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A. Małecki : INELASTIC SUM RULES FOR ELECTROMAGNETIC PROCESSES ON NUCLEAR TARGETS. -

(Nota interna : n. 365)

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ABSTRACT -

A formalism for the description of the electrodynamic processes with nuclear targets in the one photon exchange approximation is presented. The cross-section formulae for light nuclei, correct through order $1/M^2$ (M - nucleon mass), are derived. The inelastic sum rule contains no weighing factors and can be compared with the experimental sums constructed with varying as well as at constant momentum transfer. High energy electron scattering and electron pair photoproduction are discussed as examples.

1. - We consider in general an electrodynamic process connected by one photon exchange to a nuclear vertex-see Fig. 1. The circle includes an arbitrary diagram which together with the right part of Fig. 1 represents a process with probability amplitude[(1)]:

(1) $$M_{if} = e^f O_\mu A_\mu(q_\alpha)$$

The operator O_μ includes quantities describing particles in the electromagnetic part, f is an integer number, and $A_\mu(q_\alpha)$ is the four dimensional Fou-

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rier transform of the nuclear electromagnetic field.

The result of carrying out a relevant average and sum over initial and final states for the electrodynamic part and the nuclear part of the process as well, is:

(2) $$\overline{\sum_i} \sum_f |M_{if}|^2 = e^{2f} M_{\mu\nu} \overline{\sum_{|i\rangle}} \sum_{|f\rangle} A^*_\mu A_\nu ,$$

where $|i\rangle$, $|f\rangle$ are nuclear states.

We define a tensor:

(3) $$W_{\mu\nu} = \frac{q_\alpha^4}{(4\pi)^3 e^2 VT} \overline{\sum_{|i\rangle}} \sum_{|f\rangle} A^*_\mu(q_\alpha) A_\nu(q_\alpha),$$

where VT is a four dimensional normalization volume. The $W_{\mu\nu}$ tensor can be expressed through the three dimensional fourier transform of the electromagnetic nuclear current, as follows:

(4) $$\begin{cases} W_{\mu\nu} = \frac{1}{2} \overline{\sum_{|i\rangle}} \sum_{|f\rangle} J^*_{\mu fi}(\vec{q}) J_{\nu fi}(\vec{q})\, \delta(\omega - \omega_f) \\ J_{\mu fi}(x_\alpha) = \langle f | J_\mu(x_\alpha) | i \rangle = \left[Q_{fi}(x_\alpha), \vec{J}_{fi}(x_\alpha) \right] \end{cases}$$

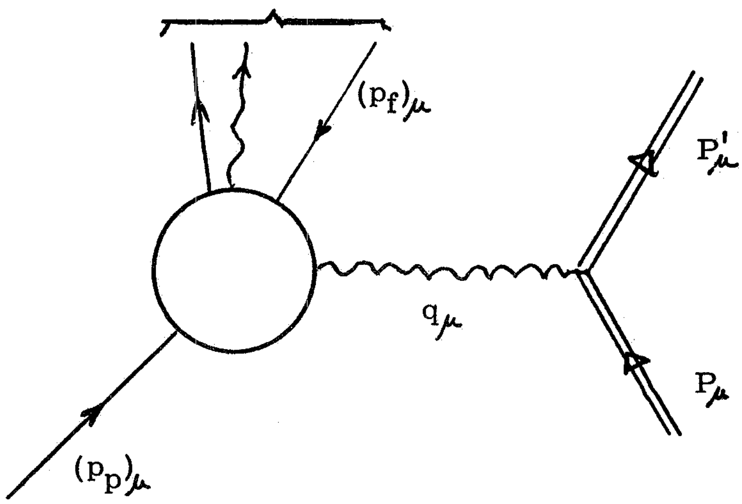
where $Q(x_\alpha)$, $\vec{J}(x_\alpha)$ are the charge and current operators of the nucleus at the point x_α $(t, \vec{x})$. We neglect the nuclear recoil and thus $\omega_f = E_f - E_i$ is the nuclear excitation energy.

It can be shown[2], using Lorentz and gauge invariance, that

(5) $$\begin{cases} W_{\mu\nu} M_{\mu\nu} = W_1 M_{\mu\mu} + W_2 M_{00} / \mathrm{LAB} \\ W_1 = -\frac{1}{4} \overline{\sum_{|i\rangle}} \sum_{|f\rangle} \delta(\omega - \omega_f) (\vec{J}^*_{fi} \cdot \vec{J}_{fi})_\perp / \mathrm{LAB} \\ W_2 = \frac{1}{2} \overline{\sum_{|i\rangle}} \sum_{|f\rangle} \delta(\omega - \omega_f) \left[\frac{q_\alpha^4}{q^4} Q^*_{fi} Q_{fi} - \frac{q_\alpha^2}{2q^2} (\vec{J}^*_{fi} \cdot \vec{J}_{fi})_\perp \right] \mathrm{LAB} \\ (\vec{J}^*_{fi} \cdot \vec{J}_{fi})_\perp = \vec{J}^*_{fi} \cdot \vec{J}_{fi} - \frac{1}{q^2} (\vec{q} \cdot \vec{J}^*_{fi})(\vec{q} \cdot \vec{J}_{fi}), \end{cases}$$

where all the quantities are to be taken in the laboratory frame.

With the aid of (2) ÷ (5) we get the following formula for the cross-section:



$(p_p)_\mu$ - projectile four momentum.

$(p_f)_\mu$ - four momenta of outgoing particles.

P_μ, $P'_\mu = P_\mu + q_\mu$ - initial and final four momentum of the nuclear target, respectively.

$q_\mu(\omega, \vec{q})$ - four momentum of the virtual photon.

FIG. 1

$$(6) \quad \left\{ \begin{aligned} &\frac{d^{2F}\sigma}{\prod_{j=1}^{F} d\varepsilon_f(j) d\Omega_f(j)} = \frac{e^{2(f+1)}}{(16\pi^3)^{F-1}} M(\omega) \overline{\sum_{|i\rangle}} \sum_{|f\rangle} \delta(\omega - \omega_f) \cdot \\ &\qquad \cdot \left[\frac{q_\alpha^4}{q^4} Q^*_{fi} Q_{fi} - \frac{1}{2}\left(\frac{M_{\mu\mu}}{M_{oo}} + \frac{q_\alpha^2}{q^2}\right)(\vec{J}^*_{fi} \cdot \vec{J}_{fi})_\perp \right]_{LAB}, \\ &M(\omega) = \frac{M_{oo}}{p_p q_\alpha^4} \prod_{j=1}^{F} p_f^{(j)}, \end{aligned} \right.$$

where p_p is the projectile momentum in the laboratory frame. The product symbol refers to all particles in the final state with four momenta $(p_f)_\mu = (\varepsilon_f, \vec{p}_f)$, which are produced in the electromagnetic part of the process. We have put V = 1 and assumed that wave functions of particles in the electrodynamic part are normalized to twice their energies while the nuclear states are normalized to unity.

2. - We neglect exchange currents and assume the charge and current operator of the nucleus to be the sum of the operators for the individual nucleons. Consistently, we use the single-particle model for the description of the nuclear states. We use the nonrelativistic form of the nuclear charge and current operators[3] including terms to the order $1/M^2$ (M-nucleon mass):

$$
(7)\quad \begin{cases}
Q(\vec{q}) = \sum_{j=1}^{A} \left\{ \left[e_j \left(1 - \frac{a}{2}\frac{q^2}{M^2} + \frac{q^2}{8M^2}\right) - \mu_j \frac{q^2}{4M^2} \right] e^{i\vec{q}\cdot\vec{r}_j} \right\} \\
\vec{J}(\vec{q}) = \sum_{j=1}^{A} \left\{ \frac{e_j}{2M} \left(\vec{p}_j e^{i\vec{q}\cdot\vec{r}_j} + e^{i\vec{q}\cdot\vec{r}_j}\vec{p}_j\right) + \right. \\
\qquad \left. + \frac{\mu_j}{2M}\, i\,(\vec{\sigma}_j \times \vec{q})\, e^{i\vec{q}\cdot\vec{r}_j} \right\},
\end{cases}
$$

where

$$
e_j = \frac{1+\tau_{3j}}{2}, \quad \mu_j = \frac{\mu_p + \mu_n}{2} + \frac{\mu_p - \mu_n}{2}\,\tau_{3j},
$$

and $\vec{r}_j$, $\vec{p}_j$, $\frac{1}{2}\vec{\sigma}_j$ are the position, momentum and spin operators for the j-th nucleon, respectively. The nucleon form factor is assumed to be $f(q_\mu{}^2) = 1 - (a/2)(q^2/M^2)$. Ref. 4 gives the value of $a = 4.82$.

Let us write:

$$
(8)\qquad M(\omega) = M(0)\left(1 + M_1 \frac{\omega}{Q} + M_2 \frac{\omega^2}{Q^2}\right),
$$

where Q is the elastic momentum transfer. We keep consistently only terms through order $1/M^2$. For nuclei with the same number of protons and neutrons in both spin states, we get:

$$
(9)\quad \begin{aligned}
& M(\omega) \sum_{|f\rangle} \delta(\omega - \omega_f) \frac{q_\alpha{}^4}{q^4} Q^*_{fi} Q_{fi} = 2M(0) \sum_{|f\rangle} \delta(\omega - \omega_f)\, \cdot \\
& \quad \cdot \left\{ \left[1 + M_1 \frac{\omega_f}{Q^2} + M_2 \frac{\omega_f^2}{Q^2} - \frac{aq^2}{M^2} + (1-2\mu_p)\frac{q^2}{4M^2} - \frac{2\omega_f^2}{q^2} \right] \cdot \right. \\
& \quad \left. \cdot \left| \langle a_p | e^{iqx} | a_h \rangle \right|^2 \right\}
\end{aligned}
$$

(9)

$$M(\omega)\frac{q\alpha^4}{q^4}Q^*_{ii}Q_{ii}\,\delta(\omega) = 4\,M(0)\left[1-\frac{aq^2}{M^2}+(1-2\mu_p-2\mu_n)\frac{q^2}{4M^2}\right]\cdot$$

$$\cdot\left|\sum_{a_h}\langle a_h|e^{iqx}|a_h\rangle\right|^2\delta(\omega)$$

$$M(\omega)\sum_{|f\rangle}\delta(\omega-\omega_f)\left(\frac{M_{\mu\mu}}{M_{oo}}+\frac{q\alpha^2}{q^2}\right)(\vec{J}^*_{fi}\cdot\vec{J}_{fi})_\perp = M(0)\left[\left(\frac{M_{\mu\mu}}{M_{oo}}\right)_o-1\right]\cdot$$

$$\cdot\sum_{|f\rangle}\delta(\omega-\omega_f)\left\{\frac{q^2}{M^2}(\mu_p^2+\mu_n^2)\left|\langle a_p|e^{iqx}|a_h\rangle\right|^2+\right.$$

$$\left.+\frac{4}{M^2}\left|\langle a_p|p_z\,e^{iqx}|a_h\rangle\right|^2\right\}\qquad (J^*_{ii}\cdot J_{ii})_\perp = 0,$$

where we chose $\vec{q}$ along the x-axis. The only final nuclear states which contribute in (9) are one particle-one hole states $|f\rangle = |\alpha_p\alpha_h\rangle$, where $|\alpha_p\rangle = |a_p\rangle/\sigma_{ap}\rangle|\tau_{ap}\rangle$ and $|\alpha_h\rangle = |a_h\rangle/\sigma_{ah}\rangle|\tau_{ah}\rangle$ are single particle states, respectively above and below the Fermi level of the nucleus. $(M_{\mu\mu}/M_{oo})_o$ indicates the value of $M_{\mu\mu}/M_{oo}$, which is in general a function of ω, at $\omega = 0$. We choose the shell model potential to be the oscillator potential well; the excitation energies of the nucleus are thus $\omega_f = n\,\omega_o = n(\alpha^2/M)$. For nuclei with filled s-shell and Z-2 protons and neutrons in p-shell we get:

(10)

$$\sum_{\substack{a_p a_h\\(n\omega_o)}}\left|\langle a_p|e^{iqx}|a_h\rangle\right|^2 = \frac{1}{6}e^{-t}\frac{t^n}{n!}\,(1-\delta_{no})\cdot$$

$$\cdot\left[3Z+(Z-2)\left(n-2t+\frac{t^2}{n+1}-\delta_{n1}\right)\right]$$

$$\sum_{\substack{a_p a_h\\(n\omega_o)}}\left|\langle a_p|p_z e^{iqx}|a_h\rangle\right|^2 = \frac{\alpha^2}{12}\,e^{-t}\,\frac{t^{n-1}}{(n-1)!}(1-\delta_{no})\Big[3Z+$$

$$+(Z-2)\left(n-2t+\frac{n+2}{n+1}\,\frac{t^2}{n}-\delta_{n1}\right)\Big]+\frac{(8-Z)(Z-2)}{24}\,\alpha^2 t\,e^{-t}\,\delta_{no}$$

$$\left|\sum_{a_h}\langle a_h|e^{iqx}|a_h\rangle\right|^2 = \frac{1}{12}\,e^{-t}\left[3Z^2-2Z(Z-2)t+2(Z-2)t^2\right],$$

where $t = (q^2/2\alpha^2)$ and all the final states one sums over have the same excitation energy $n\omega_o$.

3. - With the aid of (6), (9) and (10) one can calculate the cross-section corresponding to an excitation energy $n\omega_o$ and in particular, the elastic cross-section.

The inelastic sum rule which one obtains summing (6) over the whole nuclear spectrum, can be performed experimentally in various way. In general the momentum transfer q varies with the energy loss ω, as follows:

$$q^2 = Q^2 \left(1 + Q_1 \frac{\omega}{Q} + Q_2 \frac{\omega^2}{Q^2}\right) \tag{11}$$

Keeping consistently terms through order $1/M^2$ only, we get from (6), (9), (10) and (11) the inelastic sum rule:

$$\left\{\begin{aligned} &\frac{(16\pi^3)^{F-1}}{e^{2(f+1)} M(0)} \int_{\text{inel.}} d\omega \frac{d^{2F}\sigma}{\prod_{j=1}^{F} d\mathcal{E}_f(j)\, d\Omega_f(j)} = \\ &= \left\{1 - 2aT\frac{\alpha^2}{M^2} + (1-2_{\mu p})T\frac{\alpha^2}{2M^2} - \left[\left(\frac{M_{\mu\mu}}{M_{oo}}\right)_o - 1\right](\mu_p^2 + \mu_n^2)T\frac{\alpha^2}{2M^2}\right\} \cdot \\ &\cdot \left[Z - \left(Z + \frac{Z-2}{3}T^2\right)e^{-T}\right] + (-2+M_2)\left[5Z - 4 + 3ZT - 2(Z-2)e^{-T}\right] \cdot \\ &\cdot \frac{\alpha^2}{6M^2} - \left[\left(\frac{M_{\mu\mu}}{M_{oo}}\right)_o - 1\right]\left[5Z - 4 - 2(Z-2)(1+T)e^{-T}\right]\frac{\alpha^2}{6M^2} + \\ &+ \frac{Z\sqrt{T}}{\sqrt{2}}(Q_1 + M_1)\frac{\alpha}{M} + \frac{\alpha^2}{6M^2}\left\{(Q_2 + Q_1 M_1)\left[5Z - 4 + 6ZT - \right.\right. \\ &\left.\left. -2(Z-2)(1-T)e^{-T}\right] + Q_1^2 T\left[3Z + (Z-2)(2-T)e^{-T}\right]\right\}, \end{aligned}\right. \tag{12}$$

where:

$$T = \frac{Q^2}{2\alpha^2} .$$

4. - We end with a few examples.

a) High energy electron scattering at constant beam energy $\mathcal{E}$ and scattering angle θ.

$$(13)\quad \begin{cases} F = f = 1; \quad \left.\frac{M_{\mu\mu}}{M_{oo}}\right|_{LAB} = -2\tan^2\frac{\theta}{2}; \\ M(0) = \frac{\cos^2\theta/2}{4\mathcal{E}^2\sin^4\theta/2}; \quad M_1 = M_2 = 0; \\ Q = 2\mathcal{E}\sin\theta/2; \; Q_1 = -2\sin\theta/2; \; Q_2 = 1. \end{cases}$$

The inelastic sum rule is performed experimentally by varying the final electron energy.

b) High energy electron scattering at constant momentum transfer q and scattering angle θ.

$$(14)\quad \begin{cases} F = f = 1; \quad \left.\frac{M_{\mu\mu}}{M_{oo}}\right|_{LAB} = -2\tan^2\frac{\theta}{2}; \\ M(0) = \frac{\cos^2\theta/2}{q^2\sin^2\frac{\theta}{2}}; \quad M_1 = -2\sin\theta/2; \; M_2 = 1 + 2\sin^2\frac{\theta}{2}; \\ Q = q; \qquad Q_1 = Q_2 = 0. \end{cases}$$

The inelastic sum rule is performed experimentally by varying both the final and primary electron energies.

c) High energy electron pair photoproduction. As usually one uses the heteroenergetic bremsstrahlung photon beam, the inelastic sume rule is performed automatically.

In the case of the heteroenergetic projectile beam with a spectrum $S(\mathcal{E}_p)$ the formulae used so far have to be changed slightly. The only change needed is to divide the L.H.S. of (6), and hence the integrand function in (12), by $d\mathcal{E}_p$, and to modify the definition of $M(\omega)$ multiplying it by $S(\mathcal{E}_p)$.

We consider the symmetric pair case when the electron and positron are produced at the same angle θ with regard to the photon momentum, all the momenta are in the same plane, and both electrons have the same energy $\mathcal{E}$. To a very good approximation, over a small range in k the bremsstrahlung photon spectrum is[5] $S(k) = f(\mathcal{E}, k_{max})/k$, where k_{max} is the end point energy of the spectrum.

The formulas (we consider the Bethe-Heitler diagrams only) for this case are:

$$
(15)\quad \begin{cases}
F = f = 2; \qquad \left(\dfrac{M_{\mu\mu}}{M_{oo}}\right)_o \Bigg|_{LAB} = -2\,\mathrm{ctan}^2\,\theta/2 \\[2ex]
M(0) = \dfrac{\pi}{16}\,\dfrac{\cos^2\theta/2}{\mathcal{E}^4 \sin^6\theta/2}\; f(\mathcal{E}, kmax); \quad M_1 = -\,2(1+4\sin^2\dfrac{\theta}{2}); \\[2ex]
M_2 = 5 + 16\sin^2\dfrac{\theta}{2} + 40\sin^4\theta/2; \\[2ex]
Q = 4\mathcal{E}\sin^2\dfrac{\theta}{2}\;;\; Q_1 = 2; \quad Q_2 = 1.
\end{cases}
$$

Calculations for carbon (ref. 4 gives the oscillator potential parameter α = 121 MeV), at $\theta = 9^o36'$ and $\mathcal{E}$ = 1973 MeV give for the ratio of the inelastic sum rule to the elastic pair production the result: 14.7%. This result disagrees with the estimate quoted in [5] which suggests that the inelastic contribution is negligible.

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(1) - We use a metric such that $a_\alpha = (a_o, \vec{a})$ and $a_\alpha b_\alpha = a_o b_o - \vec{a}\cdot\vec{b}$. The magnitude of the three vector is $a = /\vec{a}/$. We also use units $c = \hbar = 1$, $e^2 = 1/137$.

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