Laboratori Nazionali di Frascati

LNF-61/44 (1961)

G. Da Prato, G. Putzolu: RADIATIVE CORRECTIONS TO $\pi^- \to \pi^0 +$ $+ e^- + \bar{v}$ DECAY.

Estratto dal: Nuovo Cimento, 21, 541 (1961)

IL NUOVO CIMENTO VOL. XXI, N. 3 1° Agosto 1961

Radiative Corrections to $\pi^- \to \pi^0 + e^- + \bar{\nu}$ Decay.

G. DA PRATO and G. PUTZOLU

Laboratori Nazionali del C.N.E.N., Frascati - Roma

(ricevuto il 16 Giugno 1961)

Summary. — We have calculated the radiative corrections to the $\pi^- \to \pi^0 + e^- + \bar{\nu}$ decay.

1. – Introduction.

FEYNMAN and GELL-MANN [1] introduced the hypothesis of conserved current to explain the absence of renormalization effects in the V part of the β-decay. In their scheme the weak vector current is identified with the $(+)$ component of the isotopic spin current $J_k^{(+)}$. One of the suggested tests of the theory is an accurate measurement of the decay rate for the leptonic decay of the pion:

$$\pi^- \to \pi^0 + e^- + \bar{\nu} \,. \tag{1}$$

In fact, neglecting electromagnetic corrections, the corresponding matrix element is given by

$$ig\sqrt{2}(\bar{e}\gamma^k[1 + i\gamma^5]\nu)(\pi^0|J_k^{(+)}|\pi^-) \tag{2}$$

and we have a simple connection between the relevant matrix element of the vector current and the electromagnetic form factor of the pion F_π:

$$(\pi^0|J_k^{(+)}|\pi^-) = \frac{1}{\sqrt{2}}(\pi_k^0 + \pi_k^-)\, F_\pi(k^2)\,, \tag{3}$$

---

[1] R. P. FEYNMANN and M. GELL-MANN: *Phys. Rev.*, 109, 193 (1958).

where $k^2 = (\pi^- - \pi^0)^2$ is the momentum transfer to the lepton pair. In the actual process (1) this momentum transfer is very small, so that one can safely put $F_\pi = 1$. In this work we propose to evaluate the radiative corrections (to order e^2) to process (1). This would be important for a comparison of an accurate experimental result and the prediction of the Feynmann and Gell-Mann theory.

Since it is difficult to introduce the pion form factor in a gauge-invariant way for vertices with virtual pion lines, we will use a local Lagrangian and a Feynman cut-off in the calculation of radiative corrections. The results will not depend critically on this cut-off, since the divergence will be found to be only logarithmic.

2. – Formulation.

In the following we shall use the notations and the conventions of the textbook of BOGOLIUBOV and SHIRKOV (2). Let p_1, p_2, p_3, p_4 be the momenta and m_1, m_2, m_3, $m_4 = 0$ the masses of the π^-, e^-, π^0, $\bar{\nu}$. We put also:

$$\cosh\theta = \frac{(p_1 p_2)}{m_1 m_2}\,. \tag{4}$$

All the calculations will be performed in the center-of-mass system.

The Lagrangian responsible for the process is

$$\mathscr{L}_1 = \{\pi^- \partial_k \pi^0 - \pi^0 \partial_k \pi^-\}(\bar{e}\gamma^k[1 + i\gamma^5]\nu) + \text{h. c.}\,. \tag{5}$$

Following the principle of the minimal electromagnetic interaction, the Lagrangian that takes into account the electromagnetic interactions as well, is obtained from the complete Lagrangian without them

$$\mathscr{L}_0 = \mathscr{L}^{(e)}_{\text{free}} + \mathscr{L}^{(\nu)}_{\text{free}} + \mathscr{L}^{(\pi)}_{\text{free}} + \mathscr{L}^{(\gamma)}_{\text{free}} + \mathscr{L}_1\,, \tag{6}$$

by the substitution

$$\partial_k \varphi(x) \to \big(\partial_k - ieA_k(x)\big)\varphi(x)\,, \tag{7}$$

where $A(x)$ and $\varphi(x)$ are respectively the field operators of the photon and of the generic charged particle that appears in the process (1).

(2) N. N. BOGOLIUBOV and D. V. SHIRKOV: *Introduction to the Theory of Quantized Fields* (New York, London, 1959).

In this way we obtain for the complete interaction Lagrangian L:

$$\mathscr{L} = \mathscr{L}_1 + e:\bar{e}\hat{A}e: + ie:[\partial^k\pi^*\cdot\pi - \pi^*\partial^k\pi]:A_k - e^2:\pi^*\pi A_k A^k: + \\ + \left[g(\bar{e}\gamma^k[1+i\gamma^5]\nu)ieA_k\pi^*\pi^0 + \text{h. c.}\right]. \tag{8}$$

The last term is a new direct interaction between the five particles π^-, π^0, e^-, $\bar{\nu}$ and γ.

3. – Feynman diagrams.

Using the Lagrangian (8) we have seven diagrams corresponding to the process (1) up to the order g^2e^2 (Fig. 1).

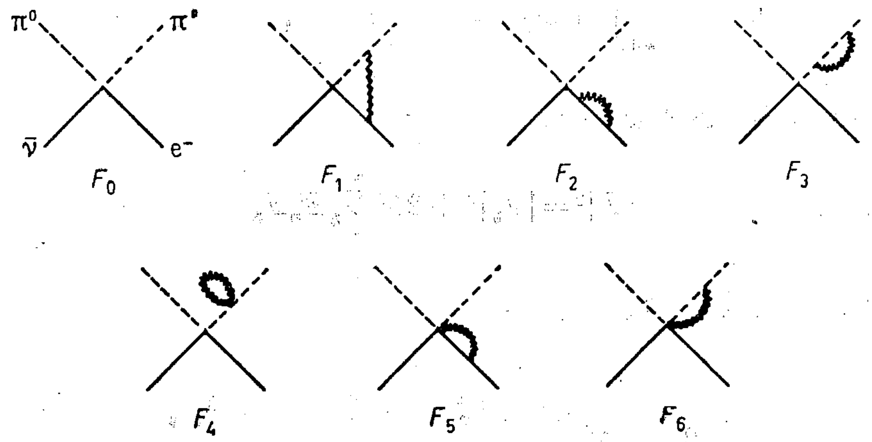

Fig. 1.

Besides them we have also to consider three diagrams (Fig. 2) relative to the same process with bremsstrahlung:

$$\pi^- \to \pi^0 + e^- + \bar{\nu} + \gamma. \tag{9}$$

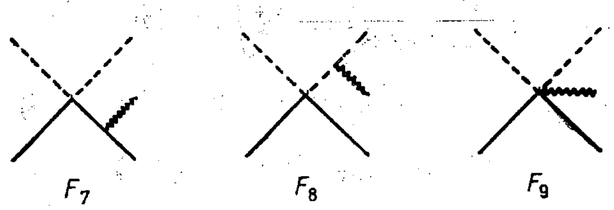

Fig. 2.

They give to the transition probability a contribution of the order g^2e^2, depending on the experimental situation (see Section 5).

4. – Transition probabilities (virtual photons).

We put

$$(e^-, \pi^0, \bar{\nu}\,|\,S-1\,|\,\pi^-) = \delta(p_1-p_2-p_3-p_4)F \tag{10}$$

and we have

$$F = \sum_{h=1}^{6} F_h\,, \tag{11}$$

where F_h's refer to the various diagrams of Fig. 1. Calling $\mathrm{d}P_1$ the corresponding differential transition probability, summed over the final spins, we have

$$\mathrm{d}P_1 = \frac{1}{2\pi}\sum_{\text{spin}} |F|^2\delta(p_1-p_2-p_3-p_4)\,\mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_2\,\mathrm{d}\boldsymbol{p}_3\,, \tag{12}$$

where to the order g^2e^2 we have

$$|F|^2 = |F_0|^2 + 2\mathscr{R}\sum_{h=1}^{6} F_0^* F_h\,. \tag{13}$$

4·1. – *Calculation of* F_0.

$$F_0 = \frac{ig}{2(2\pi)^2\sqrt{p_{10}p_{30}}}\,(\bar{v}_2^+[\hat{p}_1+\hat{p}_3][1+i\gamma^5]v_4^+)\,, \tag{14}$$

$$\sum_{\text{spin}} |F_0|^2 = \frac{g^2}{2(2\pi)^4 p_{10}p_{20}p_{30}p_{40}}\{8(p_1p_2)(p_1p_4) - 4m_2^2(p_1p_4) + [m_2^2 - 4m_1^2](p_2p_4)\}\,. \tag{15}$$

4·2. – *Calculation of* $(F_1+F_5+F_6)$.

$$\begin{aligned} F_1+F_5+F_6 = &-\frac{ge^2}{(2\pi)^6}\frac{1}{\sqrt{4p_{10}p_{30}}}\,\bar{v}_2^+\int \mathrm{d}k((2\hat{p}_1-\hat{k})(k^2-2(p_2k)) + \\ &+(4m_2-2\hat{p}_2+2\hat{k})(k^2-2(p_1k)) + (2\hat{p}_1-\hat{k})(m_2+\hat{p}_2-\hat{k})(\hat{p}_1+\hat{p}_3-\hat{k})]\cdot \\ &\cdot k^{-2}[k^2-2(p_1k)]^{-1}[k^2-2(p_2k)]^{-1}[1+i\gamma^5]v_4^+\,. \end{aligned} \tag{16}$$

This integral shows ultraviolet and infrared divergences; consequently it has been evaluated introducing a Feynman cut-off $\big(-(\lambda^2/(k^2-\lambda^2))\big)$ and a fictitious photon mass λ_m. The same will be made, where necessary, for the other diagrams.

By standard methods one obtains

$$(17)\qquad F_1+F_5+F_6=-\frac{ge^2}{(2\pi)^6}\frac{1}{\sqrt{4p_{10}p_{30}}}\bar{v}_2^+\{2I_1\hat{p}_1[\hat{p}_2+m_2][\hat{p}_1+\hat{p}_3]-$$
$$-\hat{I}_2[5m_2\hat{p}_1+m_2\hat{p}_3+\hat{p}_2\hat{p}_3+\hat{p}_2\hat{p}_1-2\hat{p}_1\hat{p}_2]-2[4(p_1p_2-m_2\hat{p}_2)]\hat{I}_2-$$
$$-2\hat{p}_1\hat{I}_2[(\hat{p}_1+\hat{p}_3+2m_2]+I_3[4\hat{p}_1+3m_2]-2\hat{I}_4\}(1+i\gamma^5)v_4^+,$$

where

$$(18)\qquad \begin{cases} I_1=\dfrac{i\pi^2}{2}j_1\,,\\ \hat{I}_2=\dfrac{i\pi^2}{2}\sum\limits_h g^{hh}\gamma^h j_2^h\,,\\ I_3=\dfrac{i\pi^2}{2}\sum\limits_h g^{hh}j_3^{hh}\,,\\ \hat{I}_4=\dfrac{i\pi^2}{2}\sum\limits_{hk} g^{hh}g^{kk}\gamma^h(\hat{p}_2-\hat{p}_1)\gamma^k j_3^{hk}\,. \end{cases}$$

j_1, j_2^h and j_3^{hk} have been defined and evaluated by BEHRENDS, FINKELSTEIN and SIRLIN (³) (formulae (7a) and following).

Substituting the expressions for the j's, we obtain

$$(19)\qquad F_1+F_5+F_6=-\frac{ge^2}{(2\pi)^6}\frac{1}{\sqrt{4p_{10}p_{30}}}\frac{i\pi^2}{2}\bar{v}_2^+(Am_2+2B\hat{p}_1)(1+i\gamma^5)v_4^+,$$

where

$$(20)\qquad \begin{cases} A=-4I_1(p_1p_2)-2I_{21}[m_1^2-(p_1p_2)]-\\ \qquad\qquad -2I_{22}[4(p_1p_2)-m_2^2-2m_1^2]+3I_3-2I_{42}\,,\\ B=4I_1(p_1p_2)-2I_{21}[3(p_1p_2)+m_1^2]-I_{22}[3m_2^2+4(p_1p_2)]+2I_3-I_{41}\,. \end{cases}$$

I_{21}, I_{22}, I_{41} and I_{42} are defined by the positions:

$$(21)\qquad \begin{cases} \hat{I}_2=I_{21}\hat{p}_1+I_{22}\hat{p}_2\,,\\ \hat{I}_4=I_{41}\hat{p}_1+I_{42}\hat{p}_2\,. \end{cases}$$

Finally the contribution of $(F_1+F_5+F_6)$ to the transition probability is

$$(22)\qquad \sum_{\text{spin}}2\mathscr{R}[(F_1+F_5+F_6)\cdot F_0^*]=-\frac{g^2}{4(2\pi)^4}\frac{\alpha}{\pi}[8B(p_1p_2)(p_1p_4)+$$
$$+2(A-B)m_2^2(p_1p_4)-(Am_2^2+4Bm_1^2)(p_2p_4)]\frac{1}{p_{10}p_{20}p_{30}p_{40}},$$

(³) R. H. BEHRENDS, R. J. FINKELSTEIN and A. SIRLIN: *Phys. Rev.*, **101**, 866 (1956).

and the fractional correction $\delta^{(1,5,6)}$ is

$$(23)\qquad \delta^{(1,5,6)} = \frac{\sum_{\text{spin}} 2\mathscr{R}[(F_1+F_5+F_6)F_0^*]}{\sum_{\text{spin}} |F_0|^2} =$$

$$= -\frac{1}{2}\frac{\alpha}{\pi}\frac{8B(p_1p_2)(p_1p_4)+2(A-B)m_2^2(p_1p_4)-(Am_2^2+4Bm_1^2)(p_2p_4)}{8(p_1p_2)(p_1p_4)-4m_2^2(p_1p_4)+(m_2^2-4m_1^2)(p_2p_4)}\,.$$

4·3. – *Calculation of F_2, F_3 and F_4.* – These diagrams describe self-energy effects; after mass and wave function renormalization the diagram F_4 does not give any contribution, while the diagrams F_2 and F_3 give the following fractional corrections to the transition probability:

$$(24)\qquad \left\{\begin{aligned} \delta^{(2)} &= -\frac{\alpha}{\pi}\left(\frac{1}{2}\ln\frac{\lambda}{m_2}+\ln\frac{\lambda_m}{m_2}+\frac{9}{8}\right),\\ \delta^{(3)} &= -\frac{\alpha}{\pi}\left(-\ln\frac{\lambda}{m_1}+\ln\frac{\lambda_m}{m_1}+1\right). \end{aligned}\right.$$

5. – Transition probabilities (real photons).

As is well known, when treating approximations to the order g^2e^2, one must consider the corrections due to real photons of energy inferior to a maximum value ε depending upon the experimental resolution, as well as the corrections due to virtual photons. In our case we have assumed $\varepsilon \ll m_2$. We put

$$(25)\qquad (e^-, \pi^0, \bar{\nu}, \gamma\,|(S-I)|\,\pi^-) = \delta(p_1-p_2-p_3-p_4-k)G\,,$$

where k is the momentum of the emitted photon, and $G = F_7+F_8+F_9$ (see Fig. 2).

Calling $\mathrm{d}P_2$ the corresponding differential transition probability, summed over the final spins and polarizations, and integrated over $\boldsymbol{k}$ with $|\boldsymbol{k}| \leqslant \varepsilon$, one obtains

$$(26)\qquad \mathrm{d}P_2 = \frac{1}{2\pi}\sum_{\text{spin}}\sum_{\text{pol}}\int_{|\boldsymbol{k}|\leqslant\varepsilon}\mathrm{d}\boldsymbol{k}\,|G|^2\,\delta(p_1-p_2-p_3-p_4-k)\,\mathrm{d}\boldsymbol{p}_2\,\mathrm{d}\boldsymbol{p}_3\,\mathrm{d}\boldsymbol{p}_4\,.$$

Because $\varepsilon \ll m_2$, we can neglect the diagram F_9, which gives corrections proportional to ε and ε^2; for the correction relative to (F_7+F_8) one obtains by standard methods

$$(27)\qquad \mathrm{d}P_2 = \delta^{(7,8)}\frac{1}{2\pi}\sum_{\text{spin}}|F_0|^2\,\delta(p_1-p_2-p_3-p_4)\,\mathrm{d}\boldsymbol{p}_2\,\mathrm{d}\boldsymbol{p}_3\,\mathrm{d}\boldsymbol{p}_4\,,$$

where $\delta^{(7,8)}$, that is consequently the fractional correction due to the bremsstrahlung, is given by

$$(28) \qquad \delta^{(7,8)} = -\frac{\alpha}{\pi}\left[2\ln\frac{2\varepsilon}{\lambda_m}(1-\theta\cdot\operatorname{ctgh}\theta) - \theta\operatorname{ctgh}\theta - 1 + \right.$$
$$\left. + (p_1 p_2)\int_{-1}^{1} dz \frac{1}{p_z^2}\frac{1}{2v_z}\ln\frac{1+v_z}{1-v_z}\right],$$

where

$$p_z = \tfrac{1}{2}[p_1(1+z) + p_2(1-z)] \qquad \text{and} \qquad v_z = \frac{|\boldsymbol{p}_z|}{p_{z_0}}.$$

6. – Total correction and approximations.

The total percentual correction is given by

$$(29) \qquad \delta = \delta^{(1,5,6)} + \delta^{(2)} + \delta^{(3)} + \delta^{(7,8)},$$

where the $\delta^{(\,)}$'s are given from (23), (24) and (28). As it must be, δ does not contain λ_m.

In the very good approximation $m_2 \ll m_1$, $\delta^{(1,5,6)}$ and $\delta^{(7,8)}$ may be written as follows:

$$(30) \qquad \begin{cases} \delta^{(4,5,6)} = -\dfrac{\alpha}{\pi}\left[\theta\operatorname{ctgh}\theta\left(\theta - 2\ln\dfrac{\lambda_m}{m_2} - 5\right) - \ln\dfrac{m_1}{m_2} + \dfrac{5}{2}\ln\dfrac{m_1}{\lambda} - \dfrac{13}{8}\right], \\ \delta^{(7,8)} = -\dfrac{\alpha}{\pi}\left[2\ln\dfrac{2\varepsilon}{\lambda_m}(1-\theta\operatorname{ctgh}\theta) + \theta\operatorname{ctgh}\theta - 1\right]. \end{cases}$$

Then

$$(31) \qquad \delta = -\frac{\alpha}{\pi}\left[\theta^2\operatorname{ctgh}\theta - 4\theta\operatorname{ctgh}\theta - \frac{1}{2}\ln\frac{m_1}{m_2} - 3\ln\frac{\lambda}{m_1} - \frac{1}{2} + \right.$$
$$\left. + \ln\frac{2\varepsilon}{m_1} + \ln\frac{2\varepsilon}{m_2}(1 - 2\theta\operatorname{ctgh}\theta)\right].$$

7. – Integral transition probability.

We call P_0 and P the integral transition probabilities to the order g^2 and g^2e^2, and $\delta p = (P - P_0)/P_0$ the corresponding fractional correction.

3115

By definition:

$$(32)\quad \begin{cases} P_0 = \dfrac{1}{2\pi}\displaystyle\int \sum_{\text{spin}} |F_0|^2\,\delta(p_1 - p_2 - p_3 - p_4)\,\mathrm{d}\boldsymbol{p}_2\,\mathrm{d}\boldsymbol{p}_3\,\mathrm{d}\boldsymbol{p}_4\,, \\ P \;= \dfrac{1}{2\pi}\displaystyle\int \sum_{\text{spin}} |F_0|^2(1+\delta)\,\delta(p_1 - p_2 - p_3 - p_4)\,\mathrm{d}\boldsymbol{p}_2\,\mathrm{d}\boldsymbol{p}_3\,\mathrm{d}\boldsymbol{p}_4\,. \end{cases}$$

Performing the integrations considering also the pion's recoil, and neglecting only terms like m_2^2 in comparison with m_1^2, one obtains

$$(33)\quad P_0 = \frac{g^2}{\pi^3 m_1}\left[\frac{1}{5}(\Delta^2 - m_2^2)^{\frac{5}{2}}(m_1 - 2\Delta) + \frac{1}{3}(\Delta^2 - m_2^2)^{\frac{3}{2}}\left(\frac{5}{4}\Delta^3 - \frac{1}{2}m_1\Delta^2 - \right.\right.$$
$$\left.\left. - \frac{13}{8}m_2^2\Delta + m_1 m_2^2\right) + \frac{1}{8}m_2^2\Delta^2(\Delta - 2m_1)(\Delta^2 - m_2^2)^{\frac{1}{2}}\right],$$

$$(34)\quad \delta_p = -\frac{\alpha}{\pi}\left[\left(\ln\frac{2\Delta}{m_2}\right)^2 - \frac{167}{30}\ln\frac{2\Delta}{m_2} + \frac{6229}{1800} + \frac{3}{2}\ln\frac{m_2}{m_1} - \right.$$
$$\left. - 3\ln\frac{\lambda}{m_1} + 2\ln\frac{2\varepsilon}{m_2}\left(\frac{107}{60} - \ln\frac{2\Delta}{m_2}\right)\right],$$

where

$$\Delta = \frac{m_1^2 - m_3^2}{2m_1}.$$

Finally one obtains numerically

$$(35)\quad P_0 = 0.552\ \text{s}^{-1}\,,$$

$$(36)\quad \delta_p = 0.027 + 0.005\ln\frac{2\varepsilon}{m_2} + 0.007\ln\frac{\lambda}{m_1}\,.$$

8. Discussion.

The value (35) has been obtained using the following numerical values:

$$(37)\quad \begin{cases} m_1 = (139.59 \pm 0.05)\ \text{MeV}\ (^4) \\ m_3 = (135.00 \pm 0.05)\ \text{MeV}\ (^4) \\ g m^2_{\text{proton}} = (1.204 \pm 0.001)\cdot 10^{-5}\ (^5)\,. \end{cases}$$

---

(4) W. H. BARKAS and A. H. ROSELFELD: UCRL 8030.
(5) M. GELL-MANN and M. LÉVY: *Nuovo Cimento*, 16, 705 (1960).

From the errors on m_1, m_3 and g follows an error of 1.6 % on P_0, essentially determinated by the incertitude on the masses; hence this error is about one half of the correction (36).

In (36) the cut-off for short wavelengths is not cancelled; however, the result depends only logarithmically on this cut-off.

Assuming for example $\varepsilon = (1/20)m_2$ and $\lambda = 10m_1$ we find

$$\delta p \simeq 0.032 . \tag{38}$$

Finally for the lifetime τ we get

$$\tau = 1.808(1 + 0.032)(1 \pm 0.016)\ \text{s} = (1.866 \pm 0.030)\ \text{s} . \tag{39}$$

* * *

We thank Professor R. GATTO and Doctor N. CABIBBO for their continuous assistance and advice. We are also grateful to Dr. S. BERMAN for some useful remarks.

RIASSUNTO

Abbiamo calcolato le correzioni relative al decadimento $\pi^- \to \pi^0 + e^- + \bar{\nu}$.