Laboratori Nazionali di Frascati

LNF-61/66 (1961)

C. Pellegrini: THE TOTAL ENERGY-MOMENTUM VECTOR OF A CLOSED SYSTEM IN GENERAL RELATIVITY.

Estratto dal: Nuovo Cimento, 22, 427 (1961)

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

The Total Energy-Momentum Vector of a Closed System in General Relativity.

C. PELLEGRINI

Laboratori Nazionali di Frascati del C.N.E.N. - Frascati

(ricevuto il 15 Giugno 1961)

The difficulties connected with the concept of energy in the general theory of relativity have been widely studied by various authors [1,3] during recent years. In particular C. MØLLER [4] concluded that it is impossible to find an energy-momentum complex, θ_i^k, satisfying the following conditions:

1) $\theta_i^k(x^r)$ in the event point x^r is an affine tensor density that depends algebrically on the metric tensor g_{ik} and on its derivatives in the same point;

2) θ_i^k, $k=0$;

3) the total energy-momentum of a closed system, defined as

$$P_i = \int\limits_{x_4=\text{cost}} \theta_i^4 \, d^3x \,,$$

is time independent; the P_i transforms as a four-vector under Lorents transformations and furthermore is unchanged under arbitrary transformations of the coordinates provided that the new system coincides with the original one at spatial infinity;

4) energy density and energy current must be a scalar and a 3-vector respectively under the group of purely spatial transformations

$$(1) \qquad \begin{cases} \bar{x}^\alpha = \bar{x}^\alpha(x^\beta) \\ \bar{x}^4 = x^4 \end{cases} \qquad \alpha = 1, 2, 3 \,.$$

As shown by MAGNUSSON [5] the conditions 1), 2), 4) determine the Møller energy-momentum complex T_i^k uniquely, but this does not satisfy the condition 3), *i.e.* it does not transform like a four-vector under Lorentz transformation. Furthermore MØLLER [4] showed that 1), 2), 3), lead uniquely to the Einstein energy-momentum complex Θ_i^k [6], but this does not satisfy 4).

It seems that the only way out of this situation is to drop or to modify at least one of the four conditions. If we take the point of viewt hat 4) is a necessary condition the only possibility

(1) J. N. GOLDBERG: *Phys. Rev.*, **111**, 315 (1958).

(2) P. G. BERGMANN: *Phys. Rev.*, **112**, 287 (1958).

(3) C. MØLLER: *Ann. Phys.* **4**, 347 (1958).

(4) C. MØLLER: *Ann. Phys.* **12**, 118 (1961).

(5) M. MAGNUSSON: *Mat. Fys. Medd. Dan. Vid. Selskab.*, **32**, 6 (1960).

(6) A. EINSTEIN: *Berlin Ber.*, 778, (1915); *Ann. Physik*, **49**, 769 (1916); *Berlin Ber.*, **24**, 448 (1918).

left is to change condition 1) or the definition of P_i.

The change of 1) means introducing some new variables to describe the gravitational field, as Møller recently did (7).

The other possibility is studied here, where an approach to the definition of a new total energy-momentum vector is made.

As assumptions 1), 2), 4) are supposed to be valid, the Møller complex $T_i^{\ k}$ will be used.

We consider the case of a closed static system.

The new definition of the total energy and momentum of the system is

$$\mathscr{P}_i = \int_\sigma \frac{T_i^{\ k}}{\sqrt{-g}} \gamma_k \, d\sigma \,. \tag{2}$$

In (2) $g = \det\{g_{ik}\}$ and σ is a three-dimensional surface orthogonal to γ_k.

γ_k is the unit vector tangent to the lines x^4 in the frame of reference in which the static closed system is at rest.

It is easy to show that $\mathscr{P}_i$ is a four-vector under Lorentz transformations. In the rest frame the metric is of the form

$$ds^2 = a(r)(d\mathring{x}_1^2 + d\mathring{x}_2^2 + d\mathring{x}_3^2) - b(r)\, d\mathring{x}_4^2 , \qquad r^2 = \mathring{x}_1^2 + \mathring{x}_2^2 + \mathring{x}_3^2 , \tag{3}$$

where asimptotically rectilinear co-ordinates are used. In this frame we have

$$\mathring{\gamma}_k = 0,\, 0,\, 0,\, \sqrt{b}\,, \tag{4}$$

and as $d\sigma_0 = \sqrt{\gamma_0}\, d^3\mathring{x}$, $\gamma_0 = \det\{\mathring{g}_{\alpha\beta}\}$

$$\mathring{\mathscr{P}}_i = \int_{x_4=\text{cost}} \mathring{T}_i^{\ 4} \, d^3\mathring{x} \,, \tag{5}$$

which coincides with the old definition.

(7) C. Møller: to be published.

Using the transformation rules for $T_i^{\ k}$ and γ_k the result

$$\mathscr{P}_i = \frac{\partial \mathring{x}_c}{\partial x_i} \mathring{\mathscr{P}}_c \,. \tag{6}$$

is obtained.

Since $\mathring{\mathscr{P}}_i$ satisfies the requirements of being time-independent and of being unchanged under arbitrary transformations which leave the co-ordinate system asimptotically invariant, and since $\mathscr{P}_i$ is a linear combination of $\mathring{\mathscr{P}}_i$ with constant coefficients it follows that condition 3) is now completely satisfied. Also condition 4) is satisfied as both $T_4^{\ k}$ and γ_k are four-vectors under the transformation (1), and so $T_4^{\ k}\gamma_k$, which plays now the role of energy density, behaves like a scalar under the same group of transformations.

The total energy and momentum defined by (2) differs from the usual P_i since the integration must be performed always on the same surface and $T_i^{\ k}$ must be saturated always with the same vector normal to this surface.

It is therefore necessary to give a geometrical definition of σ and γ_k or to define σ and γ_k in one frame of reference, which amounts to the same thing.

In the case of a closed static system γ_k coincides with the four-velocity of matter inside the matter tube.

It can be remarked that our definition of energy density is similar to that given by C. Cattaneo (8).

We hope to extend definition (2) to the case of a more general system.

* * *

I wish to thank Prof. C. Cattaneo and Prof. C. Møller.

(8) C. Cattaneo: *Nuovo Cimento*, 13, 237 (1959).

4128