LABORATORI NAZIONALI DI FRASCATI
SIS-Pubblicazioni

LNF-97/028 (P)
21 Luglio 1997

JINR E2-97-141
hep-th/9707240

NONLINEAR REALIZATIONS OF SUPERCONFORMAL AND W ALGEBRAS AS EMBEDDINGS OF STRINGS

S. Bellucci[1], V. Gribanov[2], E. Ivanov[2], S. Krivonos[2], A. Pashnev[2]
[1]) *INFN - Laboratori Nazionali di Frascati, P.O.Box 13, I-00044 Frascati, Italy*
[2]) *JINR–Bogoliubov Laboratory of Theoretical Physics,*
141980 Dubna, Moscow Region, Russia

Abstract

We propose a simple method for constructing representations of (super)conformal and nonlinear W-type algebras in terms of their subalgebras and corresponding Nambu-Goldstone fields. We apply it to $N = 2$ and $N = 1$ superconformal algebras and describe in this way various embeddings of strings and superstrings for which these algebras and their subalgebras define world-sheet symmetries. Besides reproducing the known examples, we present some new ones, in particular an embedding of the bosonic string with additional $U(1)$ affine symmetry into $N = 2$ superstring. We also apply our method to the nonlinear $W_3^{(2)}$ algebra and demonstrate that the linearization procedure worked out for it some time ago gets a natural interpretation as a kind of string embedding. All these embeddings include the critical ones as particular cases.

Submitted to Nucl. Phys. B

1 Introduction

For the last years, the study of various embeddings of strings and superstrings received much attention [1]-[7]. This activity was initiated by the paper of Berkovits and Vafa [1] who showed that ordinary bosonic strings can be regarded as a special class of vacua of $N = 1$ superstrings. Later it was found that this is a general phenomenon: a (super)string with N extended world-sheet supersymmetry can be embedded into a superstring with $N + 1$ extended supersymmetry as a particular vacuum state of the latter. Analogous embeddings were constructed for strings associated with nonlinear W type algebras and their linearizing algebras [8]-[11]. It was suggested that all the known strings and superstrings present different vacua of some hypothetical universal string theory.

An essential step towards clarifying the group-theoretical grounds of the embedding procedure was made in refs. [12,13]. On the example of the bosonic string embedded into the $N = 1$ superstring Kunitomo [12] showed that the larger symmetry ($N = 1$ superconformal) is realized with the help of Nambu-Goldstone fields, typical for the spontaneously broken supersymmetry. The vacuum stability subgroup is the Virasoro one, just the world-sheet symmetry of the bosonic string. The same results have been obtained by McArthur [13] in the framework of the standard theory of nonlinear realizations [14,15] applied to the $N = 1$ superconformal algebra (SCA). These observations support a nice interpretation of the string embeddings as one more manifestation of the universal phenomenon of spontaneous symmetry breakdown, this time of the infinite-dimensional world-sheet (super)symmetry of strings [1]. From this point of view, any embedding of the lower-symmetry string into the larger-symmetry one amounts to the choice of special background (and hence the vacuum) for the latter, such that it is given by the Nambu-Goldstone fields realizing the spontaneous breaking of the larger symmetry down to the lower symmetry. The currents generating the larger symmetry are expressed in terms of those of the lower symmetry and the Nambu-Goldstone fields. A power of the nonlinear realizations method manifests itself in that these expressions can be obtained in an algorithmic way, knowing only the structure relations of the given superconformal algebra. In accord with the general concepts of this method, all the basic characteristics of such specific representation of the larger symmetry (central charges, etc.) should be fully determined by the structure of representations of the vacuum stability symmetry. In this way, the string theory associated with the lower symmetry comes out as a spontaneously broken phase of the higher-symmetry string theory.

In this paper we present simple and universal techniques of calculations for nonlinear realizations of infinite dimensional algebras written in terms of OPEs (SOPEs) for one dimensional currents (supercurrents). The application of this techniques to $N = 1,\ 2$

superconformal algebras after taking into account quantum corrections leads to the corresponding formulas for the string embeddings.

Besides reproducing the known $N = 1 \to N = 2$ embedding in terms of $N = 1$ superfields [7], we get new self-consistent embeddings by choosing different subalgebras of $N = 2$ SCA as the vacuum stability symmetries. In particular, we describe embedding of the string associated with the product of Virasoro and $U(1)$ Kac-Moody algebras into $N = 2$ superstring. This extension of the bosonic string was recently discussed [16,17] in connection with F-theory [18]. As a by-product, we also reproduce the $N = 0 \to N = 1$ embeddings within our techniques, discuss how they are related to the $N = 2$ embeddings constructed and make comparison with the results of refs. [12,13].

We argue that the linearization procedure for some W algebras worked out some time ago [20] can also be interpreted in the embedding language, namely as an embedding of the string associated with some linear subalgebra of the given nonlinear W (super)algebra into the string associated with this W (super)algebra itself. The appropriately modified nonlinear realizations techniques prove to work in this case too. We consider the simple example of the quasi-conformal algebra $W_3^{(2)}$, but the same is apparently true for a wider class of W algebras.

In Sect. 2 we give a general characterization of our method. In Sect. 3 we apply it to $N = 2$ SCA and describe embeddings of strings associated with various subalgebras of $N = 2$ SCA into the $N = 2$ superstring. Sect. 4 is devoted to applications of our method to nonlinear algebras on the example of the $W_3^{(2)}$ algebra.

2 Method of nonlinear realizations

As a starting point, let us briefly describe the theory of nonlinear realizations [14,15] with some refinement related to the case of infinite-dimensional symmetries. This theory is a set of recipes of how to realize the given group G on the parameters of its coset G/H, H being some subgroup of G.

After choosing the subgroup H, one represents an arbitrary group element in the exponential parametrization in the following form

$$G = KH = e^k e^h \ . \tag{2.1}$$

Here $h = \xi^a h_a$ and $k = \phi^i k_i$ belong, respectively, to the algebra of H and to its complement to the full algebra of G, h_a and k_i being the appropriate generators. An infinitesimal group element $g = 1 + \epsilon$, with $\epsilon = \epsilon^i k_i + \epsilon^a h_a$, has the following action on K

$$(1 + \epsilon)e^k = e^{k+\delta k}(1 + \delta h) \ , \tag{2.2}$$

whence

$$e^{-k}\epsilon e^{k} = e^{-k}\delta e^{k} + \delta h \ . \tag{2.3}$$

Both the left- and right-hand sides of this equation can be written in terms of multiple commutators

$$e^{-k} \wedge \epsilon = \frac{1-e^{-k}}{k} \wedge \delta k + \delta h \ . \tag{2.4}$$

This is the basic equation for determining δk and δh. The definition of the symbol $\wedge$ is as follows. For the given function $f(k) = f_0 + f_1 k + f_2 k^2 + ...$ it reads:

$$f(k) \wedge \epsilon = \epsilon + f_1[k,\epsilon] + f_2[k[k,\epsilon]] + \dots \ . \tag{2.5}$$

For the coefficients of generators in $\delta k = \delta\phi^i k_i, \ \delta h = \delta\xi^a h_a$ eq. (2.4) implies the following general expressions

$$\begin{aligned}
\delta\phi^l &= \epsilon^i F_i^l + \epsilon^a F_a^l \ , \\
\delta\xi^b &= \epsilon^i F_i^b + \epsilon^a F_a^b \ .
\end{aligned} \tag{2.6}$$

Here all F's are some functions of ϕ^k. They are uniquely specified by the structure relations of the G algebra. Further, one considers a space of functions $\Phi(\phi^i)$ on which some representation of the subalgebra h is realized (it is reducible in general). In what follows we will call it the 'matter' representation and denote its generators by $h_a^{(m)}$. Then the generators of the whole algebra can be realized on this space as [13]:

$$\begin{aligned}
H_a &= -F_a^l \frac{\partial}{\partial\phi^l} + F_a^b h_b^{(m)} \ , \\
K_i &= -F_i^l \frac{\partial}{\partial\phi^l} + F_i^b h_b^{(m)} \ .
\end{aligned} \tag{2.7}$$

To prevent a possible confusion, we point out that the original generators (those appearing in eqs. (2.1) - (2.6)) are always assumed to be abstract and commuting with all coset parameters, it is the algebra of their commutators which really enters the game. On the contrary, the generators (2.7) give a particular realization of the group G on the space of these parameters and functions of them.

In the case of infinite-dimensional algebras, such as the superconformal ones, it is more convenient to deal with currents or supercurrents like $T(Z) = T(z,\theta^a)$, and with OPEs or SOPEs instead of the commutation relations. A finite number of such (super)currents collects all the infinite set of generators of the given (super)algebra. These generators appear as coefficients in the θ and z expansions of supercurrents (as usual, Laurent series is assumed for the z expansion). An element of the algebra associated with the given current is expressed as an integral over the superspace with some parameter depending on z,θ^a:

$$\mathcal{T} = \frac{1}{2\pi i}\oint dZ \phi(Z) T(Z) \ . \tag{2.8}$$

The coset parameters ϕ^m come out now as the coefficients in the expansion of $\phi(Z)$ over Z. As before, it is assumed that the original abstract currents (supercurrents) have vanishing OPEs (SOPEs) with all $\phi(z)$.

After computing group variations of these parameters-functions, the representations similar to (2.7) arise. An essentially new point compared to the customary nonlinear realizations formalism is the necessity to introduce new (super)currents $p(Z)$ which substitute the derivatives $\frac{\partial}{\partial\phi^l}$. These new currents are canonically conjugated to $\phi(Z)$. This means that the following OPE

$$p_l(z)\phi^n(w) \sim \delta_l^n \frac{1}{z-w} \; , \tag{2.9}$$

or its obvious supersymmetric counterpart are valid. The resulting specific representation for the currents of the algebra reads

$$\begin{aligned} H_a(Z) &= -F_a^m p_l(Z) + F_a^b h_b^{(m)}(Z) \; , \\ K_i(Z) &= -F_i^m p_m(Z) + F_i^b h_b^{(m)}(Z) \; . \end{aligned} \tag{2.10}$$

All F's are now functions of $\phi^l(Z)$ and their derivatives. The indices a, b label the currents generating some infinite- dimensional subalgebra of the given algebra (the stability subalgebra) while the indices i, k, l refer to the remainder of currents (the coset ones). The 'matter' currents $h_b^{(m)}$ still have vanishing OPEs with the coset parameters ϕ_k and conjugated momenta p_l. At the same time, OPEs between $h_b^{(m)}$ form a representation of the stability subalgebra. The action of (2.10) on the relevant functionals of $\phi(Z)$ (analogs of $\Phi(\phi^k)$) is defined through setting OPEs between these objects. The transformation properties of these functionals with respect to the whole (super)algebra are fully determined by their transformation properties with respect to the stability subalgebra, i.e. by fixing their OPEs with the 'matter' (super)currents $h_b^{(m)}$ (these functionals are a sort of primary (super)fileds of the stability subalgebra).

We emphasize that all the infinite-dimensional nature of the algebra and the stability subalgebra is hidden in the integrals like (2.8), while the explicitly appearing indices run over finite ranges like in the case of finite-dimensional symmetries.

One essential remark at this stage is of need. The currents (2.10) realize OPEs of the initial algebra only if we consider them as classical ones, i.e. when we keep in OPEs only single contraction. This is equivalent to using the Poisson brackets and implies that the subalgebras generated by $h_a^{(m)}(Z)$ and $H_a(Z)$ are identical. In particular, they have the same central charges. Transition to the exact quantum OPEs by keeping all contractions radically changes the situation. OPEs will not close unless some terms (quantum corrections) are added to the classical expressions for the currents. It can be shown that these addings do not include p_i's. In this case central charges in the subalgebras generated by $h_a^{(m)}$ and H_a become different and the above formalism gets suitable for description of embeddings of (super)strings along the lines of refs. [1]-[11].

This method, as it was described above, suits very well the case of linear algebras. Some additional problems arise when the W type nonlinear algebras are regarded. Nevertheless, it seems to be still applicable at least when some *linear* subalgebra of given W algebra is chosen as the vacuum stability subalgebra. In Sec. 4 we demonstrate this on the example of $W_3^{(2)}$ algebra.

3 Nonlinear realizations of $N = 2$ SCA

3.1 Structure relations

We start with some definitions. OPEs for $N = 2$ SCA in terms of real currents are given by

$$\begin{aligned}
T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w}\,,\ T(z)J(w) \sim \frac{J(w)}{(z-w)^2} + \frac{J'(w)}{z-w}\,,\\
T(z)G_{1,2}(w) &\sim \frac{3/2G_{1,2}(w)}{(z-w)^2} + \frac{G'_{1,2}(w)}{z-w}\,,\ G_{1,2}(z)G_{1,2}(w) \sim \frac{c/3}{(z-w)^3} + \frac{T(w)}{z-w}\,,\\
G_1(z)G_2(w) &\sim \frac{J(w)}{(z-w)^2} + \frac{1/2J'(w)}{z-w}\,,\ J(z)J(w) \sim -\frac{c/3}{(z-w)^2}\,,\\
J(z)G_1(w) &\sim -\frac{G_2(w)}{(z-w)}\,,\ J(z)G_2(w) \sim \frac{G_1(w)}{(z-w)}\,.
\end{aligned} \tag{3.1}$$

Note that the supercurrents $\{T, G_1\}$ and $\{T, G_2\}$ form two different $N = 1$ SCAs embedded as subalgebras into the $N = 2$ SCA.

In terms of $N = 1$ supercurrents $T(Z) = \frac{1}{\sqrt{2}}G_1(z) + \theta T(z)$ and $G(Z) = -i(J(z) + \theta\sqrt{2}G_2(z))$ these OPEs are concisely presented by the following SOPEs

$$\begin{aligned}
T(Z_1)T(Z_2) &\sim \frac{c/6}{Z_{12}^3} + \frac{3/2\theta_{12}T(Z_2)}{Z_{12}^2} + \frac{1/2\mathcal{D}T(Z_2) + \theta_{12}T'(Z_2)}{Z_{12}}\,,\\
T(Z_1)G(Z_2) &\sim \frac{\theta_{12}G(Z_2)}{Z_{12}^2} + \frac{1/2\mathcal{D}G(Z_2) + \theta_{12}G'(Z_2)}{Z_{12}}\,,\\
G(Z_1)G(Z_2) &\sim \frac{c/3}{Z_{12}^2} + \frac{2\theta_{12}T(Z_2)}{Z_{12}}\,,
\end{aligned} \tag{3.2}$$

where

$$\theta_{12} = \theta_1 - \theta_2,\ Z_{12} = z_1 - z_2 - \theta_1\theta_2,\ \mathcal{D} = \frac{\partial}{\partial\theta} + \theta\partial,\ T' \equiv \frac{\partial T}{\partial z}. \tag{3.3}$$

In the rest of this Section we apply the general method described in Sect. 2 in order to construct nonlinear realizations of $N = 2$ SCA with its different subalgebras as the vacuum stability subalgebra H. For different choices of H the coset is parametrized by different sets of fields, which gives rise to non-equivalent realizations of $N = 2$ SCA. The quantum versions of the latter are easy to construct, and they give the formulas for the corresponding embeddings. We also show that the nonlinear realizations of $N = 1$ SCA and the associated $N = 0 \to N = 1$ embeddings follow from the $N = 2$ ones upon appropriate reductions.

3.2 Stability subalgebra $H = \{T(Z)\} = \{T(z), G_1(z)\}$

We first choose H to be generated by currents $T(z), G_1(z)$, or, in the $N = 1$ superfield notation, by the spin $3/2$ fermionic supercurrent $T(Z)$. This is just $N = 1$ SCA.

The general element of the coset is parametrized as

$$K = e^{\frac{1}{2\pi i}\oint dZ \phi(Z) G(Z)} \,, \tag{3.4}$$

where we have introduced a Nambu-Goldstone fermionic $N = 1$ superfield $\phi(Z)$ with the "spin" $-1/2$. An infinitesimal element of the $N = 2$ superconformal group

$$g = 1 + \frac{1}{2\pi i}\oint dZ \{\alpha(Z)G(Z) + a(Z)T(Z)\} \,, \tag{3.5}$$

acts on K according to the following relation (cf. (2.2))

$$g e^{\frac{1}{2\pi i}\oint dZ \phi(Z) G(Z)} = e^{\frac{1}{2\pi i}\oint dZ (\phi(Z)+\delta\phi(Z)) G(Z)} \tilde{H}. \tag{3.6}$$

Under this left shift the parameter of the coset space $\phi(Z)$ changes to $\phi(Z) + \delta\phi(Z)$. The induced subgroup element $\tilde{H}$ is represented by

$$\tilde{H} = 1 + \frac{1}{2\pi i}\oint dZ \{b(Z)T(Z) + c(Z))\}. \tag{3.7}$$

The quantities $\delta\phi(Z)$, $b(Z)$ and $c(Z)$ are to be expressed in terms of the Nambu-Goldstone superfield $\phi(Z)$ and the group parameters $a(Z)$ and $\alpha(Z)$.

While applying the techniques described in the previous Section, one frequently needs to compute (anti)commutators of the operators like $\frac{1}{2\pi i}\oint dZ \phi(Z) G(Z)$. This can be done with the help of the following formula:

$$\begin{aligned} \left[\frac{1}{2\pi i}\oint dZ_1 a(Z_1)A(Z_1), \frac{1}{2\pi i}\oint dZ_2 b(Z_2)B(Z_2)\right] &= \\ \sigma\left(\frac{1}{2\pi i}\right)^2 \oint dZ_2 \oint_C dZ_1 a(Z_1)b(Z_2)A(Z_1)B(Z_2) \,. \end{aligned} \tag{3.8}$$

The contour of integration C surrounds z_2 and $A(Z_1)B(Z_2)$ in the right-hand side of (3.8) stands for SOPE of the supercurrents $A(Z_1)$ and $B(Z_2)$. Additional multiplier σ is the sign which depends on the Grassmann parity of $A(Z)$ and $B(Z)$: $\sigma = (-1)^{g(A)(g(B)+1)}$.

Taking this into account and doing the computations along the line of Sec. 2, we find the following expressions for $\delta\phi(Z)$, $b(Z)$ and $c(Z)$

$$\begin{aligned}
\delta_a\phi &= \frac{1}{2}a'\phi - a\phi' - \frac{1}{2}\mathcal{D}a\mathcal{D}\phi \ , \\
\delta_\alpha\phi &= \alpha\left\{\mathcal{D}\phi\coth\mathcal{D}\phi + \frac{\phi\phi'}{(\mathcal{D}\phi)^2}\left(1 - 2\mathcal{D}\phi\coth\mathcal{D}\phi + \frac{(\mathcal{D}\phi)^2}{\sinh^2\mathcal{D}\phi}\right)\right\} \\
&\quad + \frac{\mathcal{D}\alpha\phi}{\mathcal{D}\phi}\left(1 - \mathcal{D}\phi\coth\mathcal{D}\phi\right) \ , \\
b(Z) &= a + \alpha\frac{2\phi}{\mathcal{D}\phi}\tanh\left(\frac{\mathcal{D}\phi}{2}\right) \ , \\
c(Z) &= -\alpha\frac{c_m}{6}\left\{\mathcal{D}\phi' + \left(\frac{2\phi\phi'\mathcal{D}\phi'}{(\mathcal{D}\phi)^3} - \frac{\phi\phi''}{(\mathcal{D}\phi)^2}\right)\left(\mathcal{D}\phi - 2\tanh\left(\frac{\mathcal{D}\phi}{2}\right)\right)\right\} \ . \qquad (3.9)
\end{aligned}$$

These expressions lead to the following form of the generators:

$$\begin{aligned}
T &= -\phi'\eta - \frac{1}{2}\phi\eta' + \frac{1}{2}\mathcal{D}\phi\mathcal{D}\eta + T_m \ , \\
G &= \eta + \frac{2\phi\phi'}{(\mathcal{D}\phi)^2}\left(1 - \mathcal{D}\phi\coth\mathcal{D}\phi\right)\eta - \frac{\phi}{\mathcal{D}\phi}\left(1 - \mathcal{D}\phi\coth\mathcal{D}\phi\right)\mathcal{D}\eta \\
&\quad - \frac{c_m}{6}\left\{\mathcal{D}\phi' + \left(\frac{2\phi\phi'\mathcal{D}\phi'}{(\mathcal{D}\phi)^3} - \frac{\phi\phi''}{(\mathcal{D}\phi)^2}\right)\left(\mathcal{D}\phi - 2\tanh\left(\frac{\mathcal{D}\phi}{2}\right)\right)\right\} \\
&\quad + \frac{2\phi}{\mathcal{D}\phi}\tanh\left(\frac{\mathcal{D}\phi}{2}\right)T_m \ , \qquad (3.10)
\end{aligned}$$

where the newly introduced spin 1 bosonic superfield η is canonically conjugated to ϕ

$$\eta(Z_1)\phi(Z_2) \sim \frac{\theta_{12}}{z_{12}} \ . \qquad (3.11)$$

Thus we have obtained the realization of the $N = 2$ SCA in terms of the conjugated pair of $N = 1$ Nambu-Goldstone superfields ϕ, η and $N = 1$ SCA supercurrent $T_m(Z)$

$$\begin{aligned}
T_m(Z_1)T_m(Z_2) &\sim \frac{c_m/6}{Z_{12}^3} + \frac{3/2\theta_{12}T_m(Z_2)}{Z_{12}^2} + \frac{1/2\mathcal{D}T_m(Z_2) + \theta_{12}T_m'(Z_2)}{Z_{12}} \ , \\
T_m(Z_1)\phi(Z_2) &\sim 0 \ , \ T_m(Z_1)\eta(Z_2) \sim 0 \ .
\end{aligned}$$

At the considered classical level the central charge of $N = 2$ SCA in such a realization can be checked to coincide with the central charge of $N = 1$ SCA

$$c_{N=2} = c_m \ .$$

As was already mentioned in Introduction, the main motive for working out this approach to nonlinear realizations of (super)conformal (and W) algebras was the desire to gain a systematic method for deriving the relations which describe different embeddings of strings. According to the reasoning of refs. [12,13], the linearly realized subgroup always corresponds to the embedded string and defines the world-sheet symmetry of the latter. Thus in the present case we should get an embedding of $N = 1$ superstring into $N = 2$ superstring. This embedding in terms of $N = 1$ superfields was firstly given by Berkovits and Ohta [7] by means of guesswork. Now we show that their formulas naturally follow from the above ones obtained within a systematic procedure, with taking account of quantum corrections.

To make a comparison with the Berkovits-Ohta paper [7], we perform the canonical transformation from the superfields $\phi,\ \eta$ to the superfields $C,\ B$

$$\begin{aligned} C &= \frac{2\phi}{\mathcal{D}\phi}\tanh(\frac{\mathcal{D}\phi}{2})\ , \\ B &= \frac{1}{2}(1+\cosh\mathcal{D}\phi)\eta + \frac{\cosh^2(\frac{\mathcal{D}\phi}{2})}{(\mathcal{D}\phi)^2}\left(2-\mathcal{D}\phi\coth\left(\frac{\mathcal{D}\phi}{2}\right)\right)\phi\phi'\eta \\ &\quad + \frac{\phi}{2\mathcal{D}\phi}\left(\mathcal{D}\phi\coth\left(\frac{\mathcal{D}\phi}{2}\right) - \cosh\mathcal{D}\phi - 1\right)\mathcal{D}\eta\ . \end{aligned} \tag{3.12}$$

The generators of $N = 2$ SCA in terms of these new superfields take the form:

$$\begin{aligned} T &= -C'B - \frac{1}{2}CB' + \frac{1}{2}\mathcal{D}C\mathcal{D}B + T_m\ , \\ G &= B - \frac{1}{4}(\mathcal{D}C)^2B + \frac{1}{2}BC'C + \frac{1}{2}\mathcal{D}CC\mathcal{D}B + CT_m \\ &\quad - \frac{c_m}{6}\left(\frac{4\mathcal{D}C'}{4-(\mathcal{D}C)^2} - \frac{2CC'\mathcal{D}C'(\mathcal{D}C)^2}{(4-(\mathcal{D}C)^2)^2} - \frac{CC''\mathcal{D}C}{4-(\mathcal{D}C)^2}\right), \end{aligned} \tag{3.13}$$

and almost coincide with those of [7]. The only difference is the presence of some additional B-independent terms in the realization of ref. [7]. The origin of this difference lies in the following. SOPEs for the supercurrents (3.13) are closed on the classical level, when only one contraction is taken into account. Besides, the values of the central charge for T_m and T on the classical level are the same. The extra terms found in ref. [7] can be easily restored in our formulation by demanding the closure of the *quantum* SOPEs, when all contractions are taken into account. The final expressions for the quantum supercurrents are [7]:

$$\begin{aligned} T^q &= T + \left(\frac{C\mathcal{D}C'}{4-(\mathcal{D}C)^2}\right)', \\ G^q &= G + \frac{CC'\mathcal{D}C'}{4-(\mathcal{D}C)^2}\ . \end{aligned} \tag{3.14}$$

As was stated in [7], these expressions describe both the cases of critical and non-critical embeddings. It is a matter of direct computation to see that the central charge $c_{N=2}$ in this quantum realization is related to the central charge c_m inherent to T_m as

$$c_{N=2} = c_m - 9 \ . \tag{3.15}$$

The critical value of $c_m = 15$ yields just the critical value $c_{N=2} = 6$ for the central charge of $N = 2$ SCA.

3.3 Stability subalgebra $H = \{T(z)\}$

In our second example we take as the linearly realized subalgebra the Virasoro subalgebra. So it should give, after passing to the quantum case, the description of embedding of bosonic string into the $N = 2$ superstring. As distinct from the previous example, all supersymmetries are nonlinearly realized in the case under consideration. As a result, no superfield formalism exists, and one should deal with the component currents rather than supercurrents.

We are led to introduce three Nambu-Goldstone fields associated with the currents

$$\overline{G}(z) = \frac{1}{\sqrt{2}}(G_1(z) - iG_2(z)) \ , \ G(z) = -\frac{1}{\sqrt{2}}(G_1(z) + iG_2(z)) \ \text{ and } \ \widetilde{J}\,(z) = -iJ(z) \ .$$

We can parametrize the coset in two different ways, each leading to different representations of the algebra in terms of $T_m(z)$, Nambu-Goldstone fields $\xi(z), \overline{\xi}(z), \phi(z)$ and their conjugated momenta $\overline{\eta}(z), \eta(z), \mu(z)$ with the following OPEs:

$$\eta(z)\overline{\xi}(w) \sim \frac{1}{z-w} \ , \ \overline{\eta}(z)\xi(w) \sim \frac{1}{z-w} \ , \ \mu(z)\phi(w) \sim \frac{1}{z-w} \ . \tag{3.16}$$

So far as the classical case is concerned, these different parametrizations are obviously related by an equivalence transformation (it can still be rather complicated), but in the quantum case they can yield non-equivalent realizations. This is the reason why we quote the latter for both parametrizations.

Let us first consider the 'symmetric' parametrization

$$K = e^{\frac{1}{2\pi i}\oint dz \{\overline{\xi}(z)G(z) + \xi(z)\overline{G}(z)\}} e^{\frac{1}{2\pi i}\oint dz \phi(z)\widetilde{J}(z)} \ . \tag{3.17}$$

A straightforward computation leads to the following expressions for the supercurrents:

$$\begin{aligned} T &= T_m + \frac{3}{2}\xi'\overline{\eta} + \frac{3}{2}\overline{\xi}'\eta + \frac{1}{2}\xi\overline{\eta}' + \frac{1}{2}\overline{\xi}\eta' + \phi'\mu \ , \\ \widetilde{J} &= -\mu - \overline{\xi}\eta + \xi\overline{\eta} - \frac{c_m}{6}\phi' \ , \end{aligned}$$

$$
\begin{aligned}
G &= -\eta+\frac{2}{3}\xi\overline{\xi}'\eta+\frac{1}{3}\xi'\overline{\xi}\eta+\frac{1}{3}\xi\overline{\xi}\eta'-\frac{1}{9}\xi\xi'\overline{\xi}'\overline{\xi}\eta+\frac{2}{3}\xi\xi'\overline{\eta} \\
&+\frac{1}{2}(\xi+\frac{1}{6}\xi\xi'\overline{\xi})T_m+\frac{1}{4}\{(2\xi'+\frac{1}{6}\xi\overline{\xi}\xi''+\frac{2}{3}\xi\overline{\xi}'\xi'+2\xi\phi'+\frac{1}{3}\xi\xi'\overline{\xi}\phi')\mu \\
&+(\xi+\frac{1}{3}\xi\overline{\xi}\xi')\mu'\}+\frac{c_m}{12}\{\xi''+\frac{1}{12}(2\xi\overline{\xi}'\xi''+\xi'\overline{\xi}\xi''+\xi\overline{\xi}\xi'''+2\xi\xi'\overline{\xi}'') \\
&+\frac{1}{24}\xi\xi'\overline{\xi\xi}'\xi''+(\xi'+\frac{1}{12}\xi\overline{\xi}\xi''+\frac{1}{3}\xi\overline{\xi}'\xi')\phi'+\frac{1}{2}(\xi+\frac{1}{3}\xi\overline{\xi}\xi')\phi''\}\ , \\
\overline{G} &= -\overline{\eta}+\frac{2}{3}\overline{\xi}\xi'\overline{\eta}+\frac{1}{3}\overline{\xi}'\xi\overline{\eta}+\frac{1}{3}\overline{\xi}\xi\overline{\eta}'-\frac{1}{9}\overline{\xi\xi}'\xi'\xi\overline{\eta}+\frac{2}{3}\overline{\xi\xi}'\eta \\
&+\frac{1}{2}(\overline{\xi}+\frac{1}{6}\overline{\xi\xi}'\xi)T_m-\frac{1}{4}\{(2\overline{\xi}'+\frac{1}{6}\overline{\xi}\xi\overline{\xi}''+\frac{2}{3}\overline{\xi}\xi'\overline{\xi}'-2\overline{\xi}\phi'-\frac{1}{3}\overline{\xi\xi}'\xi\phi')\mu \\
&+(\overline{\xi}+\frac{1}{3}\overline{\xi}\xi\overline{\xi}')\mu'\}+\frac{c_m}{12}\{\overline{\xi}''+\frac{1}{12}(2\overline{\xi}\xi'\overline{\xi}''+\overline{\xi}'\xi\overline{\xi}''+\overline{\xi}\xi\overline{\xi}'''+2\overline{\xi\xi}'\xi'') \\
&+\frac{1}{24}\overline{\xi\xi}'\xi\xi'\overline{\xi}''-(\overline{\xi}'+\frac{1}{12}\overline{\xi}\xi\overline{\xi}''+\frac{1}{3}\overline{\xi}\xi'\overline{\xi}')\phi'-\frac{1}{2}(\overline{\xi}+\frac{1}{3}\overline{\xi}\xi\overline{\xi}')\phi''\}\ .
\end{aligned} \tag{3.18}
$$

In the case of the 'non-symmetric' parametrization,

$$
K=e^{\frac{1}{2\pi i}\oint dz\overline{\xi}(z)G(z)}e^{\frac{1}{2\pi i}\oint dz\xi(z)\overline{G}(z)}e^{\frac{1}{2\pi i}\oint dz\phi(z)\widetilde{J}(z)}\ , \tag{3.19}
$$

the expressions for the currents of the algebra are much simpler:

$$
\begin{aligned}
T &= T_m+\frac{3}{2}\xi'\overline{\eta}+\frac{3}{2}\overline{\xi}'\eta+\frac{1}{2}\xi\overline{\eta}'+\frac{1}{2}\overline{\xi}\eta'+\phi'\mu\ , \\
\widetilde{J} &= -\mu-\overline{\xi}\eta+\xi\overline{\eta}-\frac{c_m}{6}\phi'\ , \\
G &= -\eta\ , \\
\overline{G} &= -\overline{\eta}+2\overline{\xi}\xi'\overline{\eta}+\overline{\xi}'\xi\overline{\eta}+\overline{\xi}\xi\overline{\eta}'+\overline{\xi\xi}'\eta \\
&+\overline{\xi}T_m-\overline{\xi}'\mu-\frac{1}{2}\overline{\xi}\mu'+\overline{\xi}\phi'\mu+\frac{c_m}{12}(2\overline{\xi}''-2\overline{\xi}'\phi'-\overline{\xi}\phi'')\ .
\end{aligned} \tag{3.20}
$$

Once again, OPEs for the currents (3.18) and (3.20) are closed only on the classical level and with the same values of the central charge for T_m and T. The corresponding quantum expressions are

$$
\begin{aligned}
T^q &= T+\frac{1}{6}(\overline{\xi}''\xi'+\xi''\overline{\xi}'+\overline{\xi}'''\xi+\xi'''\overline{\xi})+\frac{1}{9}\overline{\xi}''\xi''\overline{\xi}\xi \\
&-\frac{1}{18}(\overline{\xi}'\xi'\overline{\xi}\xi''+\xi'\overline{\xi}'\xi\overline{\xi}''+\overline{\xi}'\xi\overline{\xi}\xi'''+\xi'\overline{\xi}\xi\overline{\xi}''')\ , \\
\tilde{J}^q &= \tilde{J}+\frac{13}{3}\phi'-\frac{2}{3}\overline{\xi}'\xi'-\frac{1}{3}(\overline{\xi}''\xi-\xi''\overline{\xi})\ , \\
G^q &= G-\frac{5}{3}\xi''-\frac{3}{8}\xi\xi'\overline{\xi}''+\frac{7}{24}\xi\xi''\overline{\xi}'-\frac{1}{72}\xi'\overline{\xi}\xi''+\frac{23}{108}\xi\xi'''\overline{\xi}+\frac{49}{432}\overline{\xi}'\xi'\overline{\xi}\xi\xi'' \\
&+\frac{13}{18}\phi'\xi\xi'\overline{\xi}'+\frac{13}{72}\phi'\xi\xi''\overline{\xi}+\frac{13}{36}\phi''\xi\xi'\overline{\xi}-\frac{13}{6}\phi'\xi'-\frac{13}{12}\phi''\xi\ ,
\end{aligned}
$$

$$
\begin{aligned}
\overline{G}^q &= \overline{G} - \frac{5}{3}\overline{\xi}'' - \frac{3}{8}\overline{\xi\xi}'\xi'' + \frac{7}{24}\overline{\xi\xi}''\xi' - \frac{1}{72}\overline{\xi}'\xi\overline{\xi}'' + \frac{23}{108}\overline{\xi\xi}'''\xi + \frac{49}{432}\xi'\overline{\xi}'\xi\overline{\xi\xi}'' \\
&\quad - \frac{13}{18}\phi'\overline{\xi\xi}'\xi' - \frac{13}{72}\phi'\overline{\xi\xi}''\xi - \frac{13}{36}\phi''\overline{\xi\xi}'\xi + \frac{13}{6}\phi'\overline{\xi}' + \frac{13}{12}\phi''\overline{\xi}
\end{aligned}
\tag{3.21}
$$

for the symmetric case and

$$
\begin{aligned}
T^q &= T \ , \ \tilde{J}^q = \tilde{J} + \frac{13}{3}\phi' \ , \\
G^q &= G \ , \ \overline{G}^q = \overline{G} - \frac{10}{3}\overline{\xi}'' + \frac{13}{3}\phi'\overline{\xi}' + \frac{13}{6}\phi''\overline{\xi}
\end{aligned}
\tag{3.22}
$$

for the non-symmetric one. Expressions (3.18), (3.21) and (3.20), (3.22) describe two possible embeddings of bosonic string into $N = 2$ string. To our knowledge, they were not given before in literature. In both cases one readily checks the following relation between the central charges $c_{N=2}$ and c_m

$$
c_{N=2} = c_m - 20 \ . \tag{3.23}
$$

Once again, the critical value of the Virasoro central charge $c_m = 26$ yields the critical value $c_{N=2} = 6$ for the central charge of $N = 2$ SCA (it corresponds to the bosonic critical dimension $d_{N=2} = 4$).

For completeness and for the sake of comparison with other options given below, let us remind the well-known form of BRST operator for the Virasoro algebra $H = \{T_m^q\}$

$$
Q = \frac{1}{2\pi i}\oint dz \ c(T_m^q + \frac{1}{2}T_{gh}) \tag{3.24}
$$

with

$$
T_{gh} = cb' + 2c'b \ . \tag{3.25}
$$

Condition for the nilpotency of the BRST operator (3.24) is just $c_m = 26$.

Note that the constructed embedding of $N = 0$ string into the $N = 2$ superstring simultaneously defines a chain of embeddings $N = 0 \to N = 1$ and $N = 1 \to N = 2$. As is clear from the relation (3.23), both these intermediate embeddings do not include the corresponding critical ones as particular cases, in contrast to the resulting $N = 0 \to N = 2$ embedding.

Besides two cases already considered, there exist three other subalgebras of $N = 2$ SCA which include Virasoro stress tensor T :

$$
H_1 = \{T, \tilde{J}\} \ , \quad H_2 = \{T, \overline{G}\} \ , \quad H_3 = \{T, \overline{G}, \tilde{J}\} \ . \tag{3.26}
$$

They all can be equally chosen as the linearly realized subalgebras. The corresponding expressions for the currents, both on the classical and quantum level, as well as the relations between central charges, are presented in the next Subsections. In all cases we observe a remarkable matching between the critical central charges of $N=2$ SCA and its linearly realized subalgebras. So, once again, these cases admit a nice interpretation as embeddings of some strings into the $N=2$ superstring.

3.4 Stability subalgebra $H=\{T_m,\tilde{J}_m\}$

This case corresponds to the embedding of the bosonic string with additional local $U(1)$ symmetry into the $N=2$ superstring.

A coset element reads

$$g = e^{\frac{1}{2\pi i}\oint dz\overline{\xi}(z)G(z)}e^{\frac{1}{2\pi i}\oint dz\xi(z)\overline{G}(z)}\ . \tag{3.27}$$

The currents are given by the following expressions

$$\begin{aligned}
T &= \frac{3}{2}\overline{\xi}'\eta+\frac{1}{2}\overline{\xi}\eta'+\frac{3}{2}\xi'\overline{\eta}+\frac{1}{2}\xi\overline{\eta}'+T_m\ ,\\
\tilde{J} &= -\overline{\xi}\eta+\xi\overline{\eta}+\tilde{J}_m\ ,\\
G &= -\eta\ ,\\
\overline{G} &= -\overline{\eta}+\overline{\xi\xi}'\eta+\overline{\xi}'\xi\overline{\eta}+2\overline{\xi}\xi'\overline{\eta}+\overline{\xi}\xi\overline{\eta}'+\overline{\xi}T_m+\overline{\xi}'J_m+\frac{1}{2}\overline{\xi}J_m'+\frac{c_m}{6}\overline{\xi}''\ .
\end{aligned}\tag{3.28}$$

In order to derive quantum version we have to redefine the central charge in the OPEs of the subalgebra $H=\{T^q_m,\tilde{J}^q_m\}$

$$\begin{aligned}
T^q_m(z)T^q_m(w) &\sim \frac{c_m/2}{(z-w)^4}+\frac{2T^q_m(w)}{(z-w)^2}+\frac{T^q_m{}'(w)}{z-w}\ ,\\
T^q_m(z)\tilde{J}^q_m(w) &\sim \frac{\tilde{J}^q_m(w)}{(z-w)^2}+\frac{\tilde{J}^q_m{}'(w)}{z-w}\ ,\\
\tilde{J}^q_m(z)\tilde{J}^q_m(w) &\sim \frac{(c_m-28)/3}{(z-w)^2}\ .
\end{aligned}\tag{3.29}$$

The quantum correction arises only for the current $\overline{G}$:

$$\overline{G}^q=\overline{G}-\frac{11}{3}\overline{\xi}''. \tag{3.30}$$

Now one can check that the currents

$$\begin{aligned}
T^q &= \frac{3}{2}\overline{\xi}'\eta+\frac{1}{2}\overline{\xi}\eta'+\frac{3}{2}\xi'\overline{\eta}+\frac{1}{2}\xi\overline{\eta}'+T^q_m\ ,\\
\tilde{J}^q &= -\overline{\xi}\eta+\xi\overline{\eta}+\tilde{J}^q_m\ ,\\
G^q &= -\eta\ ,\\
\overline{G}^q &= -\overline{\eta}+\overline{\xi\xi}'\eta+\overline{\xi}'\xi\overline{\eta}+2\overline{\xi}\xi'\overline{\eta}+\overline{\xi}\xi\overline{\eta}'+\overline{\xi}T^q_m+\overline{\xi}'\tilde{J}^q_m+\frac{1}{2}\overline{\xi}\tilde{J}^q_m{}'+\\
&\quad+\frac{c_m-22}{6}\overline{\xi}''
\end{aligned}\tag{3.31}$$

form closed quantum $N=2$ SCA (3.1) with

$$c_{N=2} = c_m - 22. \tag{3.32}$$

To find the critical dimension for the bosonic string with additional $U(1)$ symmetry, let us construct the BRST operator for the algebra (3.29). It can be written in the form

$$Q = \frac{1}{2\pi i}\oint dz\ [c(T_m^q + \frac{1}{2}T_{gh}) + a(\tilde{J}_m^q + \frac{1}{2}\tilde{J}_{gh})], \tag{3.33}$$

where

$$T_{gh} = cb' + 2c'b + a's\ ,\ \tilde{J}_{gh} = cs' + c's\ , \tag{3.34}$$

and the ghosts-anti-ghost pairs (c,b), (a,s) correspond to the $\{T_m^q, J_m^q\}$ currents. Condition for the nilpotency of the BRST operator (3.33) is $c_m = 28$ which gives the correct value for the critical central charge of $N=2$ superstring $c_{N=2} = 6$.

It is interesting to note that the extended bosonic algebra $H = \{T_m, \tilde{J}_m\}$ with the critical $c_m = 28$ naturally comes out as the algebra of constraints of the interacting system of string and massless particle moving in the space with two timelike dimensions [16]. It was also discussed in [17] in the context of 'universal string theory' and F-theory [18]. The embedding of the string associated with H into the $N=1$ superstring possessing some extra symmetry ($N=1$ extension of $U(1)$ symmetry) was constructed. The above relations yield an alternative critical embedding of the same bosonic string, this time into the $N=2$ superstring.

3.5 Stability subalgebra $H = \{T_m, \overline{G}_m\}$

This case describes the embedding of some string which has, besides the Virasoro symmetry, an additional local supersymmetry generated by the Grassmann-odd current $\overline{G}_m$.

The coset element is

$$g = e^{\frac{1}{2\pi i}\oint dz \overline{\xi}(z)G(z)} e^{\frac{1}{2\pi i}\oint dz \phi(z)\tilde{J}(z)}\ . \tag{3.35}$$

The expressions for currents read

$$\begin{aligned}
T &= \frac{3}{2}\overline{\xi}'\eta + \frac{1}{2}\overline{\xi}\eta' + \phi'\mu + T_m\ , \\
J &= -\mu - \overline{\xi}\eta - \frac{c_m}{6}\phi'\ , \\
G &= -\eta\ , \\
\overline{G} &= \overline{\xi}\overline{\xi}'\eta + \overline{\xi}\phi'\mu - \overline{\xi}'\mu - \frac{1}{2}\overline{\xi}\mu' + \overline{\xi}T_m + e^{\phi}\overline{G}_m + \frac{c_m}{12}(2\overline{\xi}'' - 2\overline{\xi}'\phi' - \overline{\xi}\phi'')
\end{aligned} \tag{3.36}$$

The corresponding quantum expressions are as follows:

$$
\begin{aligned}
T^q &= T - \phi'' , \\
J^q &= J + 2\phi' , \\
G^q &= G , \\
\overline{G}^q &= \overline{G} - \frac{3}{2}\overline{\xi}'' + 2\overline{\xi}'\phi' .
\end{aligned}
\tag{3.37}
$$

The currents (3.37) form quantum $N = 2$ SCA with

$$
c_{N=2} = c_m - 9 . \tag{3.38}
$$

BRST operator for the subalgebra $H = \{T^q_m, \overline{G}^q_m\}$,

$$
\begin{aligned}
T^q_m(z)T^q_m(w) &\sim \frac{c_m/2}{(z-w)^4} + \frac{2T^q_m(w)}{(z-w)^2} + \frac{T^q_m{}'(w)}{z-w} , \\
T^q_m(z)\overline{G}^q_m(w) &\sim \frac{3/2\overline{G}^q_m(w)}{(z-w)^2} + \frac{\overline{G}^q_m{}'(w)}{z-w} ,
\end{aligned}
\tag{3.39}
$$

has the following form

$$
Q = \frac{1}{2\pi i}\oint dz \, [c(T^q_m + \frac{1}{2}T_{gh}) + \alpha(\overline{G}^q_m + \frac{1}{2}\overline{G}_{gh})] , \tag{3.40}
$$

with

$$
\begin{aligned}
T_{gh} &= cb' + 2c'b - \frac{1}{2}\alpha\beta' - \frac{3}{2}\alpha'\beta , \\
\overline{G}_{gh} &= \beta'c + \frac{3}{2}\beta c' .
\end{aligned}
\tag{3.41}
$$

Condition for the nilpotency of the BRST operator (3.40) is $c_m = 15$, which again gives rise through (3.38) to the critical central charge of the $N = 2$ superstring, $c_{N=2} = 6$. A natural $2D$ field theory realization of the fermionic string associated with H of the present example is on 10 bosonic and 10 fermionic fields, thus implying the bosonic critical dimension $d_m = 10$.

3.6 Stability subalgebra $H = \{T_m, \overline{G}_m, \tilde{J}_m\}$

This case corresponds to an embedding of some string with Grassmann-odd current $\overline{G}_m$ and additional local $U(1)$ symmetry into the $N = 2$ superstring. The embedded string is a 'hybrid' of the strings associated with the stability subalgebras of two previous examples.

The OPEs of the stability subalgebra read

$$
\begin{aligned}
T_m(z)T_m(w) &\sim \frac{c_m/2}{(z-w)^4}+\frac{2T_m(w)}{(z-w)^2}+\frac{T'_m(w)}{z-w}\ ,\ T_m(z)\tilde{J}_m(w)\sim\frac{\tilde{J}_m(w)}{(z-w)^2}+\frac{\tilde{J}'_m(w)}{z-w}\ ,\\
T_m(z)\overline{G}_m(w) &\sim \frac{3/2\overline{G}_m(w)}{(z-w)^2}+\frac{\overline{G}'_m(w)}{z-w}\ ,\ \tilde{J}_m(z)\overline{G}_m(w)\sim-\frac{\overline{G}_m(w)}{(z-w)}\ ,\\
\tilde{J}_m(z)\tilde{J}_m(w) &\sim \frac{c_m/3}{(z-w)^2}\ .
\end{aligned}
\tag{3.42}
$$

The relevant coset element is defined by

$$
g=e^{\frac{1}{2\pi i}\oint dz\overline{\xi}(z)G(z)}\ . \tag{3.43}
$$

The currents are given by the expressions

$$
\begin{aligned}
T &= \frac{3}{2}\overline{\xi}'\eta+\frac{1}{2}\overline{\xi}\eta'+T_m\ ,\\
\tilde{J} &= -\overline{\xi}\eta+\tilde{J}_m\ ,\\
G &= -\eta\ ,\\
\overline{G} &= \overline{\xi}\overline{\xi}'\eta+\overline{\xi}T_m+\overline{\xi}'\tilde{J}_m+\frac{1}{2}\overline{\xi}\tilde{J}'_m+\overline{G}_m+\frac{c_m}{6}\overline{\xi}''\ .
\end{aligned}
\tag{3.44}
$$

After redefining OPEs in the subalgebra (3.42) as follows

$$
\begin{aligned}
T^q_m(z)T^q_m(w) &\sim \frac{c_m/2}{(z-w)^4}+\frac{2T^q_m(w)}{(z-w)^2}+\frac{T^q_m{}'(w)}{z-w}\ ,\\
T^q_m(z)\tilde{J}^q_m(w) &\sim \frac{-2}{(z-w)^3}+\frac{\tilde{J}^q_m(w)}{(z-w)^2}+\frac{\tilde{J}^q_m{}'(w)}{z-w}\ ,\\
T^q_m(z)\overline{G}^q_m(w) &\sim \frac{3/2\overline{G}^q_m(w)}{(z-w)^2}+\frac{\overline{G}^q_m{}'(w)}{z-w}\ ,\ \tilde{J}^q_m(z)\overline{G}^q_m(w)\sim-\frac{\overline{G}^q_m(w)}{(z-w)}\ ,\\
\tilde{J}^q_m(z)\tilde{J}^q_m(w) &\sim +\frac{(c_m-14)/3}{(z-w)^2}\ ,
\end{aligned}
\tag{3.45}
$$

we obtain the quantum correction again only for $\overline{G}$:

$$
\overline{G}_q = \overline{G}-\frac{11}{6}\overline{\xi}''\ . \tag{3.46}
$$

One can check that the currents

$$
\begin{aligned}
T^q &= \frac{3}{2}\overline{\xi}'\eta+\frac{1}{2}\overline{\xi}\eta'+T^q_m\ ,\\
\tilde{J}^q &= -\overline{\xi}\eta+\tilde{J}^q_m\ ,\\
G^q &= -\eta\ ,\\
\overline{G}^q &= \overline{\xi}\overline{\xi}'\eta+\overline{\xi}T^q_m+\overline{\xi}'\tilde{J}^q_m+\frac{1}{2}\overline{\xi}\tilde{J}'_m+\overline{G}^q_m+\frac{c_m-11}{6}\overline{\xi}''
\end{aligned}
\tag{3.47}
$$

span a quantum $N=2$ SCA with

$$c_{N=2}=c_m-11\ . \tag{3.48}$$

The BRST operator for the subalgebra (3.45) is

$$Q=\frac{1}{2\pi i}\oint dz\ [c(T^q_m+\frac{1}{2}T_{gh})+a(\tilde{J}^q_m+\frac{1}{2}\tilde{J}_{gh})+\alpha(\overline{G}^q_m+\frac{1}{2}\overline{G}_{gh})]\ , \tag{3.49}$$

where

$$\begin{aligned} T_{gh} &= cb'+2c'b+a's-\frac{1}{2}\alpha\beta'-\frac{3}{2}\alpha'\beta\ , \\ \tilde{J}_{gh} &= cs'+c's-\alpha\beta\ , \\ \overline{G}_{gh} &= \beta'c+\frac{3}{2}\beta c'-\beta a \end{aligned} \tag{3.50}$$

and the ghost-anti-ghost pairs (c,b), (a,s), (α,β) correspond to the $\{T^q_m,\tilde{J}^q_m,\overline{G}^q_m\}$ currents.

The nilpotency of the BRST operator (3.49) is achieved with $c_m=17$, that again yields, via eq. (3.48), just the critical value for the central charge of $N=2$ string $c_{N=2}=6$. In accord with the reasoning at the end of previous Subsection, the critical bosonic dimension of the string associated with the given choice of H is expected to be $d_m=12$. So this string might bear a tight relation to the hypothetical F-theory.

3.7 Stability subalgebra $H=\{T_m,G_2\}$ and $N=0\to N=1$ embeddings

In order to show the universality of our method and reveal a correspondence with the embeddings of $N=0$ string into the $N=1$ superstring in the approach of refs. [12,13], in this last Subsection we consider an embedding of the $N=1$ superstring into the $N=2$ one in the component formalism. We use the real fermionic currents G_1 and G_2 satisfying the algebra (3.1), and place them, respectively, into the coset and the stability subalgebra.

We start from the coset element

$$g=e^{\frac{1}{2\pi i}\oint dz\xi(z)G_1(z)}e^{\frac{1}{2\pi i}\oint dz\phi(z)J(z)}\ , \tag{3.51}$$

and find the following expressions for the $N=2$ SCA currents

$$\begin{aligned} T &= \frac{3}{2}\xi'\eta+\frac{1}{2}\xi\eta'+\phi'\mu+T_m\ , \\ J &= -\mu-\xi\eta\tan\phi-\frac{1}{2}\xi\xi'\mu-\xi\sec\phi G_{2m}+\frac{c_m}{12}(2\phi'+\xi\xi'\phi'+\xi\xi''\tan\phi)\ , \\ G_1 &= -\eta-\frac{1}{2}\xi\xi'\eta-\frac{1}{2}\xi\phi'\mu-\frac{1}{2}\xi T_m-\frac{c_m}{12}(\xi''-\frac{1}{8}\xi\xi'\xi'')\ , \end{aligned}$$

$$
\begin{aligned}
G_2 &= (1-\frac{1}{2}\xi\xi')\eta\tan\phi + \xi'\mu + \frac{1}{2}\xi\mu' - \frac{1}{2}\xi\phi'\mu\tan\phi - \frac{1}{2}\xi\tan\phi T_m \\
&\quad +(1-\frac{1}{4}\xi\xi')\sec\phi G_{2m} - \frac{c_m}{12}(2\phi'\xi' + \xi\phi'' + \xi''\tan\phi - \frac{3}{8}\xi\xi'\xi''\tan\phi)
\end{aligned}
\tag{3.52}
$$

in terms of the real Nambu-Goldstone currents $\eta(z), \xi(z)$ and their conjugates $\mu(z), \phi(z)$:

$$
\eta(z)\xi(w) \sim \frac{1}{z-w}, \;\; \mu(z)\phi(w) \sim \frac{1}{z-w} \; . \tag{3.53}
$$

The corresponding quantum expressions are

$$
\begin{aligned}
T^q &= T - \frac{1}{4}(\xi''\xi' + \xi'''\xi) - (\phi'\tan\phi)' \; , \\
J^q &= J + \frac{1}{4}\phi'\xi'\xi + \frac{5}{8}\xi''\xi\tan\phi + \frac{1}{2}(\tan^2\phi - 3)\phi' \; , \\
G_1^q &= G_1 + \frac{3}{4}\xi'' + \frac{1}{4}\xi''\xi'\xi - \frac{3}{4}\xi'\phi'\tan\phi + \frac{1}{8}\xi(\phi'\tan\phi)' \; , \\
G_2^q &= G_2 + \frac{1}{16}\xi''(12+5\xi'\xi)\tan\phi + \frac{1}{4}\xi'\phi'(5-\sec^2\phi) \\
&\quad + \frac{1}{8}\xi\left(\phi'' + \phi''\sec^2\phi + 3\phi'\phi'\tan\phi\sec^2\phi\right) \; .
\end{aligned}
\tag{3.54}
$$

Let us now consider the embedding of $N=0$ string into $N=1$ string within our approach and its relationship with the $N=1 \to N=2$ embedding just presented.

We choose the $N=1$ SCA formed by the currents T, G_1. Taking as a stability subalgebra $H = \{T_m\}$ and parametrizing the coset space as

$$
g = e^{\frac{1}{2\pi i}\oint dz \xi(z) G_1(z)} \; , \tag{3.55}
$$

we find the following expressions for the classical currents

$$
\begin{aligned}
T &= \frac{3}{2}\xi'\eta + \frac{1}{2}\xi\eta' + T_m \; , \\
G_1 &= -\eta - \frac{1}{2}\xi\xi'\eta - \frac{1}{2}\xi T_m - \frac{c_m}{12}(\xi'' - \frac{1}{8}\xi\xi'\xi'') \; .
\end{aligned}
\tag{3.56}
$$

After the redefinitions

$$
G_1 = \frac{1}{\sqrt{2}}\tilde{G} \; , \; \xi = -\sqrt{2}\tilde{\xi} \; , \; \eta = -\sqrt{2}\tilde{\eta} \; , \tag{3.57}
$$

the currents T and $\tilde{G}$ precisely coincide with those deduced by McArthur [13]. Coincidence with the result of Kunitomo [12] at the classical level can be further achieved by decoupling the matter and putting the central charge equal to zero [1].

[1] In ref. [12] the classical versions of $N=1$ SCA and Virasoro algebra were assumed to have a zero central charge. Actually, it is not necessary to require this: central charges can be switched on already at the classical level through, e.g., Feigin-Fuks terms.

Total expressions for the currents contain quantum corrections:

$$\begin{aligned}
\hat{T}^q &= T_m + \frac{3}{2}\xi'\eta + \frac{1}{2}\xi\eta' - \frac{1}{4}(\xi''\xi' + \xi'''\xi) \;, \\
\hat{G}_1^q &= -\eta - \frac{1}{2}\xi\xi'\eta - \frac{1}{2}\xi T_m - \frac{c_m - 11}{12}\xi'' + \frac{c_m - 26}{96}\xi\xi'\xi'' \;.
\end{aligned} \tag{3.58}$$

The redefinitions (3.57) bring these expressions into those given by Berkovits and Vafa [1] and Berkovits and Ohta [7].

Let us compare $\hat{T}^q$, $\hat{G}_1^q$ with the currents T^q, G_1^q from the $N = 2$ set (3.54). They also form $N = 1$ SCA as a subalgebra of $N = 2$ SCA and so are expected to admit a truncation to the expressions (3.58).

Despite the fact that the coset space (3.55) formally can be derived from the coset space (3.51) in the limit $\phi = 0$, the relation between the corresponding quantum currents of $N = 1$ SCA is not so simple due to the presence of conjugate variable $\mu(z)$

$$\mu(z)\phi(w) \sim \frac{1}{z - w}$$

in the expressions (3.52). The currents T^q and G_1^q can be brought into the following form

$$\begin{aligned}
T^q &= (T_m + \phi'\mu - (\phi'\tan\phi)') + \frac{3}{2}\xi'\eta + \frac{1}{2}\xi\eta' - \frac{1}{4}(\xi''\xi' + \xi'''\xi) \;, \\
G_1^q &= -\eta - \frac{1}{2}\xi\xi'\eta - \frac{1}{2}\xi(T_m + \phi'\mu - (\phi'\tan\phi)') - \frac{c_m - 9}{12}\xi'' + \frac{c_m - 24}{96}\xi\xi'\xi'' \\
&\quad - \frac{3}{4}\xi'\phi'\tan\phi - \frac{3}{8}\xi(\phi'\tan\phi)' \;.
\end{aligned} \tag{3.59}$$

We observe that after passing to $\tilde{T}_m = T_m + \phi'\mu - (\phi'\tan\phi)'$, the currents T^q and G_1^q coincide with those given by eqs. (3.58), up to the change $T_m \to \tilde{T}_m$ and the presence of two extra terms in G_1^q:

$$\begin{aligned}
T^q &= \hat{T}^q \;, \quad G_1^q = \hat{G}_1^q + \delta G_1^q \;, \\
\delta G_1^q &= -\frac{3}{4}\xi'\phi'\tan\phi - \frac{3}{8}\xi(\phi'\tan\phi)' \;.
\end{aligned} \tag{3.60}$$

It is easy to check that the addition δG_1^q possesses the following OPEs

$$T^q\;\delta G_1^q \sim 0,\; G_1^q\;\delta G_1^q + \delta G_1^q\;G_1^q \sim 0,\; \delta G_1^q\;\delta G_1^q \sim 0 \;. \tag{3.61}$$

Though the presence of this term in G_1^q is absolutely necessary for the closure of $N = 2$ superconformal algebra at the quantum level, it is unessential from the standpoint of its $N = 1$ subalgebra formed by T^q and G_1^q (as follows from (3.61), it is a 'null field' with respect to this $N=1$ SCA). If one does not care about the whole $N=2$ SCA, then, as a consequence of the relations (3.61), there is a one-parameter freedom in the definition of G_1^q:

$$G_1^q(a) = \hat{G}_1^q + a\delta G_1^q. \tag{3.62}$$

It means that the last term can be consistently omitted from G_1^q. Then the resulting expressions for the $N=1$ SCA currents coincide with the expressions (3.58) in which T_m is replaced by $\tilde{T}_m$. The presence of extra terms in $\tilde{T}_m$ also leads to the shift of its central charge to $\tilde{c}_m = c_m + 2$. Thus, the final expressions for the currents T^q, G_1^q reduced in this way coincide with the expressions (3.58) in which T_m, c_m are replaced by $\tilde{T}_m, \tilde{c}_m$.

We end this Section with the following comment. At any value of a in (3.62) the currents T^q, $G_1^q(a)$ generate $N=1$ SCA with the standard relation between the $N=1$ and $N=0$ central charges $c_{N=1} = \tilde{c}_m - 11$. But only at $a=0$ one gets a minimal realization solely in terms of η, ξ and $\tilde{T}_m$ that can be interpreted in the language of embedding of $N=0$ string into the $N=1$ superstring.

4 Linearization of $W_3^{(2)}$ as a string embedding

In this part of the article we demonstrate how the described techniques work in the case of nonlinear algebras. As an example we take a $W_3^{(2)}$ algebra. The spin content of currents of this algebra $2, \frac{3}{2}, \frac{3}{2}, 1$ is the same as in $N=2$ SCA. The basic difference between these two algebras is the Grassmann parity of the spin $\frac{3}{2}$ currents. In the case of $W_3^{(2)}$ they are bosonic and this fact leads to the appearance of nonlinear terms in the OPEs of the algebra

$$\begin{aligned}
T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w}\,, \\
T(z)J(w) &\sim \frac{J(w)}{(z-w)^2} + \frac{J'(w)}{z-w}\,, \quad T(z)G^{\pm}(w) \sim \frac{3/2\; G^{\pm}(w)}{(z-w)^2} + \frac{G'^{\pm}(w)}{z-w}\,, \\
J(z)J(w) &\sim -\frac{c/9}{(z-w)^2}\,, \quad J(z)G^{\pm}(w) \sim \pm\frac{G^{\pm}(w)}{(z-w)}\,, \quad G^{\pm}(z)G^{\pm}(w) \sim 0\,, \\
G^{+}(z)G^{-}(w) &\sim \frac{-c/3}{(z-w)^3} + \frac{3J(w)}{(z-w)^2} - \frac{T(w) + 18/c\; J^2(w) - 3/2\; J'(w)}{z-w}\,.
\end{aligned} \tag{4.1}$$

The algebra (4.1) is nonlinear because there is a quadratic (in J) term on the right-hand side of the OPE G^+G^-. One way to construct a coset realization of this algebra along the lines of the previous Sections is to convert it into an infinite-dimensional *linear* algebra using the trick proposed in [19]. It consists in treating all the composite currents (including J^2) as some new independent currents. We use here another, more economic version of this approach which will allow us to deal only with the original finite set of $W_3^{(2)}$ currents at each step of calculations.

We choose as the stability subalgebra the maximal linear subalgebra of $W_3^{(2)}$, that is the set

$$H = \{G^+, J, T, c\}, \tag{4.2}$$

as well as all the composite currents constructed out of it, like J^2, JT, cJ, cT, ... For consistency, we need to treat the central charge of the whole algebra also as an independent current and to include it from the beginning into the set (4.2) on equal footing with other currents.

The remaining current G^- is placed into the coset. Respectively, we will interpret the quantity

$$K = e^{\frac{1}{2\pi i}\oint dz \xi(z) G^-(z)} \tag{4.3}$$

as a representative of this coset. We will try to realize the whole $W_3^{(2)}$ as left shifts of the coset current $\xi(z)$ making use of the general formalism described in Sect. 2. An essentially new feature compared to the case of linear algebras will be the appearance of composite currents in the induced elements h on the right hand side of the general relation (2.3) specialized to the present case. We will give a self-consistent explicit prescription of how to treat such objects with preserving the original algebraic structure.

Calculating as above the left action of the infinitesimal element

$$\alpha = \frac{1}{2\pi i}\oint dz \{\alpha_+(z)G^+(z) + \alpha_-(z)G^-(z) + \beta(z)J(z) + a(z)T(z)\}\ , \tag{4.4}$$

on (4.3), we find the variation of $\xi(x)$

$$\delta\xi = \alpha_- + \alpha_+\xi\xi' - \alpha_+'\xi^2 - \beta\xi + \frac{1}{2}a'\xi - a\xi'\ , \tag{4.5}$$

and induced infinitesimal element of the subalgebra to the right of K

$$\begin{aligned} \delta h \ = \ & \frac{1}{2\pi i}\oint dz\{\beta J + aT + \alpha_+ G^+ - \frac{c}{6}\alpha_+\xi'' - 3\alpha_+\xi' J \\ & - \frac{3}{2}\alpha_+\xi J' - \alpha_+\xi T - \frac{18}{c}\alpha_+\xi J^2 + \frac{18}{c}\alpha_+\xi^2 J G^- - \frac{6}{c}\alpha_+\xi^3 G^- G^-\}\ . \end{aligned} \tag{4.6}$$

Since the currents J, T and G^- form a linear subalgebra in $W_3^{(2)}$, no problems occur while obtaining the expressions for them within the coset realizations approach. Just like in the cases considered in the previous Section, one should replace, in the relevant terms in (4.6), the involved stability subalgebra generators J, T, c by their 'matter' representation J_m, T_m, c_m (c_m is assumed to be a constant, and this is why we are allowed to divide by c in (4.6)). Also, one introduces the current $p(z)$ canonically conjugated to $\xi(z)$

$$p(z)\xi(w) \sim \frac{1}{z-w}\ . \tag{4.7}$$

As a result, the following expressions for J, T and G^- can be extracted from eqs. (4.5) and (4.6)

$$J = \xi p + J_m \ , \quad T = \frac{3}{2}\xi' p + \frac{1}{2}\xi p' + T_m \ , \quad G^- = -p \ . \tag{4.8}$$

A new characteristic feature of (4.6) is the appearance of the composite currents J^2, JG^-, G^-G^- in the expression for the induced element of the subalgebra corresponding to the left shift by the generator G^+ (the coefficient before the parameter α_+ in the r.h.s of (4.6)). This comes about just due to the presence of nonlinear term in the last of OPEs (4.1), so one can expect that such a phenomenon is typical for nonlinear algebras. This is a crucial difference compared to the previously considered linear case, and one should give a recipe of how to treat such composite currents within the nonlinear realizations formalism.

Once again, in accord with the general prescriptions of Sect. 2, we should replace the stability subalgebra generators by their 'matter' representation $\{G_m^+, J_m, T_m, c_m\}$ wherever they appear, on their own or as building blocks of composite currents. What concerns the composite currents containing G^-, namely JG^- and G^-G^-, a natural idea is to replace the current G^- in them by its already found coset expression (4.8). In this way we get

$$\begin{aligned} G^+ &= G_m^+ - 3\xi\xi' p - \xi^2 p' - \frac{c}{6}\xi'' - 3\xi' J_m - \frac{3}{2}\xi J_m' - \xi T_m - \frac{18}{c}\xi J_m^2 \\ &\quad - \frac{18}{c}\xi^2 J_m p - \frac{6}{c}\xi^3 p^2 \ . \end{aligned} \tag{4.9}$$

It is straightforward to verify that the currents (4.8), (4.9) satisfy just the OPEs (4.1) with $c = c_m$.

A consistency check of our procedure of deriving the coset representation for the $W_3^{(2)}$ currents goes as follows. We act on the coset element (4.3) from the left by composite currents constructed out of the original abstract currents J, T, $G^\pm$ and find their coset realization by pulling them through (4.3) with the use of OPEs (4.1) and finally making the changes $\{J, T, G^+, c\} \to \{J_m, T_m, G_m^+, c_m\}$ and $G^- \to -p$. We explicitly found such a realization for the composite currents J^2, G^-G^-, JG^+. In all cases, the resulting expressions are given by the appropriate products of the currents (4.8) and (4.9).

Note that this modified nonlinear (coset) realizations scheme seems to work, with minor further modifications, in the case of other nonlinear algebras too.

Similarly to the $N = 2$ SCA realizations constructed in Sect. 3, the formulas (4.8), (4.9) give a realization of the $W_3^{(2)}$ generators in terms of Nambu-Goldstone fields ξ, p and a closed set of 'matter' currents J_m, T_m, G_m^+. It can be checked that these expressions coincide with the classical limit of the expressions for the $W_3^{(2)}$ currents obtained in [20] within the procedure of conformal linearization of $W_3^{(2)}$. The currents J_m, T_m, G_m^+ together with the 'ghost-antighost' pair ξ, p form just the classical version of the linearizing algebra for $W_3^{(2)}$. Thus at the classical level the 'linearization' of $W_3^{(2)}$ amounts to constructing its above coset realization, with the Nambu-Goldstone field ξ (and its conjugate momenta p) playing the role of additional currents which allow to linearize $W_3^{(2)}$ in the spirit of ref. [20]. One encounters here a surprising situation when two nonlinearities (the intrinsic nonlinearity of $W_3^{(2)}$ algebra and the nonlinearity of the coset realization procedure) 'interfere' to yield a linearity in the end. It would be interesting to apply the same techniques to construct linearizing algebras for those W algebras for which this construction is still lacking, e.g., $SO(N)$ Knizhnik-Bershadsky algebra.

It is tempting to make a step further and to wonder whether this linearization procedure through a nonlinear realization of $W_3^{(2)}$ admits an interpretation in terms of string embeddings, like nonlinear realizations of $N = 2$ SCA constructed in the previous Section. It does! To see this, one needs, first of all, to pass [20] to the quantum counterparts of eqs. (4.8), (4.9)

$$\begin{aligned} J^q &= J, \quad T^q = T + \frac{9}{9-c} J^{q\prime}, \quad G^{-q} = G^- \ , \\ G^{+q} &= G_m^+ - \xi T^q + \frac{18}{9-c}\xi J^q J^q + \frac{3(21-c)}{2(9-c)}\xi J^{q\prime} - \frac{18}{9-c}\xi^2 p J^q \\ &\quad + \frac{6}{9-c}\xi^3 p^2 + 3\xi\xi' p + \frac{9+c}{9-c}\xi^2 p' - 3\left(\xi J^q - \frac{9-c}{18}\xi'\right)' \ . \end{aligned} \tag{4.10}$$

An essential peculiarity of this case originating from the nonlinear nature of $W_3^{(2)}$ and having no analog in the case of $N = 2$ SCA (and other linear conformal algebras) is that not only the central charges of $W_3^{(2)}$ and the subalgebra $\{T_m, J_m, G_m^+\}$ become different after passing to the quantum case, but also the structure relations of the latter algebra are modified. Namely, its quantum OPEs are as follows

$$\begin{aligned} T_m(z_1)T_m(z_2) &\sim \frac{9+4c-c^2}{2(9-c)}\frac{1}{z_{12}^4} + \frac{2T_m}{z_{12}^2} + \frac{T_m'}{z_{12}} \ , \\ T_m(z_1)G_m^+(z_2) &\sim \left[\frac{3}{2} + \frac{9}{9-c}\right]\frac{G_m^+}{z_{12}^2} + \frac{G_m^{+\prime}}{z_{12}} \ , \ T_m(z_1)J_m(z_2) \sim \frac{J_m}{z_{12}^2} + \frac{J_m'}{z_{12}} \ , \\ J_m(z_1)J_m(z_2) &\sim \frac{9-c}{9z_{12}^2} \ , \ J_m(z_1)G_m^+(z_2) \sim \frac{G_m^+}{z_{12}} \ , \end{aligned} \tag{4.11}$$

and they differ from the classical ones in that the current G_m^+ acquires an anomalous conformal dimension $\frac{3}{2} + \frac{9}{9-c}$. Actually, this is none other than the algebra W_3^{lin} which linearizes another kind of nonlinear algebra, Zamolodchikov's W_3 algebra [20]. Now, we wish to interpret eqs. (4.10) as describing an embedding of some string with the symmetry algebra W_3^{lin} in the matter sector into a string associated with $W_3^{(2)}$. A consistency check for such an interpretation is to inquire whether the critical central charges of these two algebras match with each other. They can be parametrized in terms of a single parameter c as follows [20]

$$c_{(TT)} = \frac{(7+c)c}{c-9}\ ,\ c_m = \frac{9+4c-c^2}{9-c}\ . \tag{4.12}$$

The BRST operator for W_3^{lin} was constructed in [8], it is nilpotent at $c = 18$ and, accordingly, $c_m = 27$. On the other hand, as noted in the same paper [8], the nilpotency condition for the BRST charge operator for $W_3^{(2)}$ [21] requires $c_{(TT)} = 50$. But this is just the value that arises upon substitution, into c_{TT} in (4.12), of $c = 18$ which simultaneously produces the critical value for the central charge c_m. Thus the remarkable matching between critical central charges which is a characteristic feature of embeddings in the case of linear superconformal algebras extends to this nonlinear case as well. This is a strong indication that in the present case we indeed deal with an embedding of string associated with the linear algebra W_3^{lin} into the $W_3^{(2)}$ string [2]. Note that an analogous relation between critical central charges was found in [8] while considering an embedding of the bosonic string (associated with the Virasoro subalgebra T_m) into the $W_3^{(2)}$ string. It would be interesting to check whether such an embedding can be reproduced within our nonlinear realization approach.

5 Conclusions

In this paper we presented a modified nonlinear realization method directly applicable to superconformal and some W type algebras in the formulation based on OPEs or SOPEs of the relevant currents or supercurrents. This provides a systematic way of deducing the relations describing various embeddings of bosonic and fermionic strings. The embedded string or superstring (its matter sector, to be precise) always corresponds to the vacuum stability subalgebra of the given nonlinear realization while for the (super)currents of the embracing algebra one algorithmically gets the expressions in terms of the appropriate coset (super)fields, their conjugate momenta and the generators of the stability subalgebra. The method as it stands is limited to the classical algebras (although with non-zero central charges, as distinct from the approach of [12], say). Nonetheless, it is straightforward to explicitly find the quantum corrections and to get the genuine relations describing the string embeddings. We reproduced in this way some known examples ($N = 0 \to N = 1$ [1], $N = 1 \to N = 2$ [7]) and constructed new embeddings of the bosonic string and some of its extensions into the $N = 2$ string. It would be interesting to reveal possible

[2] In order to rigorously prove this conjecture, one should show that the cohomology of the BRST operator for $W_3^{(2)}$ in this specific realization coincides with that of the BRST operator for W_3^{lin}.

physical implications of such extended strings and to identify their place in the modern string realm. All the embeddings constructed include as a particular case the corresponding critical embedding.

We also applied our method to an example of nonlinear W type algebras, $W_3^{(2)}$ algebra, choosing as the stability subalgebra the maximal linear subalgebra of $W_3^{(2)}$. It coincides with the algebra W_3^{lin} introduced in [20] as the linearizing algebra for Zamolodchikov's W_3. Surprisingly, our approach immediately leads to the relations of ref. [20] describing the linearization of $W_3^{(2)}$. Thus the linearization procedure for this particular W algebra can be as well understood as the embedding of the W_3^{lin} string into the $W_3^{(2)}$ string. The critical central charges in this case nicely match with each other, similarly to other examples considered. An interesting problem is to treat a wider class of W type algebras from the point of view presented here and to see whether the linearization procedure for them [22]-[24] always admits an interpretation in terms of proper string embeddings.

As a prospect for further developments we mention possible applications of our method to other superconformal algebras, e.g. 'small' and 'large' $N=4$ SCAs. Some string embeddings related to these algebras were described in [7,25]. Our method will hopefully allow to list all possible such embeddings by choosing various subalgebras of $N=4$ SCAs as the vacuum stability ones. It would be extremely interesting to generalize our method to more complicated extended objects like p-branes, i.e. to learn how to construct nonlinear realizations of the relevant infinite-dimensional world-volume gauge symmetries and to describe embeddings of such objects in this universal language [3].

Finally, let us remind that nonlinear realizations of $1D$ superconformal algebras and W algebras were previously considered from various points of view in a number of works, e.g. in [26,19], [27]-[29]. It is of interest to understand how these approaches are related with the one presented here.

Acknowledgments

Two of us (S.K. and A.P.) are indebted to D. Lüst and C. Preitschopf for hospitality at the Humboldt University in Berlin, where a part of this work has been done. This investigation has been supported in part by the Fondo Scambi Internazionali Convenzione Particellare INFN-Dubna, the Russian Foundation of Fundamental Research, grant RFFR 96-02-17634, joint grant RFFR-DFG 96-02-00186G, INTAS grants 93-127 and 94-2317 and by a grant of the Dutch NWO Organization.

[3] See a recent preprint [17] for an attempt to apply the embedding ideas in the context of F-theory.

References

[1] N. Berkovits and C. Vafa, Mod. Phys. Lett. **A 9** (1994) 653

[2] J.M. Figueroa-O'Farrill, Phys. Lett. **B 321** (1994) 344

[3] H. Ishikawa and M. Kato, Mod. Phys. Lett. **A 9** (1994) 725

[4] F. Bastianelli, Phys. Lett. **B 322** (1994) 340

[5] N. Ohta and J.L. Petersen, Phys. Lett. **B 325** (1994) 67

[6] F. Bastianelli, N. Ohta and J.L. Petersen, Phys. Lett. **B 327** (1994) 35

[7] N. Berkovits and N. Ohta, Phys. Lett. **B 334** (1994) 72

[8] F. Bastianelli and N. Ohta, Phys. Lett. **B 348** (1995) 411

[9] N. Berkovits, M. Freeman and P. West, Phys. Lett. **B 325** (1994) 63

[10] H. Kunitomo, M. Sakaguchi and A. Tokura, Int. J. Mod. Phys. **A 10** (1995) 827

[11] E. Bergshoeff, H.J. Boonstra and M. de Roo, Phys. Lett. **B 346** (1995) 269

[12] H. Kunitomo, Phys. Lett. **B 343** (1995) 144

[13] I.N. McArthur, Phys. Lett. **B 342** (1995) 94

[14] S. Coleman, J. Wess and B. Zumino, Phys. Rev. **177** (1969) 2239

[15] D. Volkov, Sov. J. Part. Nucl. Phys. **4** (1973) 3

[16] I. Bars and C. Kounnas, 'String and Particle with Two Times', CERN-TH/97-108, USC-97/HEP-B3, LPTENS-97/25, hep-th/9705205

[17] J.M. Figueroa-O'Farrill, 'F-theory and the universal string theory', hep-th/9704009

[18] C. Vafa, Nucl. Phys. **B 469** (1996) 403

[19] E. Ivanov, S. Krivonos and A. Pichugin, Phys. Lett. **B 284** (1992) 260

[20] S. Krivonos and A. Sorin, Phys. Lett. **B 335** (1994) 45

[21] Z. Khviengia and E. Sezgin, Phys. Lett. **B 326** (1994) 243

[22] S. Bellucci, S. Krivonos and A. Sorin, Phys. Lett. **B 392** (1997) 350

[23] S. Bellucci, S. Krivonos and A. Sorin, Phys. Lett. **B 366** 1996) 104

[24] S. Bellucci, S. Krivonos and A. Sorin, Phys. Lett. **B 347** (1995) 260

[25] N. Ohta and T. Shimizu, Phys. Lett. **B 355** (1995) 127

[26] S. Bellucci and E. Ivanov, Mod. Phys. Lett. **A 6** (1991) 1269

[27] E. Sezgin and K.S. Stelle, Class. Quantum Grav. **10** (1993) 19

[28] S. Bellucci, E. Ivanov and S. Krivonos, Class. Quantum Grav. **10** (1993) 1711

[29] S. Bellucci, V. Gribanov, S. Krivonos and A. Pashnev, Phys. Lett. **A 191** (1994) 216