

$$10.23. \quad \begin{cases} \theta_{A2} = -2Q \cdot \gamma \\ \theta_{A3} = -\frac{\pi^2}{2\gamma Q^3} \end{cases} \quad \text{if } \gamma \gg \pi/Q^2$$

$$10.24. \quad \theta_{A4} = \theta_{A3} = -Q\gamma \quad \text{if } \gamma = \pi/Q^2$$

$$10.25. \quad \theta_{A2}, \theta_{A3} \rightarrow \text{complex conjugates} \quad \text{if } \gamma < \pi/Q^2$$

In App. IX we deduce an explicit expression for the factor B, in the case of a vanishing  $\gamma$ :

$$10.27. \quad B = f \cdot 8 \cdot e^{-3\alpha T} \text{senh } T \left(1 + \text{tg}^2 \frac{\theta}{2}\right)^{-2} \left\{ \begin{aligned} & -\text{sen } \xi (1 - \text{cosh } \alpha T)(\cos \delta - \text{cosh } \alpha T) + 2 \text{tg} \frac{\theta}{2} \cos \xi \text{ senh } \alpha T \\ & (1 - \text{cosh } \alpha T) - 2 \text{tg}^2 \frac{\theta}{2} \text{sen } \xi \cosh \alpha T (1 - \cos \delta) - 2 \text{tg}^3 \frac{\theta}{2} \cos \xi \text{ senh } \alpha T (1 + \text{cosh } \alpha T) + \\ & + \text{tg}^4 \frac{\theta}{2} \text{sen } \xi (1 + \text{cosh } \alpha T)(\cos \delta + \text{cosh } \alpha T) \end{aligned} \right\}$$

B vanishes for (see App. IX)

$$10.28. \quad \theta_B \simeq - \left( \frac{\delta^2}{2\pi} + \frac{\pi}{2Q^2} \right)$$

Let us last remember that coefficient D of 10.19 is the zero-load value of F. If  $\gamma$  vanishes two initial roots have modulus 1 and because of 10.4 and 10.5  $G(1+a_4, a_1, a_2, a_3, a_4)$  (i. e. D) vanishes.

In fig. 2 we plot, as functions of  $\theta$  and at fixed  $\gamma$ ,  $\delta$  and  $Q$ , the two values of  $(\alpha_c q_B)/CU_0$ , which annihilate F as defined in 10.7, i. e.:

$$10.29. \quad r_1 = \frac{-B \pm \sqrt{B^2 - 4AD}}{2A}$$

We can see from 10.19 that these two roots determine (at fixed  $\theta$  and varying load) the stability (and instability) regions. Since the case  $q_B = 0$  is stable ( $\gamma$  non zero), it is easy to determine the stability (and instability) regions (see fig. 2 and following).

Fig. 2 shows that the stability region corresponding to the 1<sup>st</sup> root goes from zero to some positive value. The stability limits are much higher when  $\theta < 0$ . That is qualitatively as in (1, 2, 3).

The 2<sup>nd</sup> root is negative when  $\theta < \theta_{A1}$  (10.22.) and goes from  $-\infty$  to  $+\infty$  when  $\theta$  crosses the value  $\theta_{A1}$ , that is when A vanishes. For  $\theta > \theta_{A1}$  the 2<sup>nd</sup> root exceeds the 1<sup>st</sup> one so that with increasing load we have first stability these instability, and then stability again.

Fig. 3, 4 show the some graphs but for different values of  $\delta$  and  $Q$ . In the case of fig. 4 the  $\theta_{A3}$  (10.25) value is eight times smaller than the corresponding value of fig. 2 and we have an asymptote for the 1<sup>st</sup> root too.

Let us last consider the limiting case of zero  $\gamma$  (that is the case of a proton storage ring).

In this case D vanishes and one of the roots of 10.29 vanishes also. The other one is:

$$10.30. \quad r = -\frac{B}{A}$$

Formula 10.30 is plotted in fig. 5. It has an asymptotic behaviour for  $\theta = \theta_{A1}$  (10.22) and goes through zero for  $\theta = \theta_B$  (10.28). It also has approximately the same behaviour as the 1<sup>st</sup> root (see fig. 2) for  $\theta < \theta_B$ , and as the 2<sup>nd</sup> root for  $\theta > \theta_B$ .

How this behaviour is obtained with continuity from fig. 2, can be understood by the following argument. Let us assume (by continuity) that also for  $\gamma \neq 0$  there exists a  $\theta_B$  for which B of 10.20 is zero. In that case 10.29 gives

$$10.31. \quad r_{1,2} = \pm \sqrt{\frac{D}{A}}$$

When  $\gamma$  (hence D) decreases the two roots (10.31) vanish. As the 1<sup>st</sup> and 2<sup>nd</sup> roots go to zero, to do the right and left halves of the curves respectively. By continuity the right part of the 2<sup>nd</sup> curve becomes the continuation of the left part right of the 1<sup>st</sup> curve. By continuity it is also possible to determine the stability regions.

From 10.21 and 10.27 we obtain an approximate of 10.30 (see App. IX):

$$10.32. \quad \frac{\alpha_c q_B}{C U_0} \approx \frac{\pi^2}{Q^2} \left( \theta + \frac{\delta^2}{2\pi} + \frac{\pi}{2Q^2} \right) / \theta$$

When such high values of the load are considered that they can approach the asymptotic of the stability region, the limitation of app. XIII, must be taken into account.

## CONCLUSION -

1) - With the hypothesis we say in introduction the further essential approximations in our work are that of considering small oscillations (about the equilibrium) and that the perturbations produced by beam and by supply on the cavity are distinct with respect to the time.

2) From figures 2, 3, 4 and 5 we draw the general conclusion that the stability range is greater for negative  $\theta$ 's than for positive ones; it is therefore useful to operate the cavity at a higher frequency than the natural one. This agrees with the results of previous papers<sup>(1, 2, 3)</sup>.

From the same figures it appears that there are stability regions for positive  $\theta$ 's and for a load greater than a given value; it is not possible through to reach them without going through instability regions.

3) - This work is mainly concerned with the equilibrium conditions. Nothing can be exactly said about the transient of injection during which the hypothesis of a rigid structure of the beam and the small-oscillations approximation do not hold. However we may do qualitative considerations. Because the delay of the beam with respect to the synchronous period is proportional to the  $\Delta p/p$  and the overexcitation due to the crossing beam is proportional to the delay and to the load, the coupling between the beam and the cavity decreases as the energy. Therefore it is useful to inject and to work with higher energies.

In the figures we said before, we draw also the range of stability corresponding to negative values of the load. Such situation that looks extremely theoretical really happens during the injection: in fact many of the injected particles before being lost may give disorderly energy to the cavity helping the beam-cavity system to go out of equilibrium.



4) - From the above mentioned figures one can see that for a negative  $\theta$ , the stability limit is reached when  $(\alpha_c q_B / CU_0)$  lies between  $10^{-7}$  and  $10^{-6}$  (for more details on diagram 4 see § 10. In the worst case, i. e. vanishing  $\gamma$  (figure 5), we obtain from 10.32 that the order of magnitude of  $(\alpha_c q_B / CU_0)$  for reaching the instability is  $\pi^2 / Q^2$ , where  $Q$  is the quality factor of the cavity.

For Adone  $q_B$  is the charge of a bunch multiplied by 2 since both electrons and positrons pass through the cavity.

$$q_B = 2 \cdot N \cdot e = 2 \frac{2 \cdot 10^{11}}{3} \cdot 1,6 \cdot 10^{-19} = 2,14 \cdot 10^{-8} \text{ coulombs}$$

$$\alpha_c = 6,12 \cdot 10^{-2} \text{ is the momentum compaction}$$

$$C = 10^3 \text{ pF the cavity capacity}$$

$$U_0 = 3,5 \cdot 10^8 \text{ volt the energy of machine during the injection}$$

and one therefore obtains

$$\frac{\alpha_c q_B}{CU_0} = 3,75 \cdot 10^{-9}$$

a value smaller than  $10^{-7}$  by a factor of 30.

This result may still not be satisfactory because of what said under point 3 and because generalization to the case of many cavities and bunches is not straight forward. The work of Henry deals with this last point but does not ready a definite conclusion.

5) - There is a qualitative agreement with the conclusions of preceedings works. For a quantitative comparison let us consider only paper<sup>(2)</sup> because it is the only one that compares with ours, in that on 4<sup>th</sup> degree polynomial is considered and on inequality for the stability attained.

If  $x$  and  $y$  are solutions of the polynomials the behaviour of our variables is as  $x^n$  whereas in (2) it is  $e^{\gamma t}$ . Then one must make a comparison between the logarithm of the solutions of our polynomial and the solution of the polynomial in<sup>(2)</sup> multiplied by  $T$  (period). One can easily verify that, for zero  $\gamma$ , the four non perturbed solutions are equivalent.

On the contrary there is no agreement on the four perturbed solutions. This can be shown by comparing the final formulae that give the limits of stability.

In ref. (2) the condition for the limits of stability is

$$0 < \text{sen} 2\phi_\gamma < 2 \frac{V}{V_B} \cos \psi_B$$

where

$$\text{tg} \phi_\gamma = - \frac{Q}{\pi} \theta$$

and

$$\text{sen} 2\phi_\gamma = \frac{2 \text{tg} \phi_\gamma}{1 + \text{tg}^2 \phi_\gamma} = - \frac{2(Q/\pi)\theta}{1 + (\frac{Q}{\pi}\theta)^2}$$

$$V \cos \psi_B = \left( \frac{\omega_s}{f_0} \right)^2 \frac{U_0}{2\pi\alpha_c} = \delta^2 \frac{U_0}{2\pi\alpha_c}$$

Finally  $V_B$  is proportional to the load and one approximately has (3.4.)

$$V_B = \frac{Q}{\pi} \frac{q_B}{C}$$

Substituting one obtains

$$0 < \frac{-2(Q/\pi)\theta}{1+(\frac{Q}{\pi}\theta)^2} < \frac{\delta^2}{Q} / \left( \frac{\alpha_c q_B}{CU_o} \right)$$

which may be written as

$$\frac{\alpha_c q_B}{CU_o} < - \frac{\delta^2}{Q^2} \frac{\pi^2 + Q^2 \theta^2}{2\pi\theta}$$

The last formula must be compared with (10.32) that gives our approximate solution in the case of vanishing  $\gamma$

$$\frac{\alpha_c q_B}{CU_o} < \frac{\pi^2}{Q^2} \frac{\theta + (\delta^2/2\pi) + (\pi/2Q^2)}{\theta}$$

The disagreement, is remarkable. (see fig. 5).

6) - Let us remember that in ref. (3) the case that the cavity is not equivalent to an RLC circuit is considered. In the frame of the present work we do not know how to deal with such case and we hope that the RF experts will rule its occurrence out.

7) - In our work we show that the rigid beam structure is practically described by one parameter only, i. e. by a positive angle  $\psi$ . Generally  $\psi$  (see § 8) can be assumed to be zero as long as the beam length is short with respect to the RF wavelength.

8) - In the present work we did consider the radiation damping of synchrotron oscillations accurately. When  $\gamma = 2,52 \cdot 10^{-7}$  ( $e^{-\gamma}$  is the attenuation of coherent synchrotron oscillations from one passage to the next) the limits of stability gain a factor of 2 or 3 over the case of vanishing is predicted. The  $\gamma$  value of  $2,52 \cdot 10^{-7}$  that was used in computing curves 2, 3, 4 corresponds for Adone to an energy of 350 MeV and a third of a full machine turn.

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APPENDIX 1 -

Eq. 1. 1, using the definitions 1. 2 may be written:

$$A1. 1. \quad \frac{d^2 x}{dt^2} + 2\alpha \frac{dx}{dt} + (\beta^2 + \alpha^2) x = 0$$

This eq. has a general solution which is

$$A. 1. 2. \quad x = e^{-\alpha t} [a \cos \beta t + b \sin \beta t]$$

a and b are two arbitrary constants.

Let us express a and b using the initial values  $x(0)$  and  $(dx/dt)_t = 0$ .

From eq. A. 1. 2 we obtain:

$$A. 1. 3. \quad \dot{x} = e^{-\alpha t} [\cos \beta t (-\alpha a + b\beta) - \sin \beta t (\alpha b + \beta a)]$$

$\dot{x}$  is the time derivative of  $x(t)$ .

From A. 1. 3. we deduce:

$$x(0) = a$$

$$\dot{x}(0) = b\beta - \alpha x(0)$$

$$A. 1. 4. \quad b = \frac{\dot{x}(0)}{\beta} + \frac{\alpha}{\beta} x(0)$$

$$a\beta + \alpha b = \beta x(0) + \frac{\alpha}{\beta} \dot{x}(0) + \frac{\alpha^2}{\beta} x(0) = \frac{\alpha}{\beta} \dot{x}(0) + \frac{\beta_0^2}{\beta} x(0)$$

Using these, eq. s A. 1. 2. and A. 1. 3. become:

$$(t) = e^{-\alpha t} \left[ (\cos \beta t + \frac{\alpha}{\beta} \sin \beta t) x(0) + \frac{\sin \beta t}{\beta} \dot{x}(0) \right]$$

A. 1. 5.

$$\dot{x}(t) = e^{-\alpha t} \left[ -\frac{\beta_0^2}{\beta} \sin \beta t x(0) + (\cos \beta t - \frac{\alpha}{\beta} \sin \beta t) \dot{x}(0) \right]$$

APPENDIX II -

In eq. 3. 1. appears the sum of the geometrical progression

$$S_n = \sum_{l=0}^n e^{(\alpha - \beta J) l T} = \frac{e^{(n+1)(\alpha - \beta J) T} - 1}{e^{(\alpha - \beta J) T} - 1}$$

and rationalising

$$S_n = \frac{[e^{(n+1)(\alpha - \beta J) T} - 1] [e^{(\alpha + \beta J) T} - 1]}{[e^{(\alpha - \beta J) T} - 1] [e^{(\alpha + \beta J) T} + 1]} =$$

$$= \frac{e^{(n+2)\alpha T - n\beta J T} - e^{(n+1)(\alpha - \beta J) T} - e^{(\alpha + \beta J) T} + 1}{e^{2\alpha T} - e^{\alpha T} (e^{\beta J T} + e^{-\beta J T}) + 1} =$$

$$= \frac{e^{n(\alpha - \beta J)T} e^{\alpha T} - e^{n(\alpha - \beta J)T} e^{-\beta J T} - e^{\beta J T} + e^{-\alpha T}}{2 [\cosh \alpha T - \cos \beta T]}$$

$$= \frac{(e^{\alpha T} - e^{-\beta J T}) e^{n(\alpha - \beta J)T} + (e^{-\alpha T} - e^{\beta J T})}{2 [\cosh \alpha T - \cos \beta T]}$$

Using this expression in eq. 3.1 we deduce:

$$\vec{v}(t) = e^{(-\alpha + \beta J)t} \left\{ \vec{v}(0) + \frac{[(e^{\alpha T} - e^{-\beta J T})(e^{n(\alpha - \beta J)T} \vec{v}_A + e^{m(\alpha - \beta J)(T+\tau)} \vec{v}_B)]}{2 [\cosh \alpha T - \cos \beta T]} + \frac{[(e^{-\alpha T} - e^{\beta J T})(\vec{v}_A + e^{(\alpha - \beta J)\tau} \vec{v}_B)]}{2 [\cosh \alpha T - \cos \beta T]} \right\}$$

This last formula allows us to deduce eq. s 3.3. and 3.4.

### APPENDIX III -

Let's first consider the term:

$$p_1 = e^{(-\alpha + \beta J)\tau(n+1)} \left\{ \vec{v}_c^+ [\chi(n), T] - \vec{v}_c^+ [\tau(n+1), T] \right\}$$

From eq. 3.4., we deduce:

$$p_1 = \frac{e^{\alpha T} - e^{-\beta J T}}{2 [\cosh \alpha T - \cos \beta T]} \left\{ e^{(-\alpha + \beta J)(\tau(n+1) - \tau(n))} - 1 \right\} e^{(-\alpha + \beta J)T} \vec{v}_B$$

and expanding in series to first order:

$$p_1 = (\alpha - \beta J) \frac{e^{-\alpha T} - e^{\beta J T}}{2 [\cosh \alpha T - \cos \beta T]} [\chi(n+1) - \chi(n)] \cdot \vec{v}_B$$

see eq. s 6.5 and 6.7.

The other terms:

$$p_2 = \vec{v}_c^+ [\tau(n+1), \tau(n+1)] - \vec{v}_c^+ [\tau_s, \tau_s]$$

again from eq. 3.4.:

$$p_2 = \frac{e^{\alpha T} - e^{-\beta J T}}{2 [\cosh \alpha T - \cos \beta T]} [e^{(-\alpha + \beta J)\tau(n+1)} - e^{(-\alpha + \beta J)\tau_s}] \vec{v}_A$$

and expanding to first order:

$$P_2 = (-\alpha + \beta J) \frac{(e^{\alpha T} - e^{-\beta J T}) e^{(-\alpha + \beta J) \tau_s}}{2 [\cosh \alpha T - \cos \beta T]} \chi^{(n+1)} \vec{v}_A$$

see eq. s 6.6. e 6.8.

#### APPENDIX IV -

Using definition 6.5., eq. 1.9. and 1.2. we obtain:

$$\begin{aligned} P_B &= \frac{\beta_0}{2 [\cosh \alpha T - \cos \theta]} [\text{I sen } \psi - \text{J cos } \psi] [I(e^{-\alpha T} - \cos \theta) - \text{J sen } \theta] = \\ &= \frac{\beta_0}{2 [\cosh \alpha T - \cos \theta]} \left\{ I [(e^{-\alpha T} - \cos \theta) \text{sen } \psi - \cos \psi \text{ sen } \theta] - \text{J} [(e^{-\alpha T} - \cos \theta) \cos \psi + \right. \\ &\quad \left. + \text{sen } \psi \cos \theta] \right\} = \\ &= \frac{\beta_0}{2 [\cosh \alpha T - \cos \theta]} \left\{ I [e^{-\alpha T} \text{sen } \psi - \text{sen}(\theta + \psi)] + \text{J} [\cos(\theta + \psi) - e^{-\alpha T} \cos \psi] \right\}. \end{aligned}$$

From 1.11.:

$$\begin{aligned} P_B &= \frac{\beta_0}{2 \cos \psi (\cosh \alpha T - \cos \theta)} \left[ \begin{array}{l} (e^{-\alpha T} \text{sen } \psi - \text{sen}(\theta + \psi)) \cos \psi - \\ - (\cos(\theta + \psi) - e^{-\alpha T} \cos \psi) \text{sen } \psi \end{array} \right] - [\cos(\theta + \psi) - e^{-\alpha T} \cos \psi] Z_0 \\ &= \frac{\beta_0}{2 \cos \psi (\cosh \alpha T - \cos \theta)} \left[ \begin{array}{l} \frac{\cos(\theta + \psi) - e^{-\alpha T} \cos \psi}{Z_0} \\ e^{-\alpha T} \text{sen } \psi - \text{sen}(\theta + \psi) \cos \psi + (\cos(\theta + \psi) - e^{-\alpha T} \cos \psi) \text{sen } \psi \end{array} \right] \\ &= \frac{\beta_0}{2 \cos \psi (\cosh \alpha T - \cos \theta)} \left[ \begin{array}{l} e^{-\alpha T} \text{sen} 2\psi - \text{sen}(2\psi + \theta) \\ \frac{\cos(\theta + \psi) - e^{-\alpha T} \cos \psi}{Z_0} \end{array} \right] - [\cos(\theta + \psi) - e^{-\alpha T} \cos \psi] Z_0 \\ &= \frac{\beta_0}{2 \cos \psi (\cosh \alpha T - \cos \theta)} \left[ \begin{array}{l} e^{-\alpha T} \text{sen} 2\psi - \text{sen}(2\psi + \theta) \\ \frac{\cos(\theta + \psi) - e^{-\alpha T} \cos \psi}{Z_0} \end{array} \right] - \text{sen } \theta \end{aligned}$$

eq. s 6.13 follow.

In the same way we deduce from eq. 6.6:

$$P_A = \frac{\beta_0 e^{-\alpha \tau_s}}{2 [\cosh \alpha T - \cos \theta]} \left\{ [-\text{I sen } \psi + \text{J cos } \psi] [I(e^{\alpha T} \cos \beta \tau_s - \cos \beta (T - \tau_s)) + \text{J}(e^{\alpha T} \text{sen } \beta \tau_s + \right. \\ \left. + \text{sen } \beta (T - \tau_s))] \right\}$$

and defining:

$$\phi_s = \beta \tau_s$$

$$P_A = \frac{\beta_0 e^{-\alpha \tau_s}}{2 [\cosh \alpha T - \cos \theta]} \left\{ -I \left[ \text{sen} \psi (e^{\alpha T} \cos \phi_s - \cos(\theta - \phi_s)) + \cos \psi (e^{\alpha T} \text{sen} \phi_s + \text{sen}(\theta - \phi_s)) \right] + \right. \\ \left. + J \left[ \cos \psi (e^{\alpha T} \cos \phi_s - \cos(\theta - \phi_s)) - \text{sen} \psi (e^{\alpha T} \text{sen} \phi_s + \text{sen}(\theta - \phi_s)) \right] \right\} \\ P_A = \frac{\beta_0 e^{-\alpha \tau_s}}{2 [\cosh \alpha T - \cos \theta]} \left\{ -I \left[ e^{\alpha T} \text{sen}(\phi_s + \psi) + \text{sen}(\theta - \phi_s \psi) \right] + J \left[ e^{\alpha T} \cos(\psi + \phi_s) - \cos(\phi_s + \psi - \theta) \right] \right\}$$

from 1. 11 we obtain:

$$P_A = \frac{\beta_0 e^{-\alpha \tau_s}}{2 [\cosh \alpha T - \cos \theta] \cos \psi} \left| \begin{array}{l} -e^{+\alpha T} \text{sen}(\phi_s + 2\psi) + \text{sen}(\phi_s + 2\psi - \theta) \quad \left[ \cos(\phi_s + \psi - \theta) - e^{\alpha T} \cos(\phi_s + \psi) \right] Z_0 \\ - \left[ \cos(\phi_s + \psi - \theta) - e^{\alpha T} \cos(\psi + \phi_s) \right] \frac{1}{Z_0} \quad \text{sen}(\phi_s - \theta) - e^{\alpha T} \text{sen} \phi_s \end{array} \right|$$

which gives eq. 6. 14.

#### APPENDIX V -

Let us define

$$p_1 = \cosh \alpha t_s$$

$$p_2 = \text{senh} \alpha t_s$$

$$q_1 = \cos \beta t_s$$

$$q_2 = \text{sen} \beta t_s$$

We obtain

$$V. 1. \quad p_1^2 - p_2^2 = 1$$

$$V. 2. \quad q_1^2 + q_2^2 = 1$$

and

$$\cosh \alpha(t - t_s) = p_1 \cosh \alpha t - p_2 \text{senh} \alpha t$$

$$\text{senh} \alpha(t - t_s) = p_1 \text{senh} \alpha t - p_2 \cosh \alpha t$$

$$\cos \beta(t - t_s) = q_1 \cos \beta t + q_2 \text{sen} \beta t$$

$$\text{sen} \beta(t - t_s) = q_1 \text{sen} \beta t - q_2 \cos \beta t$$

If we further define

$$A_{11}^{12} = \int_{t_1}^{t_2} \rho(t) \begin{Bmatrix} \cosh \alpha t \cdot \cos \beta t \\ \sinh \alpha t \cdot \sin \beta t \end{Bmatrix} dt$$

we obtain (8.3.)

$$V. 3. \left\{ \begin{array}{l} A_{11} = p_1 q_1 A'_{11} + p_1 q_2 A'_{12} - p_2 q_1 A'_{21} - p_2 q_2 A'_{22} = \sum_{i,j} p_i q_j A'_{ij} (-1)^{i+j} \\ A_{12} = p_1 q_1 A'_{12} - p_1 q_2 A'_{11} - p_2 q_1 A'_{22} + p_2 q_2 A'_{21} = \sum_{i,j} p_i q_{j+1} A'_{ij} (-1)^{i+j} \\ A_{21} = p_1 q_1 A'_{21} + p_1 q_2 A'_{22} - p_2 q_1 A'_{11} - p_2 q_2 A'_{12} = \sum_{i,j} p_{i+1} q_j A'_{ij} (-1)^{i+j} \\ A_{22} = p_1 q_1 A'_{22} - p_1 q_2 A'_{21} - p_2 q_1 A'_{12} + p_2 q_2 A'_{11} = \sum_{i,j} p_{i+1} q_{j+1} A'_{ij} (-1)^{i+j} \end{array} \right.$$

In V. 3. the indexes are modulus 2. We have for  $Q_1$ .

$$\begin{aligned} Q_1 &= (A_{11}^2 + A_{12}^2) - (A_{22}^2 + A_{21}^2) = \sum_{i,j,m,n} A'_{ij} A'_{mn} [p_i p_m q_n (-1)^{i+m} + \\ &+ p_i q_{j+1} p_m q_{n+1} (-1)^{i+j+m+n} - p_{i+1} q_j p_{m+1} q_n (-1)^{i+m} - \\ &- p_{i+1} q_{j+1} p_{m+1} q_{n+1} (-1)^{i+j+m+n}] = \\ &= \sum_{i,j,m,n} A'_{ij} A'_{mn} \left\{ q_j q_n (-1)^{i+m} [p_i p_m - p_{i+1} p_{m+1}] + \right. \\ &+ q_{j+1} q_{n+1} (-1)^{i+m+j+n} [p_i p_m - p_{i+1} p_{m+1}] \left. \right\} = \\ &= \sum_{i,j,m,n} A'_{ij} A'_{mn} \left\{ [q_j q_n + q_{j+1} q_{n+1} (-1)^{j+n}] \cdot \right. \\ &\left. \cdot [p_i p_m - p_{i+1} p_{m+1}] (-1)^{i+m} \right\} \end{aligned}$$

From V. 1. and V. 2. we obtain

$$\begin{aligned} q_j q_n + q_{j+1} q_{n+1} (-1)^{j+n} &= \delta_{jn} \\ p_i p_m - p_{i+1} p_{m+1} &= \delta_{im} (-1)^{i+1} \end{aligned}$$

and from this

$$V. 4. \quad Q_1 = \sum_{i,j} (A'_{ij})^2 (-1)^{i+1}$$

In the same way we have for  $Q_2$ , using the V. 3

$$\begin{aligned}
Q_2 &= 2(A_{11}A_{22} - A_{12}A_{21}) = 2 \sum_{i, j, m, n} A'_{ij} A'_{m, n} \left[ p_i q_j p_{m+1} q_{n+1} (-1)^{i+m+n+1} - \right. \\
&\quad \left. - p_i q_{j+1} p_{m+1} q_n (-1)^{i+m+j+1} \right] = \\
&= 2 \sum_{i, j, m, n} A'_{ij} A'_{m, n} p_i p_{m+1} (-1)^{i+m+1} \left[ q_j q_{n+1} (-1)^n - q_{j+1} q_n (-1)^j \right] = \\
&= 2 \sum_{i, j, m, n} A'_{ij} A'_{m+1, n+1} p_i p_m (-1)^{i+m} \left[ q_j q_n (-1)^{n+1} - q_{j+1} q_{n+1} (-1)^j \right] = \\
&= 2 \sum_{i, j, m, n} A'_{ij} A'_{m+1, n+1} p_i p_m (-1)^{i+m+1} \left[ q_j q_n (-1)^n + q_{j+1} q_{n+1} (-1)^j \right]
\end{aligned}$$

the terms in square brackets gives  $(-1)^j \delta_{jn}$ , therefore we obtain

$$\begin{aligned}
Q_2 &= 2 \sum_{i, j, m} A'_{ij} A'_{m+1, j+1} p_i p_m (-1)^{i+m+j+1} = \\
&= 2 \sum_{i, m} p_i p_m (-1)^{i+m+1} \sum_j A'_{ij} A'_{m+1, j+1} (-1)^j
\end{aligned}$$

the second sum is different from zero only when  $i = m$  and then

$$\begin{aligned}
Q_2 &= 2 \sum_{ij} (-1)^{j+1} p_i^2 A'_{ij} A'_{i+1, j+1} = 2 \sum_i p_i^2 \left[ A'_{ii} A'_{i+1, i+1} (-1)^{i+1} - \right. \\
V. 5. \quad &\quad \left. - A'_{i, i+1} A'_{i+1, i} (-1)^{i+1} \right] =
\end{aligned}$$

$$= 2(A'_{11}A'_{22} - A'_{12}A'_{21}) \sum_i p_i^2 (-1)^{i+1} = 2(A'_{11}A'_{21} - A'_{12}A'_{21})$$

The expressions V. 4. and V. 5. indicate that  $Q_1$  and  $Q_2$  are always equal to the values obtained at  $t_s = 0$ .

#### APPENDIX VI -

From eq. s 8. 7. and 8. 9. we deduce:

$$VI. 1. \quad \operatorname{tg} \psi = 2 \frac{(A_{11}A_{22} - A_{12}A_{21})}{(A_{11}^2 + A_{12}^2) - (A_{22}^2 + A_{21}^2)}$$

From eq. 8. 3. it follows by choosing  $t_s$  appropriately, we can make either  $A_{12}$  or  $A_{21}$  zero, so that (assuming  $A_{21}$  is zero)

$$VI. 2. \quad \operatorname{tg} \psi = \frac{2A_{11}A_{22}}{A_{11}^2 + A_{12}^2 - A_{22}^2}$$

As the bunch width is small compared with the radiofrequency wavelength and function  $\rho(t)$  is roughly symmetrical around the value of  $t_s$  which annihilates  $A_{21}$ , it fol-



lows that:

$$\begin{aligned} \text{VI. 3.} \quad & \sinh \alpha (t-t_s) \sin \beta (t-t_s) > 0 \\ & \cosh \alpha (t-t_s) \cos \beta (t-t_s) > 0 \\ & A_{11} \gg A_{22} \\ & A_{12} \simeq 0 \end{aligned}$$

so that VI. 2. becomes

$$\text{VI. 4.} \quad \psi \simeq 2 \frac{A_{22}}{A_{11}} = 2 \frac{\int_{t_1}^{t_2} \rho(t) \sinh \alpha (t-t_s) \sin \beta (t-t_s) dt}{\int_{t_1}^{t_2} \rho(t) \cosh \alpha (t-t_s) \cos \beta (t-t_s) dt}$$

and expanding VI. 4.

$$\text{VI. 5.} \quad \psi \simeq 2 \alpha / \beta \frac{\int_{t_1}^{t_2} \rho(t) (t-t_s)^2 dt}{\int_{t_1}^{t_2} \rho(t) dt}$$

If we refer the time to  $t_s$ , indicating with  $t_B$  the bunch transit time and applying Schwartz inequality:

$$\begin{aligned} \int_{t_1}^{t_2} \rho(t) (t-t_s)^2 dt & \simeq \int_{-t_B/2}^{+t_B/2} \rho(t+t_s) t^2 dt \leq \left[ \int_{-t_B/2}^{+t_B/2} \rho^2(t+t_s) dt \int_{-t_B/2}^{+t_B/2} t^4 dt \right]^{1/2} = \\ & = \left[ \bar{\rho}^2 t_B \frac{2}{5} \left( \frac{t_B}{2} \right)^5 \right]^{1/2} = \bar{\rho}^{1/2} \frac{t_B^3}{4 \sqrt{5}} \end{aligned}$$

On the other side:

$$\int_{t_1}^{t_2} \rho(t) dt = \bar{\rho} \cdot t_B$$

and for eq. s 1. 2. :

$$2 \alpha / \beta = \beta_0^2 \sin 2 \psi \simeq 2 \cdot \beta_0^2 \psi$$

so that indicating with:

$$\delta_B = \beta_0 t_B$$

we have

$$\text{VI. 6.} \quad \psi = \frac{\psi}{2 \sqrt{5}} \frac{\bar{\rho}^{1/2}}{\bar{\rho}} \delta_B^2$$

## APPENDIX VII -

From the second and fourth lines of 10. 3, assuming  $(y_2 + \frac{1}{y_2})$  and  $(y_4 + \frac{1}{y_4})$  as unknown, we obtain

$$(y_2 + \frac{1}{y_2}) = \frac{\frac{b_3}{y_1} - b_1 y_1}{y_1^2 - y_3^2}$$

A. VII. 1

$$(y_4 + \frac{1}{y_4}) = \frac{\frac{b_3}{y_3} - b_1 y_3}{y_3^2 - y_1^2}$$

and from the expression of  $b_2$ , we obtain by substitution,

$$b_2 = (y_1^2 + y_3^2) - y_1 y_3 \frac{\frac{b_3^2}{y_1 y_3} - b_1^2 y_1 y_3 - b_1 b_3 (\frac{y_3}{y_1} - \frac{y_1}{y_3})}{(y_1^2 - y_3^2)^2}$$

Again, using the last one of eq. s 10. 3:

$$\text{A. VII. 2} \quad (y_1^6 + y_3^6) - b_2 (y_1^4 + y_3^4) - (b_4 - b_1 b_3) (y_1^2 + y_3^2) + [2b_2 b_4 - b_3^2 - b_1^2 b_4] = 0$$

$$\text{As} \quad (y_1^6 + y_3^6) = (y_1^2 + y_3^2)^3 - 3b_4 (y_1^2 + y_3^2)$$

$$(y_1^4 + y_3^4) = (y_1^2 + y_3^2)^2 - 2b_4$$

we finally deduce

$$\text{A. VII. 3.} \quad (y_1^2 + y_3^2)^3 - b_2 (y_1^2 + y_3^2) - (4b_4 - b_1 b_3) (y_1^2 + y_3^2) + [4b_2 b_4 - b_3^2 - b_1^2 b_4] = 0$$

from which eq. s 10. 4 and 10. 5 follow.

APPENDIX VIII -

Remembering eq. s 7. 5 and 7. 6, we have:

$$P_0(0) \neq 0 \quad \Delta P(0) = 0$$

and therefore:

$$\text{A. VIII. 1.} \quad P(0) \neq 0$$

Considering the value 1, we have:  $P_0(1) \neq 0$  and as (see eq. 8. 11):

$$\begin{aligned} \text{A. VIII. 2.} \quad \Delta a_1 + \Delta a_2 + \Delta a_3 &= 0 \\ \Delta P(1) &= 0; \end{aligned}$$

and therefore

A. VIII. 3.

$$P(1) \neq 0$$

A. VIII. 1 and A. VIII. 2 make clear that the two values 0 and 1 cannot be solutions of the perturbed polynomial.

If a complex conjugate root becomes real in the interval (0, 1), it cannot move out of that interval and then it cannot become unstable by reasons of continuity.

Anyway a couple of roots could become real in the interval (-1, 0) and one of its values could become unstable crossing the value -1.

Let us show that we are far away from this possibility.

Using Descartes rule of the signs, polynomial 10.1 may have negative real roots only when the coefficient signs are not perfectly alternated.

As we deduce from eq. 7.5, the values of the coefficients are initially, with good approximation, 1, -4, 6, -4, 1. To break the sign order, one of the three coefficients should change sign.

Considering eq. 9.8, the strongest limitation is due to the third coefficient and so it must be:

A. VIII. 4.

$$|\Delta a_2| < 6$$

From 10.15, 10.16, 10.17, 10.18 we obtain, neglecting second order terms in  $\alpha T$ ,  $\varphi$  and  $\theta$

A. VIII. 5.

$$|\Delta a_2| \approx \frac{\alpha_c q_B}{C U_0} \frac{2\theta}{\alpha^2 T^2 + \theta^2} 2\pi$$

and, posing:

A. VIII. 6.

$$\theta = -\frac{\pi}{Q} \operatorname{tg} \phi_\gamma = -\alpha T \operatorname{tg} \phi_\gamma$$

where

$\phi_\gamma$  phase angle of the radiofrequency impedance.

We obtain

A. VIII. 7.

$$|\Delta a_2| \approx \frac{2\alpha_c \cdot q_B}{C U_0} Q \operatorname{sen} 2\phi_\gamma$$

If now we assume:

$$\alpha_c = 6,12 \cdot 10^{-2}$$

$$Q = 5 \cdot 10^3$$

$q_B/C$ , voltage pulse due to the beam = 20 V,

$U_0$ , machine operating voltage =  $3,5 \times 10^8$  V

$$\phi_\gamma = 5^\circ$$

we obtain

$$|\Delta a_2| \approx 6 \cdot 10^{-6}$$

so that eq. A. VIII. 4 is well satisfied.

APPENDIX IX -

For A (10. 20) from 10. 17 we get for the 1<sup>st</sup> factor

$$(c_1 - c_3) = f \left\{ 2 e^{-\alpha T} \text{sen } \xi - \text{sen}(\xi + \theta) - e^{-2\alpha T} \text{sen}(\xi - \theta) \right\} =$$

$$= e^{-\alpha T} f \left\{ 2 \text{sen } \xi - \text{sen } \xi \cos \theta (e^{\alpha T} + e^{-\alpha T}) - \cos \xi \text{sen } \theta (e^{\alpha T} - e^{-\alpha T}) \right\}$$

remembering that

$$\left. \begin{aligned} e^{\alpha T} + e^{-\alpha T} &= 2 \cosh \alpha T \\ e^{\alpha T} - e^{-\alpha T} &= 2 \sinh \alpha T \\ \cos \theta &= \frac{1 - \text{tg}^2(\theta/2)}{1 + \text{tg}^2(\theta/2)} \\ \text{sen } \theta &= \frac{2 \text{tg}(\theta/2)}{1 + \text{tg}^2(\theta/2)} \end{aligned} \right\}$$

A. IX. 1.

we obtain

$$\text{A. IX. 2. } (c_1 - c_3) = \frac{2e^{-\alpha T} f}{(1 + \text{tg}^2 \frac{\theta}{2})} \left\{ \text{tg}^2 \frac{\theta}{2} \text{sen } \xi (1 + \cosh \alpha T) - 2 \text{tg} \frac{\theta}{2} \cos \xi \sinh \alpha T + \text{sen } \xi (1 - \cosh \alpha T) \right\}$$

In a similar way we get for the 2<sup>nd</sup> factor of A

$$(c_3 - c_1 a_4) = f \left\{ e^{-2\alpha T} \text{sen}(\xi - \theta) - e^{-\alpha T} \text{sen } \xi - e^{-(3\alpha T + 2\gamma)} \text{sen } \xi + e^{-(2\alpha T + 2\gamma)} \text{sen}(\xi + \theta) \right\} =$$

$$= f e^{-(2\alpha T + \gamma)} \left\{ \text{sen } \xi \cos \theta (e^{\gamma} + e^{-\gamma}) + \cos \xi \text{sen } \theta (e^{\gamma} - e^{-\gamma}) - \text{sen } \xi \left[ e^{(\alpha T + \gamma)} + e^{-(\alpha T + \gamma)} \right] \right\}$$

and by means of A. IX. 1 and similar formulas

$$\text{A. IX. 3. } (c_3 - c_1 a_4) = \frac{-2f e^{-(2\alpha T + \gamma)}}{(1 + \text{tg}^2 \frac{\theta}{2})} \left\{ \text{sen } \xi \text{tg}^2 \frac{\theta}{2} \left[ \cosh(\alpha T + \gamma) + \cos \gamma \right] + 2 \text{tg} \frac{\theta}{2} \cos \xi \sinh \gamma + \right.$$

$$\left. + \text{sen } \xi \left[ \cosh(\alpha T + \gamma) - \cos \gamma \right] \right\}$$

Finally from A. IX. 2 and A. IX. 3 we obtain 10. 21.

Let us now consider B (10. 20) in the  $\gamma$  zero case. From 7. 5 we obtain

$$\left. \begin{aligned} a_1 &= -2 \left[ \cos \delta + e^{-\alpha T} \cos \theta \right] \\ a_3 &= -2 \left[ e^{-2\alpha T} \cos \delta + e^{-\alpha T} \cos \theta \right] = a_1 + 2(1 + a_4) \cos \delta \\ a_4 &= e^{-2\alpha T} \end{aligned} \right\}$$

A. IX. 4.

from which

$$\begin{aligned}
B &= (1-a_4)^2(c_1+c_3)+a_1(c_1+c_3)(1+a_4)+2(1-a_4)\cos\delta c_1(1+a_4)-2a_1(c_3+c_1a_4) - \\
&\quad -4c_3\cos\delta(1-a_4) = \\
&= (1-a_4)\left\{(1-a_4)(c_1+c_3)-2\cos\delta(c_3-c_1a_4)-2e^{-\alpha T}\cos\theta(c_1-c_3)\right\} \\
B &= \frac{(1-e^{-2\alpha T})}{(1+\operatorname{tg}^2\frac{\theta}{2})}\left\{(1-e^{-2\alpha T})(c_1+c_3)(1+\operatorname{tg}^2\frac{\theta}{2})-2\cos\delta(1+\operatorname{tg}^2\frac{\theta}{2})(c_3-c_1a_4)-\right. \\
&\quad \left.-2e^{-\alpha T}(1-\operatorname{tg}^2\frac{\theta}{2})(c_1-c_3)\right\}
\end{aligned}$$

A. IX. 5.

In the last expression  $(c_1-c_3)$  and  $(c_3-c_1a_4)$  are an estimate already. We must calculate  $(c_1+c_3)$ . From 10. 17 we have

$$\begin{aligned}
c_1+c_3 &= f\left[e^{-2\alpha T}\operatorname{sen}(\xi-\theta)-\operatorname{sen}(\xi+\theta)\right] = -fe^{-\alpha T}\left[-\operatorname{sen}\xi\cos\theta(e^{\alpha T}-e^{-\alpha T})-\right. \\
&\quad \left.-\cos\xi\operatorname{sen}\theta(e^{\alpha T}+e^{-\alpha T})\right]
\end{aligned}$$

and finally

$$(c_1+c_3) = -\frac{2e^{-\alpha T}f}{(1+\operatorname{tg}^2\frac{\theta}{2})}\left\{\operatorname{tg}^2\frac{\theta}{2}\operatorname{sen}\xi\operatorname{senh}\alpha T-2\operatorname{tg}\frac{\theta}{2}\cos\xi\cosh\alpha T-\operatorname{sen}\xi\operatorname{senh}\alpha T\right\}$$

Now, with substitution of A. IX. 2, A. IX. 3 and A. IX. 6 in A. IX. 5 we have

$$\begin{aligned}
B &= 2\frac{(1-e^{-2\alpha T})e^{-2\alpha T}f}{(1+\operatorname{tg}^2\frac{\theta}{2})^2}\left\{2\operatorname{senh}\alpha T(1+\operatorname{tg}^2\frac{\theta}{2})\left[\operatorname{tg}^2\frac{\theta}{2}\operatorname{sen}\xi\operatorname{senh}\alpha T-2\operatorname{tg}\frac{\theta}{2}\cos\xi\cosh\alpha T-\operatorname{sen}\xi\operatorname{senh}\alpha T\right]+ \right. \\
&\quad +2\cos\delta(1+\operatorname{tg}^2\frac{\theta}{2})\left[\operatorname{tg}^2\frac{\theta}{2}\operatorname{sen}\xi(\cosh\alpha T+1)+\operatorname{sen}\xi(\cosh\alpha T-1)\right]- \\
&\quad \left.-2(1-\operatorname{tg}^2\frac{\theta}{2})\left[\operatorname{tg}^2\frac{\theta}{2}\operatorname{sen}\xi(1+\cosh\alpha T)-2\operatorname{tg}\frac{\theta}{2}\cos\xi\operatorname{senh}\alpha T+\operatorname{sen}\xi(1-\cosh\alpha T)\right]\right\}
\end{aligned}$$

and finally

$$B = \frac{8\operatorname{senh}\alpha Te^{-3\alpha T}f}{(1+\operatorname{tg}^2\frac{\theta}{2})^2}\left\{\operatorname{tg}^4\frac{\theta}{2}\left[(1+\cosh\alpha T)(\cosh\alpha T+\cos\delta)\right]\operatorname{sen}\xi -\right.$$

$$\begin{aligned}
\text{A. IX. 7.} \quad & -2\operatorname{tg}^3\frac{\theta}{2}\operatorname{senh}\alpha T(1+\cosh\alpha T)\cos\xi -2\operatorname{tg}^2\frac{\theta}{2}\cosh\alpha T(1-\cos\delta)\operatorname{sen}\xi + \\
& +2\operatorname{tg}\frac{\theta}{2}\operatorname{senh}\alpha T(1-\cosh\alpha T)\cos\xi -\operatorname{sen}\xi(\cos\delta-\cosh\alpha T)(1-\cosh\alpha T)\left\}
\end{aligned}$$

In order to obtain the values of  $\operatorname{tg}(\theta/2)$  for which A vanishes, we observe that this is very simple for the 1<sup>st</sup> factor. We have for the 2<sup>nd</sup> factor from A. IX. 3.

$$\operatorname{tg} \frac{\theta}{2} = \frac{-\cos \xi \operatorname{senh} \gamma \pm \sqrt{\cos^2 \xi \operatorname{sen}^2 h \gamma - \operatorname{sen}^2 \xi [\cos^2(\alpha T + \gamma) - \cos^2 \gamma]}}{\operatorname{sen} \xi [\cosh(\alpha T + \gamma) + \cos \gamma]} =$$

$$= \frac{-\cos \xi \operatorname{senh} \gamma \pm \sqrt{(\cos^2 \xi + \operatorname{sen}^2 \xi) \operatorname{sen}^2 h \gamma - \operatorname{sen}^2 \xi \operatorname{sen}^2 h(\alpha T + \gamma)}}{\operatorname{sen} \xi [\cosh(\alpha T + \gamma) + \cos \gamma]}$$

If

$$\operatorname{senh} \gamma \gg \operatorname{sen} \xi \operatorname{sen}(\alpha T + \gamma) \approx \frac{\pi}{Q^2}$$

the radical quantity is positive and we obtain

$$\frac{\theta_{A2}}{2} \approx \operatorname{senh} \gamma \frac{-\cos \xi - 1}{2 \operatorname{sen} \xi} \approx -Q \gamma$$

and

$$\operatorname{tg} \frac{\theta_{A3}}{2} = \operatorname{senh} \gamma \frac{-\cos \xi + (1 - \frac{\operatorname{sen}^2 \xi \operatorname{sen}^2 h(\alpha T + \gamma)}{2 \operatorname{sen}^2 \gamma})}{2 \operatorname{sen} \xi} \approx \frac{\gamma}{2 \xi} \left[ \frac{\xi^2}{2} -$$

A. IX. 9

$$- \frac{\xi^2}{2} \frac{(\alpha T + \gamma)^2}{\gamma^2} \right] \approx - \frac{\pi^2}{4 \gamma Q^3}$$

from which 10. 23 follows.

If the radical quantity vanishes we obtain easily 10. 24. Finally in order the zero 10. 28 of B we observe that B (A. IX. 7) is approximately

$$B \approx \frac{8 \operatorname{senh} \alpha T f}{(1 + \operatorname{tg}^2 \frac{\theta}{2})} \left\{ \frac{4}{Q} \operatorname{tg}^2 \frac{\theta}{2} - 2 \frac{\pi}{Q} \operatorname{tg}^3 \frac{\theta}{2} - \frac{\delta^2}{Q} \operatorname{tg}^2 \frac{\theta}{2} - \frac{\pi^3}{2 Q^3} \operatorname{tg} \frac{\theta}{2} - \frac{\pi^2}{8 Q^3} (\delta^2 + \frac{\pi^2}{Q^2}) \right\}$$

If

$$\theta < \frac{1}{10 \cdot Q}$$

$$\delta \approx \frac{1}{Q}$$

A. IX. 10.

$$B \approx - \frac{2f}{Q^4} \pi^4 \left[ \theta + \left( \frac{\delta^2}{2\pi} + \frac{\pi^2}{2Q^2} \right) \right]$$

from which 10. 28 follows.

In a similar way we obtain from A. IX. 2 and A. IX. 3 in the  $\gamma$  zero case

$$\text{A. IX. 11.} \quad A \approx +4f^2 \left[ \theta \frac{\pi}{Q} \right] \left[ \frac{\pi^2}{2Q^3} \right] = 2f^2 \frac{\pi^3}{Q^4} \theta$$

From A. IX. 10 and A. IX. 11 we obtain 10. 32, of vanishing  $\gamma$  is predicted. The  $\gamma$  value of  $2.52 \cdot 10^{-7}$  that was used in computing curves 2, 3, 4 corresponds for Adone to an energy of 350 MeV and a third of a full machine turn.

## REFERENCES -

- (1) - Irvin Henry: "Phase oscillations in high current synchrotrons". J. of Applied Phys. 31, n. 8 (1960).
- (2) - Kenneth W. Robinson: "Stability of beam in radiofrequency system". CEAL 1010 - February (1964).
- (3) - Auslander et al. : "Phase instability of intense electron beams in storage rings". (Preprint of a paper given at the International Conference on Accelerators, Frascati 1965).

Meaning of the symbols used in figures 2, 3, 4, 5.

$\gamma$  } damping factor and phase variation of synchrotron oscillations from a crossing to  
 $\delta$  } the next one in the absence of coupling.  $\delta$  is assumed to be load independent.

$\alpha T$  } damping factor and phase variation of the cavity overexcitation from a crossing to  
 $2\pi + \theta$  } the next one in the absence of coupling.

$$\theta = \beta T - 2\pi$$

$$\beta = \left( \frac{1}{LC} - \frac{1}{4R^2C^2} \right)^{1/2} \text{ radiant frequency of free solutions of cavity.}$$

$T$  = period of synchronous particle.

$$\alpha T = \pi/Q \quad \left\{ \begin{array}{l} \alpha = 1/2RC \text{ damping factor of cavity} \\ Q = \text{quality factor of capacity} \end{array} \right.$$

$$r = \frac{\alpha_c q_B}{C U_0}$$

$\alpha_c$  = "momentum compaction".

$q_B$  = charge of the bunch (coulomb).

$C$  = cavity capacity (faraday).

$U_0$  = energy of machine.

The values of the curves in logarithmical scale labelled with a plus or a minus sign are to be considered positive or negatives ones. The same curves are qualitatively shown in linear scale on the right.

Fig. 5 shows also the Robinson's curve.



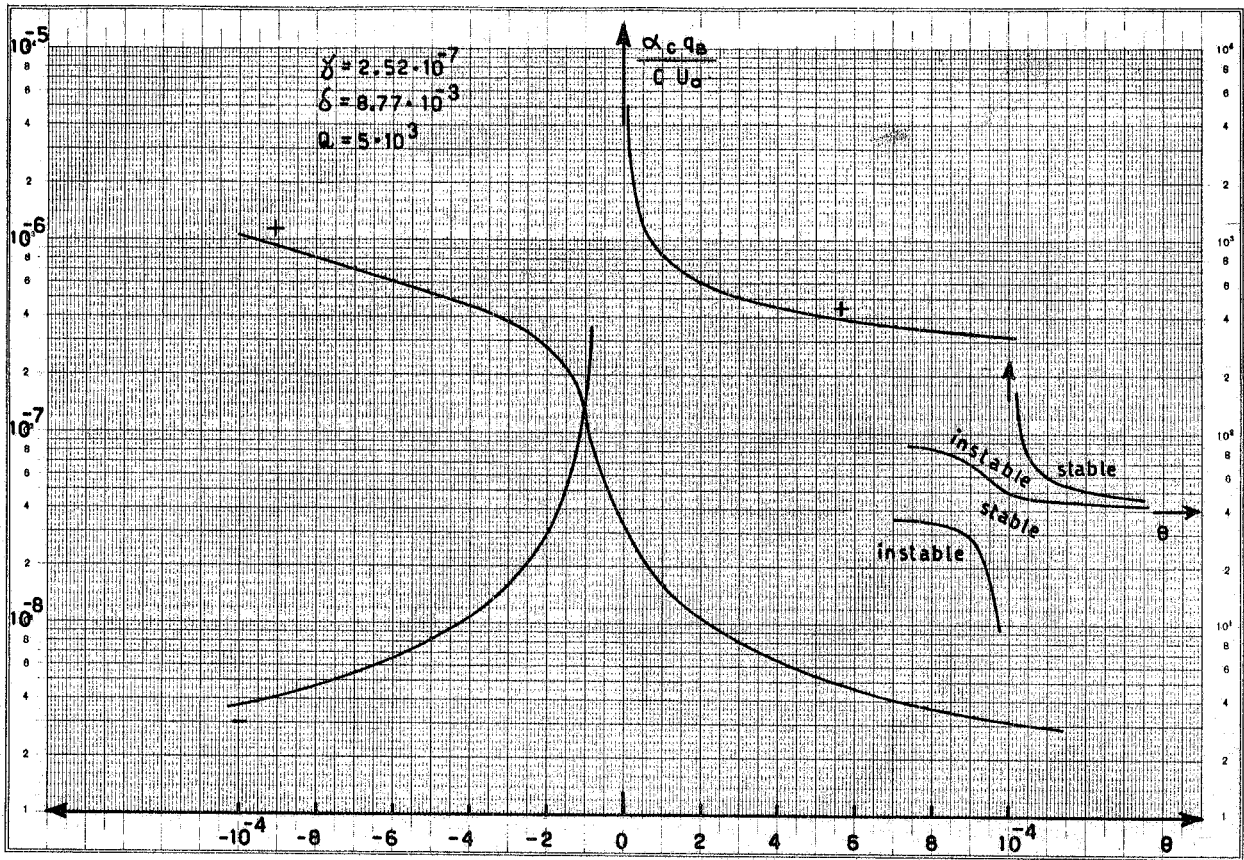


FIG. 2

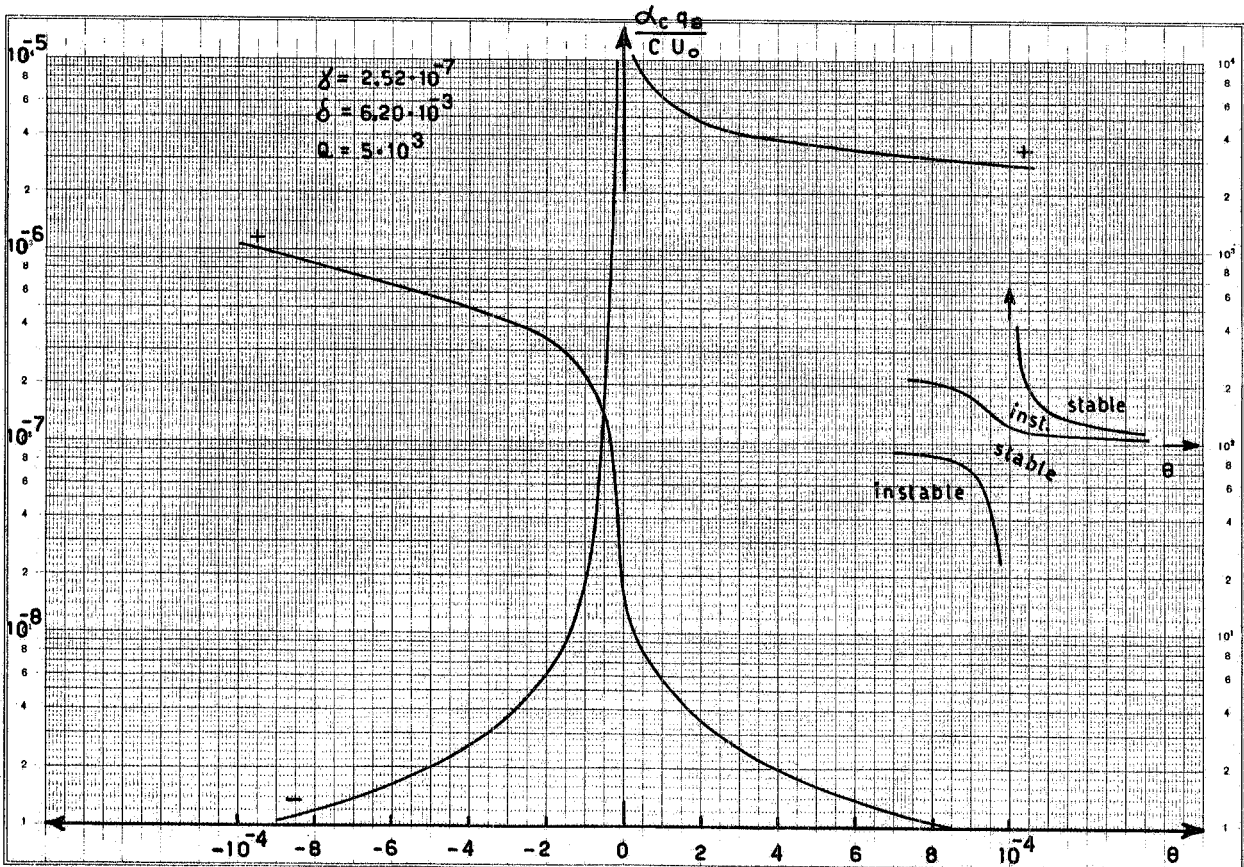


FIG. 3

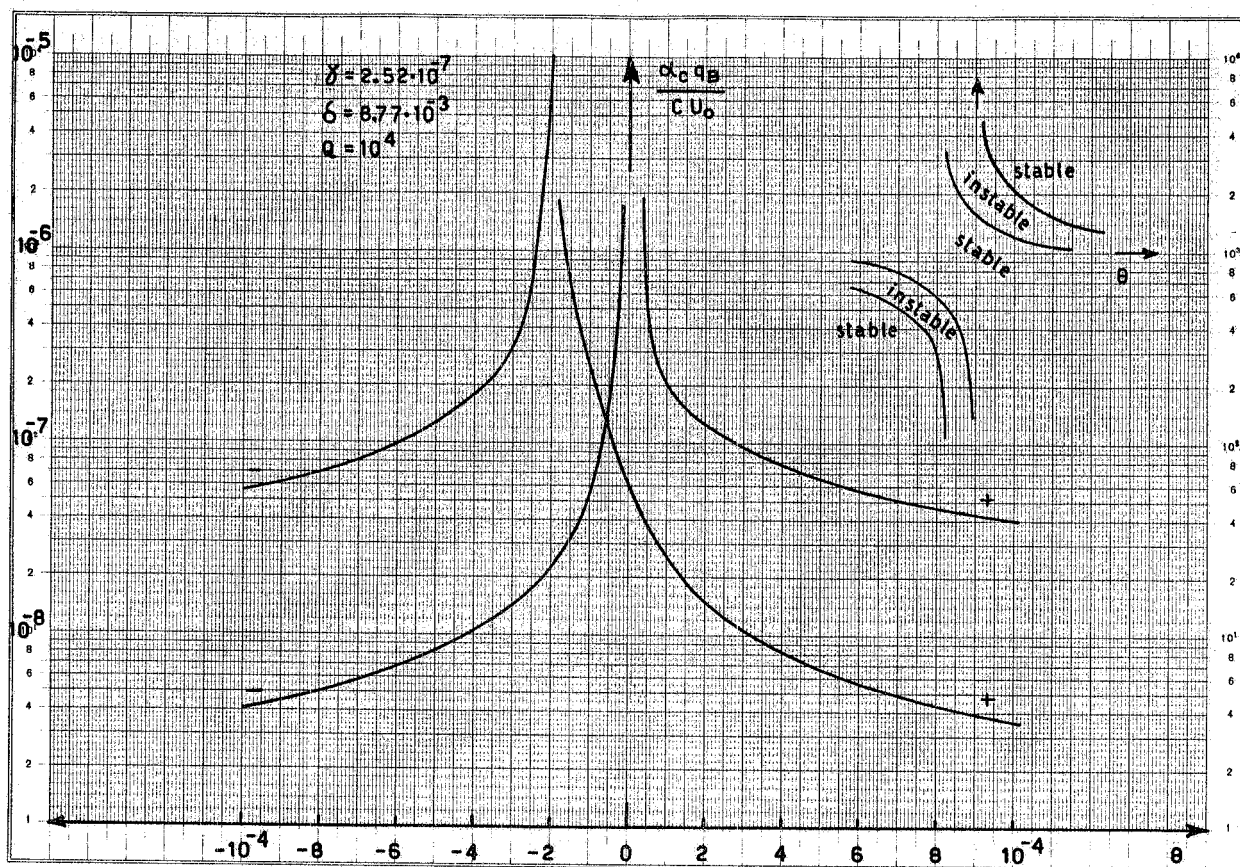


FIG. 4

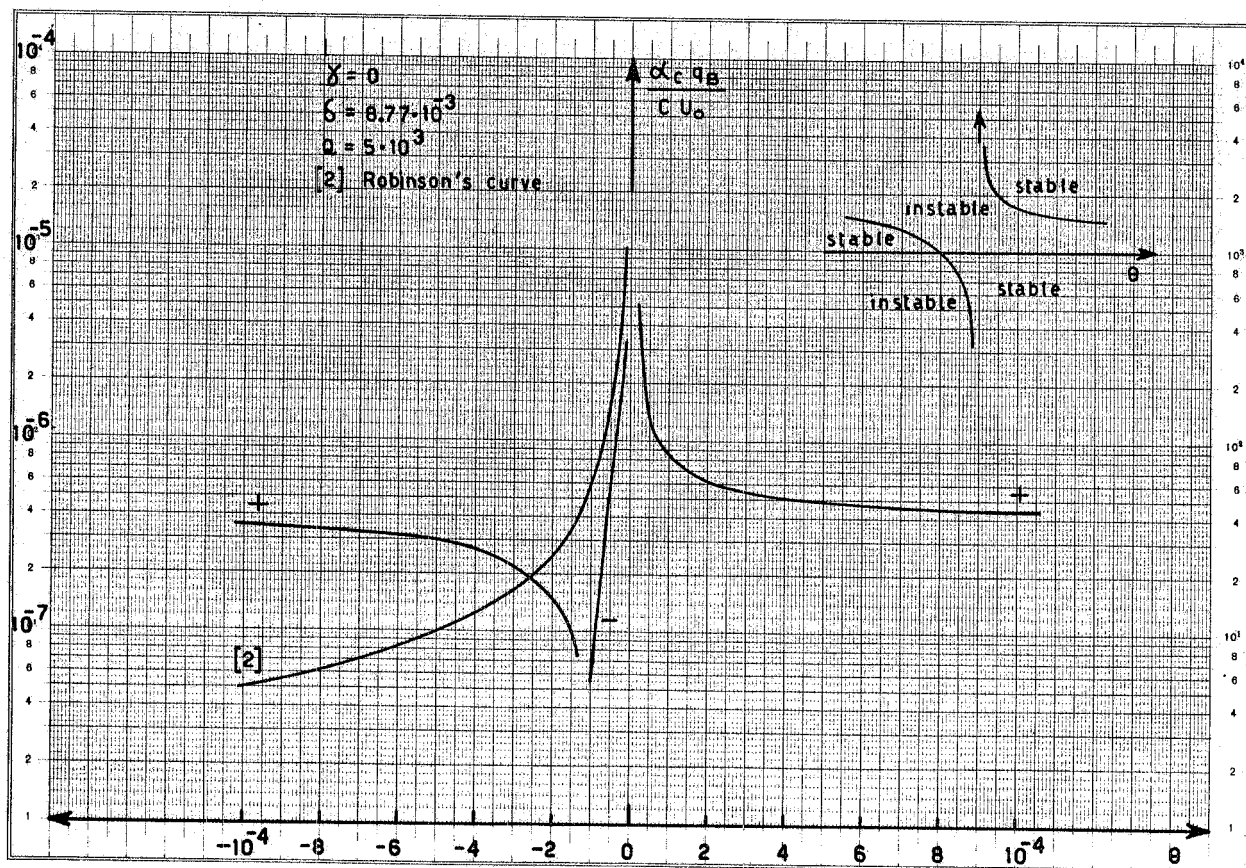


FIG. 5