LABORATORI NAZIONALI DI FRASCATI
SIS – Pubblicazioni

LNF-96/036 (P)
29 Luglio 1996

INFNNA-IV-96-29

CP Violation in Kaon Decays*

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Abstract

We review the Standard Model predictions of CP violation in kaon decays. We present an elementary introduction to Chiral Perturbation Theory, four–quark effective hamiltonians and the relation among them. Particular attention is devoted to $K \to 3\pi$, $K \to 2\pi\gamma$ and $K \to \pi \bar{f} f$ decays.

PACS N.: 13.20.Eb; 11.30.Er

Submitted to Int. Jou. Mod. Phys. A

* Work partially supported by EEC Human Capital and Mobility Program –CT920026

- The global symmetry G can be promoted to a local one modifying the transformation laws of l_μ and r_μ in

$$\begin{aligned} r_\mu = v_\mu + a_\mu &\xrightarrow{G} g_R r_\mu g_R^{-1} + i g_R \partial_\mu g_R^{-1}, \\ l_\mu = v_\mu - a_\mu &\xrightarrow{G} g_L l_\mu g_L^{-1} + i g_L \partial_\mu g_L^{-1}. \end{aligned} \tag{4.15}$$

  By this way the gauge fields of electro–weak interactions (§ sect. 3) are automatically included in v_μ and a_μ:

$$\begin{aligned} v_\mu &= -eQA_\mu - \frac{g}{\cos\theta_W}\left[Q\cos(2\theta_W) - \frac{1}{6}\right] Z_\mu - \frac{g}{2\sqrt{2}}\left(T_+ W_\mu^+ + \text{h.c.}\right), \\ a_\mu &= +\frac{g}{\cos\theta_W}\left[Q - \frac{1}{6}\right] Z_\mu + \frac{g}{2\sqrt{2}}\left(T_+ W_\mu^+ + \text{h.c.}\right), \end{aligned} \tag{4.16}$$

$$\text{with} \qquad Q = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \text{and} \qquad T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.17}$$

  As a consequence, Green functions for processes with external photons, Z or W bosons, can be simply obtained as functional derivatives of $Z(v, a, \hat{s}, \hat{p})$.

The chiral realization of $\mathcal{L}_{QCD}(v, a, \hat{s}, \hat{p})$, at the lowest order in the derivative expansion, is obtained by $\tilde{\mathcal{L}}_S^{(2)}$ including external sources in a chiral invariant way. For what concerns spin–1 sources this is achieved by means of the 'minimal substitution':

$$\partial_\mu U \to D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu. \tag{4.18}$$

Non–minimal couplings, which could be built with the tensors

$$\begin{aligned} F_L^{\mu\nu} &= \partial^\mu l^\nu - \partial^\nu l^\mu - i\left[l^\mu, l^\nu\right], \\ F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i\left[r^\mu, r^\nu\right], \end{aligned} \tag{4.19}$$

are absent at the lowest order since[d]

$$\begin{aligned} U &\qquad O(p^0), \\ u^\mu, a^\mu, v^\mu &\qquad O(p^1), \\ F_{L,R}^{\mu\nu} &\qquad O(p^2). \end{aligned} \tag{4.20}$$

For what concerns spin–0 sources, it is necessary to establish which is the order, in the derivative expansion, of $\hat{s}$ and $\hat{p}$. The most natural choice is given by[26,27]:

$$\hat{s}, \hat{p} \qquad O(p^2). \tag{4.21}$$

As we shall see in the following, this choice is well justified a posteriori by the Gell–Mann–Okubo relation.

[d] From now on, we shall indicate with $O(p^n) \sim O(\partial^n \phi)$ terms of order n in the derivative expansion.

4.2.1 The strong lagrangian.

Now we are able to write down the most general lagrangian invariant under G, of order p^2, which includes pseudoscalar mesons and external sources:

$$\mathcal{L}_S^{(2)} = \frac{F^2}{4}(D_\mu U D^\mu U^\dagger + \chi U^\dagger + U\chi^\dagger), \qquad \chi \doteq 2B(\hat{s} + i\hat{p}). \tag{4.22}$$

The two arbitrary constants (not fixed by the symmetry) F and B, which appear in $\mathcal{L}_S^{(2)}$, are related to two fundamental quantities: the pion decay constant F_π, defined by

$$\langle 0|\bar{\psi}\gamma^\mu\gamma_5\psi|\pi^+(p)\rangle \doteq i\sqrt{2}F_\pi p^\mu, \tag{4.23}$$

and the quark condensate $\langle 0|\bar{\psi}\psi|0\rangle$. Indeed, differentiating with respect to the external sources, we obtain

$$F_\pi = -i\frac{p_\mu}{\sqrt{2}p^2}\langle 0|\frac{\delta\mathcal{L}_S^{(2)}}{\delta a_\mu}|\pi^+(p)\rangle = F, \tag{4.24}$$

$$\langle 0|\bar{\psi}\psi|0\rangle = -\langle 0|\frac{\delta\mathcal{L}_S^{(2)}}{\delta\hat{s}}|0\rangle = -F^2B. \tag{4.25}$$

It is important to remark that relations (4.24–4.25) are exactly valid only in the chiral limit, in the real case ($m_q \neq 0$) are modified at order p^4 (§ sect. 4.3).

The pion decay constant is experimentally known from the process $\pi^+ \to \mu^+\nu$: $F_\pi = 92.4$ MeV, on the other hand the products Bm_u, Bm_d and Bm_s are fixed by the following identities:

$$\begin{aligned} M_{\pi^+}^2 &= (m_u + m_d)B, \\ M_{K^+}^2 &= (m_u + m_s)B, \\ M_{K^0}^2 &= (m_d + m_s)B. \end{aligned} \tag{4.26}$$

The analogous equation for $M_{\eta_8}^2$ contains no free parameter and give rise to a consistency relation:

$$3M_{\eta_8}^2 = 4M_K^2 - M_\pi^2, \tag{4.27}$$

the well–known Gell–Mann–Okubo relation[6,7].

Assuming that the quark condensate does not vanish in the chiral limit, i.e. that $B(m_q \to 0) \neq 0$, from relations (4.26) it is easier to understand why we have chosen $\hat{s} \sim O(p^2)$. This choice is justified a posteriori by Eq. (4.27) and a priori by lattice calculations of the ratio B/F (see references cited in Ref.[95]), nevertheless it is important to remark that it is an hypothesis which go beyond the fundamental assumptions of CHPT. Eq. (4.21) has also the big advantage to facilitate the power counting in the derivative expansion (this choice avoids Lorentz–invariant terms of order p^{2n+1}). The approach of Stern and Knecht[99], i.e. the hypothesis that the quark condensate could be very small, or even vanishing, in the chiral limit (so that $O(m_q^2)$ corrections to Eqs. (4.26) cannot

be neglected) gives rise to a large number of new operators for any fixed power of p and strongly reduces the predictive power of the theory.[e]

4.2.2 The non–leptonic weak lagrangian.

The lagrangian (4.22) let us to calculate at order p^2 Green functions for weak and electromagnetic transitions, beyond the strong ones, in processes with external gauge fields, like semileptonic kaon decays. However, the lagrangian (4.22) is not sufficient to describe non–leptonic decays of K mesons, since, as shown in sect. 3.2, in this case is not possible to trivially factorize strong–interaction effects. The correct procedure to describe these processes, is to build the chiral realization of the effective hamiltonian (3.30).

Under $SU(3)_L \times SU(3)_R$ transformations, the operators of Eq. (3.29) transform linearly in the following way:

$$\begin{array}{ll} O_1, O_2, O_9 & (8_L, 1_R) + (27_L, 1_R), \\ O_1^c, O_2^c, O_3, O_4, O_5, O_6 & (8_L, 1_R), \\ O_7, O_8 & (8_L, 8_R). \end{array} \tag{4.28}$$

Analogously to the case of light–quark mass terms, chiral operators which transform like the O_i can be built introducing appropriate scalar sources. As an example, to build the $(8_L, 1_R)$ operators, we introduce the source

$$\hat{\lambda} \xrightarrow{G} g_L \hat{\lambda} g_L^{-1} \tag{4.29}$$

and we consider all the operators, invariant under G, linear in λ (operators bilinear in λ correspond to terms of order G_F^2 in the effective hamiltonian). Successively, fixing the source to the constant value

$$\hat{\lambda} \to \lambda = \frac{1}{2}(\lambda_6 - i\lambda_7), \tag{4.30}$$

we select the $\Delta S = 1$ component of all possible $(8_L, 1_R)$ operators. For $(27_L, 1_R)$ terms the procedure is very similar, the only change is the source structure. On the other hand for $(8_L, 8_R)$ operators, generated by electromagnetic–penguin diagrams (§ sect. 3.2), is necessary to introduce two sources, corresponding to charged and neutral currents. The lowest order operators obtained by this procedure are[100,101]:

$$\begin{array}{lll} W_8^{(2)} = \langle \lambda L_\mu L^\mu \rangle & (8_L, 1_R) & O(p^2), \\ W_{27}^{(2)} = (L_\mu)_{23}(L^\mu)_{11} + \frac{2}{3}(L_\mu)_{21}(L^\mu)_{13} & (27_L, 1_R) & O(p^2), \\ W_8^{(0)} = F^2 \langle \lambda U^\dagger Q U \rangle & (8_L, 8_R) & O(p^0), \end{array} \tag{4.31}$$

where $L_\mu = u^\dagger u_\mu u$. Whereas singlets under $SU(3)_R$ are of order p^2, the $(8_L, 8_R)$ operator is of order p^0. This however is not a problem, since electromagnetic–penguin operators at the quark level (O_7 and O_8 in the basis (3.29)) are suppressed by a factor e^2 with respect

[e] See Ref.[95] for a complete discussion about the relation between 'standard' CHPT ($\hat{s} \sim O(p^2)$) and 'generalized' CHPT ($\hat{s} \sim O(p)$).

Contents

1 Introduction

Since 1949, when K mesons were discovered[1], kaon physics has represented one of the richest sources of information in the study of fundamental interactions.

One of the first ideas, originated by the study of K meson production and decays, was the Gell–Mann[2] and Pais[3] hypothesis of the 'strangeness' as a new quantum number. Almost at the same time, the famous '$\theta - \tau$ puzzle'[4] was determinant in suggesting to Lee and Yang[5] the revolutionary hypothesis of parity violation in weak interactions. Lately, in the sixties, K mesons played an important role in clarifying global symmetries of strong interactions, well before than QCD was proposed[6–8]. In the mean time they had a relevant role also in the formulation of the Cabibbo theory[9], which unified weak interactions of strange and non–strange particles. Finally, around 1970, the suppression of flavor changing neutral currents in kaon decays was one of the main reason which pushed Glashow, Iliopoulos and Maiani[10] to postulate the existence of the 'charm'. Hypothesis which was lately confirmed opening the way to the unification of quark and lepton electro–weak interactions.

In 1964 a completely unexpected revolution was determined by the Christenson, Cronin, Fitch and Turlay observation of $K_L \to 2\pi$ decay[11], i.e. by the discovery of a very weak interaction non invariant under CP transformations. Even if more than thirty years have passed by this famous experiment, the phenomenon of CP violation is still not completely clear and is one of the aspects which makes still very interesting the study of K mesons, both from the experimental and the theoretical point of view.

To date, in the framework of nuclear and subnuclear physics, there are no evidence of CP violation but in K_L decays, and within these processes all the observables indicate clearly only a CP violation in the mixing $K^0 - \bar{K}^0$. Nevertheless, as shown by Sakharov[12] in 1967, also the asymmetry between matter and antimatter in the universe can be considered as an indication of CP violation. Thus the study of this phenomenon has fundamental implications not only in particle physics but also in cosmology[13].

As it is well known, strong and electro–weak interactions seem to be well described within the so–called 'Standard Model', i.e. in the framework of a non–abelian gauge theory based on the $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry group (§ sect. 3). CP violation can be naturally generated in this model, both in the strong and in the electro–weak sector.

CP violation in the strong sector, though allowed from a theoretical point of view[14,15], from the experimental analysis of the neutron dipole moment turns out to be very suppressed. This suppression, which does not find a natural explanation in the Standard Model, is usually referred as the 'strong CP problem'[16]. One of the most appealing hypothesis to solve this problem is to extend the model including a new symmetry, which forbids (or drastically reduces) CP violation in the strong sector. However, all the proposals formulated in this direction have not found any experimental evidence yet[16].

CP violation in the electro–weak sector is generated by the Kobayashi–Maskawa mechanism[17]. This mechanism explains qualitatively the CP violation till now observed in the $K^0 - \bar{K}^0$ system but predicts also new phenomena not observed yet: CP violation in $|\Delta S| = 1$ transitions (measured with precision only in $K \to 2\pi$ decays, where turns out to be compatible with zero within two standard deviations[18]) and in B decays.

In the next years a remarkable experimental effort will be undertaken, both in K and in B meson physics, in order to seriously test the Standard Model mechanism of CP violation. Assuming there is no CP violation in the strong sector, in the framework of this model all the observables which violate CP depend essentially on one parameter (the phase of the Cabibbo–Kobayashi–Maskawa matrix). As a consequence, any new experimental evidence of CP violation could lead to interesting conclusions (even in minimal extensions of the model new phases are introduced and the relations among the observables are modified, see e.g. Refs.[19,20]). Obviously, to test the model seriously, is necessary to analyze with great care, from the theoretical point of view, all the predictions and the relative uncertainties for all the observables which will be measured.

In the framework of K mesons is not easy to estimate CP violating observables with great precision, since strong interactions are in a non–perturbative regime. In the most interesting case, i.e. in $K \to 2\pi$ decays, this problem has been partially solved by combining analytic calculations of the four–quark effective hamiltonian[21,22] with non–perturbative information on the matrix elements[23,24]. The latter have been obtained from lattice QCD results[24] or combining experimental information on $K \to 2\pi$ amplitudes and $1/N_c$ predictions[23]. Nevertheless in other channels, like $K \to 3\pi$ and $K \to 2\pi\gamma$ decays, the theoretical situation is less clear and there are several controversial statements in the literature.

The purpose of this review is to analyze in detail all the predictions for the observables which will be measured in the next years, trying, where possible, to relate them with those of $K \to 2\pi$. The predictions will be analyzed assuming the Cabibbo–Kobayashi–Maskawa matrix as the unique source of CP violation in the model (i.e. we shall assume that exists a symmetry which forbids CP violation in the strong sector). The tool that we shall use to relate between each other the different observables is the so–called Chiral Perturbation Theory[25–27] (§ sect. 4). This Theory, based on the hypothesis that the eight lightest pseudoscalar bosons (π, K and η) are Goldstone bosons[28], in the limit of vanishing light quark masses ($m_u = m_d = m_s = 0$), can be considered as the natural complement of lattice calculations. From one side, indeed, relates matrix elements of different processes, on the other side allows to calculate in a systematic way the absorptive parts of the amplitudes (typically not accessible from lattice simulations).

The paper is organized as follows: in the first section we shall discuss CP violation in kaon decays in a very phenomenological way, outlining the general features of the problem; in the second section the mechanism of CP violation in the Standard Model will be analyzed, both in K and in B mesons, with particular attention to the estimates of $K \to 2\pi$ parameters ϵ and ϵ'; in the third section we shall introduce Chiral Perturbation Theory. These first three sections represent three independent and complementary introductions to the problem. In the following four sections we shall discuss the estimates of several

CP violating observables in kaon decays different than $K \to 2\pi$. The results will be summarized in the conclusions.

2 Phenomenology of CP violation in kaon decays.

2.1 Time evolution of the $K^0 - \bar{K}^0$ system.

The state $|\Psi\rangle$ which describes a neutral kaon is in general a superposition of $|K^0\rangle$ and $|\bar{K}^0\rangle$ states, with definite strangeness, eigenstates of strong and electromagnetic interactions[a].

Introducing the vector $\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}$, so that $|\Psi\rangle = \Psi_1|K^0\rangle + \Psi_2|\bar{K}^0\rangle$, the time evolution of $|\Psi\rangle$, in the particle rest frame, is given by:

$$i\frac{\partial}{\partial t}\Psi(t) = H\Psi(t) = (M - \frac{i}{2}\Gamma)\Psi(t), \tag{2.1}$$

where M and Γ are 2×2 hermitian matrices with positive eigenvalues. Denoting by $\lambda_\pm$ and $\Psi_\pm$ the eigenvalues and eigenvectors of H, respectively, we have:

$$|\Psi(t)\rangle = c_+ e^{-i\lambda_+ t}|\Psi_+\rangle + c_- e^{-i\lambda_- t}|\Psi_-\rangle. \tag{2.2}$$

Under the discrete symmetries P, C and T (parity, charge conjugation and time reversal), strangeness eigenstates transform in the following way[29,31]:

$$\begin{aligned} P|K^0\rangle &= -|K^0\rangle, & P|\bar{K}^0\rangle &= -|\bar{K}^0\rangle, \\ C|K^0\rangle &= e^{i\alpha_c}|\bar{K}^0\rangle, & C|\bar{K}^0\rangle &= e^{-i\alpha_c}|K^0\rangle, \\ T|K^0\rangle &= e^{i(\theta-\alpha_c)}|K^0\rangle, & T|\bar{K}^0\rangle &= e^{i(\theta+\alpha_c)}|\bar{K}^0\rangle, \end{aligned} \tag{2.3}$$

where α_c and θ are arbitrary phases[b]. Since strangeness is conserved in strong and electromagnetic interactions, is possible to redefine $|K^0\rangle$ and $|\bar{K}^0\rangle$ phases in the following way:

$$|K^0\rangle \longrightarrow e^{-i\alpha\hat{S}}|K^0\rangle = e^{-i\alpha}|K^0\rangle \qquad |\bar{K}^0\rangle \longrightarrow e^{-i\alpha\hat{S}}|\bar{K}^0\rangle = e^{i\alpha}|\bar{K}^0\rangle, \tag{2.4}$$

where $\hat{S}$ is the operator which define the strangeness[c]. Thus the evolution matrix H is defined up to the transformation

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} H_{11} & e^{2i\alpha}H_{12} \\ e^{-2i\alpha}H_{21} & H_{22} \end{bmatrix}. \tag{2.5}$$

Choosing $\alpha = (\pi - \alpha_c)/2$, the transformations of $|K^0\rangle$ and $|\bar{K}^0\rangle$ under CP are given by:

$$CP|K^0\rangle = |\bar{K}^0\rangle, \tag{2.6}$$

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[a] For excellent reviews about the arguments presented in this section and, more in general, about CP violation in kaon decays see Refs.[29–39]

[b] Note that $T(\eta|\Psi\rangle) = \eta^* T|\Psi\rangle$.

[c] $\hat{S}|K^0\rangle = +|K^0\rangle$, $\hat{S}|\bar{K}^0\rangle = -|\bar{K}^0\rangle$, $\hat{S}|\text{non-strange particles}\rangle = 0$.

whereas those under $\Theta = CPT$ remain unchanged:

$$\Theta|K^0\rangle = -e^{i\theta}|\bar{K}^0\rangle, \qquad \Theta|\bar{K}^0\rangle = -e^{i\theta}|K^0\rangle. \tag{2.7}$$

Using this phase choice, the transformation laws of H under CP and CPT are given by:

$$CP \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} (CP)^{-1} = \begin{bmatrix} H_{22} & H_{21} \\ H_{12} & H_{11} \end{bmatrix}, \tag{2.8}$$

$$\Theta \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \Theta^{-1} = \begin{bmatrix} H_{22} & H_{12} \\ H_{21} & H_{11} \end{bmatrix}. \tag{2.9}$$

From Eqs. (2.8-2.9) and (2.5) we can easily deduce the conditions for H to be invariant under CP and CPT. The time evolution matrix is invariant under CPT if $H_{11} = H_{22}$, i.e.[d]

$$M_{11} = M_{22} \qquad \text{and} \qquad \Gamma_{11} = \Gamma_{22}, \tag{2.10}$$

no matter of the phase choice in (2.5). Eq. (2.10) is a necessary but not sufficient condition to have invariance under CP. To insure CP invariance is necessary to constraint also the off–diagonal elements of H. The condition following from (2.8), i.e. $M_{12} - i\Gamma_{12}/2 = M_{12}^* - i\Gamma_{12}^*/2$, depends on the phase choice in Eq. (2.5), indeed we can always choose α in such a way to make M_{12} or Γ_{12} real. The supplementary condition to insure CP invariance, independently from the phase choice in (2.5), is:

$$\arg\left(\frac{M_{12}}{\Gamma_{12}}\right) = 0. \tag{2.11}$$

For now on we shall consider the $K^0 - \bar{K}^0$ system assuming CPT invariance. In this case the matrix H can be written as

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{11} \end{bmatrix} \tag{2.12}$$

and solving the eigenvalue equation one gets:

$$\lambda_\pm = H_{11} \pm \sqrt{H_{12}H_{21}}, \qquad \Psi_\pm \propto \begin{bmatrix} 1 \\ \pm\sqrt{H_{21}/H_{12}} \end{bmatrix}. \tag{2.13}$$

In the limit where also CP is an exact symmetry of the $K^0 - \bar{K}^0$ system, i.e. the CP operator commutes with H, from Eq. (2.11) follows that

$$\sqrt{\frac{H_{12}}{H_{21}}} = \sqrt{\frac{M_{12} - i\Gamma_{12}/2}{M_{12}^* - i\Gamma_{12}^*/2}} \tag{2.14}$$

[d] Note that M_{11} and Γ_{11} are real and positive, since M and Γ are hermitian and positive.

is just a phase factor and, with an opportune phase transformation (2.5), the normalized eigenvectors (conventionally called $|K_1\rangle$ e $|K_2\rangle$) áre given by:

$$|K_1\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle + |\bar{K}^0\rangle\right), \tag{2.15}$$

$$|K_2\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\bar{K}^0\rangle\right). \tag{2.16}$$

On the other hand, if CP is an approximate symmetry of the $K^0 - \bar{K}^0$ system, the eigenvectors of H are usually written as

$$\begin{aligned} |K_S\rangle &= \frac{1}{\sqrt{1+|\tilde{\epsilon}|^2}}\left(|K_1\rangle + \tilde{\epsilon}|K_2\rangle\right) \\ &= \frac{1}{\sqrt{2(1+|\tilde{\epsilon}|^2)}}\left((1+\tilde{\epsilon})|K^0\rangle + (1-\tilde{\epsilon})|\bar{K}^0\rangle\right), \\ |K_L\rangle &= \frac{1}{\sqrt{1+|\tilde{\epsilon}|^2}}\left(|K_2\rangle + \tilde{\epsilon}|K_1\rangle\right) \\ &= \frac{1}{\sqrt{2(1+|\tilde{\epsilon}|^2)}}\left((1+\tilde{\epsilon})|K^0\rangle - (1-\tilde{\epsilon})|\bar{K}^0\rangle\right), \end{aligned} \tag{2.17}$$

where $\tilde{\epsilon}$ is given by:

$$\frac{1+\tilde{\epsilon}}{1-\tilde{\epsilon}} = \sqrt{\frac{M_{12} - i\Gamma_{12}/2}{M_{12}^* - i\Gamma_{12}^*/2}}\,. \tag{2.18}$$

The $\tilde{\epsilon}$ parameter is not an observable quantity and indeed is phase convention dependent. On the other hand, the quantity

$$\frac{\Re e(\tilde{\epsilon})}{1+|\tilde{\epsilon}|^2} = \frac{\Im m(\Gamma_{12})\Re e(M_{12}) - \Im m(M_{12})\Re e(\Gamma_{12})}{4|M_{12}|^2 + |\Gamma_{12}|^2}, \tag{2.19}$$

that vanishes if Eq. (2.11) is satisfied[e], is phase independent and possibly observable.

The two eigenstates of H have different eigenvalues also if CP is an exact symmetry[f]:

$$\begin{aligned} \lambda_S &= M_{11} - \frac{i}{2}\Gamma_{11} + \left(M_{12} - \frac{i}{2}\Gamma_{12}\right)\left(\frac{1+\tilde{\epsilon}}{1-\tilde{\epsilon}}\right) \\ &\simeq M_{11} + \Re e M_{12} - \frac{i}{2}\left(\Gamma_{11} + \Re e\Gamma_{12}\right), \end{aligned} \tag{2.20}$$

$$\begin{aligned} \lambda_L &= M_{11} - \frac{i}{2}\Gamma_{11} - \left(M_{12} - \frac{i}{2}\Gamma_{12}\right)\left(\frac{1+\tilde{\epsilon}}{1-\tilde{\epsilon}}\right) \\ &\simeq M_{11} - \Re e M_{12} - \frac{i}{2}\left(\Gamma_{11} - \Re e\Gamma_{12}\right). \end{aligned} \tag{2.21}$$

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[e] In the r.h.s. of Eq. (2.19) we neglect terms of order $\Re e(\tilde{\epsilon})^2$.

[f] In the r.h.s. of Eqs. (2.20–2.21) we neglect the imaginary parts of M_{12} and Γ_{12}.

From the experimental data[18] follows:

$$\begin{array}{rcccl} M_{11} & = & (M_L + M_S)/2 & = & (497.672 \pm 0.031)\ \text{MeV}, \\ -\Re e M_{12} & \simeq & (M_L - M_S)/2 & = & (1.755 \pm 0.009) \times 10^{-12}\ \text{MeV}, \\ \Gamma_{11} + \Re e \Gamma_{12} & \simeq & \Gamma_S & = & (7.374 \pm 0.010) \times 10^{-12}\ \text{MeV}, \\ \Gamma_{11} - \Re e \Gamma_{12} & \simeq & \Gamma_L & = & (1.273 \pm 0.010) \times 10^{-14}\ \text{MeV}. \end{array} \tag{2.22}$$

The big difference among M_{11} and the other matrix elements of H can be simply explained: whereas M_{11} is dominated by the strong self–interaction of a strange meson, the other terms, connecting states with different strangeness, are due to weak interactions. M_{12}, in particular, is determined by the $|\Delta S| = 2$ transition amplitude which connects K^0 and $\bar{K}^0$, whereas Γ_{11} and Γ_{12} are given by the product of two $|\Delta S| = 1$ weak transitions (those responsible of K^0 and $\bar{K}^0$ decays). Assuming that $|\Delta S| = 2$ transitions are generated by the product of two $|\Delta S| = 1$ transitions, then is natural to expect[9] (§ sect. 3)

$$M_{12} \sim \Gamma_{11} \sim \Gamma_{12} \sim \frac{G_F^2 M_P^4 \sin^2\theta}{(2\pi)^4} M_K \sim 10^{-12}\text{MeV}. \tag{2.23}$$

2.2 $K \to 2\pi$ decays.

If CP was an exact symmetry of the kaon system, then the $|K_2\rangle$ (CP-negative eigenstate) should not decay in a two–pion final state (eigenstate of CP with positive eigenvalue). However, experiments have shown that both eigenstates of H decay into two–pion final states[11,18]. The $|K_S\rangle$ decays into $|2\pi\rangle$ almost 100% of the times, whereas the $|K_L\rangle$ has a branching ratio in this channel about 3 order of magnitude smaller.

To analyze these decays more in detail, is convenient to introduce the isospin decomposition of $K \to 2\pi$ amplitudes. Denoting by S_{strong} the S–matrix of strong interactions (which we assume invariant under isospin transformations), we define the S–wave 're–scattering' phases of a two–pion state with definite isospin I as:

$${}_{\text{out}}\langle 2\pi, I | 2\pi, I\rangle_{\text{in}} = \langle 2\pi, I | S_{\text{strong}} | 2\pi, I\rangle \doteq e^{i2\delta_I}. \tag{2.24}$$

With this definition, $K^0 \to 2\pi$ transition amplitudes can be written in the form

$${}_{\text{out}}\langle 2\pi, I | H_{\text{weak}} | K^0\rangle_{\text{in}} \doteq \sqrt{\frac{3}{2}} A_I e^{i\delta_I}, \tag{2.25}$$

where H_{weak} indicates the weak hamiltonian of $|\Delta S| = 1$ transitions. Assuming CPT invariance, from Eqs. (2.24-2.25) it follows[32]:

$${}_{\text{out}}\langle 2\pi, I | H_{\text{weak}} | \bar{K}^0\rangle_{\text{in}} = \sqrt{\frac{3}{2}} A_I^* e^{i\delta_I}. \tag{2.26}$$

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[9] We denote by G_F, M_P and θ the Fermi constant, the proton mass and the Cabibbo angle, respectively.

The two–pion isospin states allowed in S–wave are $I = 0$ and $I = 2$, defining $A(K \to 2\pi) = {}_{\rm out}\langle 2\pi|H_{\rm weak}|K\rangle_{\rm in}$ and using the appropriate Clebsh–Gordan coefficients, we find:[h]

$$\begin{aligned} A(K^0 \to \pi^+\pi^-) &= A_0 e^{i\delta_0} + \frac{1}{\sqrt{2}} A_2 e^{i\delta_2}, \\ A(K^0 \to \pi^0\pi^0) &= A_0 e^{i\delta_0} - \sqrt{2} A_2 e^{i\delta_2}, \\ A(K^+ \to \pi^+\pi^0) &= \frac{3}{2} A_2 e^{i\delta_2}. \end{aligned} \tag{2.27}$$

$A(\bar{K}^0 \to 2\pi)$ amplitudes are obtained from (2.28) with the simple substitution $A_I \to A_I^*$. If CP commuted with $H_{\rm weak}$ then we should have, up to a phase factor, $A(\bar{K}^0 \to 2\pi) = A(K^0 \to 2\pi)$. Thus, analogously to Eq. (2.11), which states the CP–invariance condition of $\Delta S = 2$ amplitudes respect to $\Delta S = 1$ ones, the CP–invariance condition among the two A_I amplitudes is given by:

$$\arg\left(\frac{A_2}{A_0}\right) = 0. \tag{2.28}$$

At this point is convenient to make a choice on the arbitrary phase of the weak amplitudes. Historically, a very popular choice is the famous Wu–Yang convention[40]:

$$\arg(A_0) = 0. \tag{2.29}$$

Since A_0 is dominant with respect to the other $A(K^0, \bar{K}^0 \to f)$ amplitudes, this convention is useful because from the unitarity relation

$$\Gamma_{12} = 2\pi \sum_n {}_{\rm in}\langle \bar{K}^0|H_{\rm weak}|n\rangle_{\rm out}\, {}_{\rm out}\langle n|H_{\rm weak}|K^0\rangle_{\rm in}\delta(M_K - E_n) \tag{2.30}$$

$$\simeq 2\pi \sum_{2\pi(I=0)} {}_{\rm in}\langle \bar{K}^0|H_{\rm weak}|2\pi\rangle_{\rm out}\, {}_{\rm out}\langle 2\pi|H_{\rm weak}|K^0\rangle_{\rm in}\delta(M_K - E_{2\pi}), \tag{2.31}$$

it follows

$$\arg(\Gamma_{12}) \simeq 2\arg(A_0), \tag{2.32}$$

thus Eq. (2.29) implies also $\arg(\Gamma_{12}) \simeq 0$. By this way all weak phases are 'rotated' on suppressed amplitudes, like A_2 and M_{12}. From (2.18–2.19) follows also

$$\begin{aligned} \arg\left(\widetilde{\epsilon}|_{\rm WY}\right) &= -\arctan\left[2\frac{\Re e(M_{12})}{\Re e(\Gamma_{12})}\right] \\ &= \arctan\left[2\frac{M_L - M_S}{\Gamma_L - \Gamma_S}\right] = (43.6 \pm 0.1)^\circ. \end{aligned} \tag{2.33}$$

However, in the following we shall not adopt the Wu–Yang phase convention, which is not the most natural choice in the Standard Model (§ sect. 3), but we shall impose only

$$\arg(A_0) \ll 1, \tag{2.34}$$

[h] We assume $\Delta I \leq 3/2$.

in order to treat as perturbations weak-amplitude phases.

From the experimental information on $\Gamma(K_S \to 2\pi)$ and $\Gamma(K^+ \to \pi^+\pi^0)$[18], neglecting CP violating effects, one gets[i]

$$\Re e(A_0) = (2.716 \pm 0.007) \times 10^{-4} \text{ MeV}, \tag{2.35}$$

$$\omega^{-1} = \frac{\Re e(A_0)}{\Re e(A_2)} = (22.2 \pm 0.1) \quad \text{and} \quad (\delta_0 - \delta_2) = (45 \pm 6)^\circ. \tag{2.36}$$

As anticipated, $\Re e(A_0) \gg \Re e(A_2)$, i.e. the $\Delta I = 1/2$ transition amplitude is dominant with respect to the $\Delta I = 3/2$ one ('$\Delta I = 1/2$ rule').

Conventionally, in order to study CP violating effects, the following parameters are introduced:

$$\eta_{+-} \doteq \frac{A(K_L \to \pi^+\pi^-)}{A(K_S \to \pi^+\pi^-)} \doteq \epsilon + \epsilon', \tag{2.37}$$

$$\eta_{00} \doteq \frac{A(K_L \to \pi^0\pi^0)}{A(K_S \to \pi^0\pi^0)} \doteq \epsilon - 2\epsilon'. \tag{2.38}$$

Using Eqs. (2.17) and (2.27), and neglecting terms proportional to $\Im m(A_I)^2 / \Re e(A_I)^2$ and $\omega^2 \Im m(A_I)/\Re e(A_I)$, ϵ ed ϵ' are given by:

$$\epsilon = \tilde{\epsilon} + i\frac{\Im m(A_0)}{\Re e(A_0)} = \tilde{\epsilon}|_{\text{WY}} \tag{2.39}$$

$$\epsilon' = i\frac{e^{i(\delta_2 - \delta_0)}}{\sqrt{2}}\omega \left[\frac{\Im m(A_2)}{\Re e(A_2)} - \frac{\Im m(A_0)}{\Re e(A_0)}\right]. \tag{2.40}$$

Both ϵ and ϵ' are measurable quantities with definite phases:

$$\arg(\epsilon) = \arg\left(\tilde{\epsilon}|_{\text{WY}}\right) = (43.6 \pm 0.1)^\circ, \tag{2.41}$$

$$\arg(\epsilon') = (\delta_2 - \delta_0 + \frac{\pi}{2}) = (45 \pm 6)^\circ, \tag{2.42}$$

and both vanish in the exact-CP limit: ϵ vanishes if Eq. (2.11) if satisfied, whereas ϵ' vanishes if Eq. (2.28) is satisfied. The ϵ parameter is referred as the 'indirect' CP–violating parameter, since Eq. (2.11) could be violated by a non–zero weak phase in $|\Delta S| = 2$ amplitudes only. On the other hand, ϵ' is called the 'direct' CP–violating parameter, since a non–zero weak phase in $|\Delta S| = 1$ amplitudes is necessary to violate Eq. (2.28).

It is interesting to note that the first identity in Eq. (2.41) could be violated if CPT was not an exact symmetry. Thus, given that $\arg\left(\tilde{\epsilon}|_{\text{WY}}\right) \simeq \arg(\epsilon')$, a large value of $\Im m(\epsilon'/\epsilon)$ ($\Im m(\epsilon'/\epsilon) \gtrsim \Re e(\epsilon'/\epsilon)$) would be a signal of CPT violation.[39]

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[i] Due to small isospin–breaking effects, amplified by the suppression of A_2 (§ sect. 3.3), the value of $(\delta_2 - \delta_0)$ extracted by $K \to 2\pi$ data is not reliable. We shall adopt the prediction of Gasser and Meißner[41] that is in good agreement with the value of $(\delta_2 - \delta_0)$ extrapolated from $\pi - N$ data[42].

From the analysis of experimental data on $K_L \to 2\pi$ decays[18] follows:

$$|\epsilon| = (2.266 \pm 0.017) \times 10^{-3} \qquad \arg(\epsilon) = (44.0 \pm 0.7)^\circ, \tag{2.43}$$

$$\Re e\left(\frac{\epsilon'}{\epsilon}\right) = (1.5 \pm 0.8) \times 10^{-3} \qquad \Im m\left(\frac{\epsilon'}{\epsilon}\right) = (1.7 \pm 1.7) \times 10^{-2}. \tag{2.44}$$

Whereas $|\epsilon|$ is definitely different from zero, $|\epsilon'|$ is compatible with zero within two standard deviations. This implies that the so–called 'super–weak' model proposed by Wolfenstein more than 30 years ago[43,44], a model where CP violation is supposed to be generated by a very weak ($\sim G_F^2 M_p^4 \sin^2\theta/(2\pi)^4$) new $|\Delta S| = 2$ interaction, is still compatible with the experimental data. Of course there is no confirmation of the Standard Model mechanism (§ sect. 3), which predicts also direct CP violation. However, as we shall see in the next section, the fact that $|\epsilon'|$ is much smaller of $|\epsilon|$ could be explained also in the Standard Model where, for large values of the top quark mass, the weak phases of A_0 and A_2 accidentally tend to cancel. Then, within the Standard Model, the fundamental condition for the observability of direct CP violation tends to lack in $K \to 2\pi$ transitions:

[1] *A necessary condition in order to observe direct CP violation in $(K^0, \bar{K}^0) \to f$ transitions, is the presence of at least two weak amplitudes with different (weak) phases.*

2.3 Semileptonic decays.

Up to now CP violation has been observed in the following processes: $K_L \to 2\pi$, $K_L \to \pi^+\pi^-\gamma$ and $K_L \to \pi l \nu$ ($l = \mu, e$).

As we shall discuss better in sect. 6, the amount of CP violation till now observed in $K_L \to \pi^+\pi^-\gamma$ is generated only by the bremsstrahlung of the corresponding non–radiative transition ($K_L \to \pi^+\pi^-$) and does not carry any new information.

In the $K_L \to \pi l \nu$ case, the selection rule $\Delta S = \Delta Q$, predicted by the Standard Model and in perfect agreement with the data, allows the existence of a single weak amplitude.[j] Consequently, from the condition **[1]**, we deduce that within the Standard Model is impossible to observe direct CP violation in these channels.

The selection rule $\Delta S = \Delta Q$ implies:

$$A(K^0 \to \pi^+ l^- \bar{\nu}_l) = A(\bar{K}^0 \to \pi^- l^+ \nu_l) = 0, \tag{2.45}$$

thus defining

$$A(K^0 \to \pi^- l^+ \nu_l) \doteq A_l, \tag{2.46}$$

from CPT follows $A(\bar{K}^0 \to \pi^+ l^- \bar{\nu}_l) = A_l^*$ and the decay amplitudes for K_L and K_S are given by:

$$A(K_S \to \pi^+ l^- \bar{\nu}_l) = -A(K_L \to \pi^+ l^- \bar{\nu}_l) = \frac{1 - \tilde{\epsilon}}{2(1 + |\tilde{\epsilon}|^2)} A_l^*, \tag{2.47}$$

$$A(K_S \to \pi^- l^+ \nu_l) = A(K_L \to \pi^- l^+ \nu_l) = \frac{1 + \tilde{\epsilon}}{2(1 + |\tilde{\epsilon}|^2)} A_l. \tag{2.48}$$

---

[j] $\Delta S = -\Delta Q$ transitions are allowed in the Standard Model only at $O(G_F^2)$; explicit calculations[45,46] show a suppression factor of about $10^{-6} - 10^{-7}$.

If CP was an exact symmetry of the system we should have

$$\begin{aligned} |A(K_L \to \pi^+ l^- \bar{\nu}_l)| &= |A(K_L \to \pi^- l^+ \nu_l)| \\ &= |A(K_S \to \pi^+ l^- \bar{\nu}_l)| = |A(K_S \to \pi^- l^+ \nu_l)|, \end{aligned} \tag{2.49}$$

thus CP violation in these channel can be observed via the charge asymmetries

$$\delta_{L,S} = \frac{\Gamma(K_{L,S} \to \pi^- l^+ \nu_l) - \Gamma(K_{L,S} \to \pi^+ l^- \bar{\nu}_l)}{\Gamma(K_{L,S} \to \pi^- l^+ \nu_l) - \Gamma(K_{L,S} \to \pi^+ l^- \bar{\nu}_l)} = \frac{2\Re e(\tilde{\epsilon})}{1 + |\tilde{\epsilon}|^2} \tag{2.50}$$

sensible to indirect CP violation only.

The experimental data on δ_L is[18]:

$$\delta_L = (3.27 \pm 0.12) \times 10^{-3}, \tag{2.51}$$

in perfect agreement with the numerical value of $\Re e(\epsilon)$ obtained form Eq. (2.43).

2.4 Charged–kaon decays.

Charged–kaon decays could represent a privileged observatory for the study of direct CP violation since there is no mass mixing between K^+ and K^-.

Denoting by f a generic final state of K^+ decays, with $\bar{f}$ the charge conjugate and defining

$$_{\text{out}}\langle f|H_{\text{weak}}|K^+\rangle_{\text{in}} \doteq A_f \tag{2.52}$$

$$_{\text{out}}\langle \bar{f}|H_{\text{weak}}|K^-\rangle_{\text{in}} \doteq A_{\bar{f}}, \tag{2.53}$$

if CP commuted with H_{weak} then $|A_f| = |A_{\bar{f}}|$. Thus CP violation in this channels can be observed via the asymmetry

$$\Delta_f = \frac{|A_f| - |A_{\bar{f}}|}{|A_f| + |A_{\bar{f}}|} \tag{2.54}$$

which, differently form (2.50), is sensible to direct CP violation only.

To analyze more in detail under which conditions Δ_f can be different from zero, let's consider the case where $|f\rangle$ is not an eigenstate of strong interactions, but a superposition of two states with different re–scattering phases. In this case, separating strong and weak phases analogously to Eq. (2.25), we can write

$$A_f = a_f e^{i\delta_a} + b_f e^{i\delta_b}. \tag{2.55}$$

In addition, imposing CPT invariance, it is found

$$A_{\bar{f}} = a_f^* e^{i\delta_a} + b_f^* e^{i\delta_b}, \tag{2.56}$$

thus the charge asymmetry Δ_f is given by:

$$\Delta_f = \frac{2\Im m(a_f^* b_f)\sin(\delta_a - \delta_b)}{|a_f|^2 + |b_f|^2 + 2\Re e(a_f^* b_f)\cos(\delta_a - \delta_b)}. \tag{2.57}$$

As it appears from Eq. (2.57), in order to have Δ_f different from zero, is not only necessary to have two weak amplitudes (a_f e b_f) with different weak phases (as already stated in [1]), but is also necessary to have different strong re–scattering phases.

[2] *A necessary condition, in order to observe (direct) CP violation in $K^{\pm} \to (f, \overline{f})$ transitions, is the presence of at least two weak amplitudes with different (weak) phases and different (strong) re–scattering phases.*

As we shall see in the next sections, the condition [2] is not easily satisfied within the Standard Model, indeed CP violation in charged–kaon decays has not been observed yet. To be more specific, the most frequent decay channels (99,9% of the branching ratio) of $K^{\pm}$ mesons are the lepton channels $|l\nu\rangle$ and $|\pi l\nu\rangle$, together with the non–leptonic states $|2\pi\rangle$ and $|3\pi\rangle$. In the first two channels there is no strong re–scattering. In the $|2\pi\rangle$ case the re–scattering phase is unique (is a pure $I = 2$ state). Finally in the $|3\pi\rangle$ case there are different re–scattering phases but are suppressed, since the available phase space is very small (§ sect. 5).

3 CP violation in the Standard Model.

The 'Standard Model' of strong and electro–weak interactions is a non–abelian gauge theory based on the $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry group[a].

The $SU(3)_C$ subgroup is the symmetry of strong (or color) interactions, realized á la Wigner–Weyl, i.e. with a vacuum state invariant under $SU(3)_C$. On the other hand, the $SU(2)_L \times U(1)_Y$ symmetry, which rules electro–weak interactions[52–54], is realized á la Nambu–Goldstone[55,28], with a vacuum state invariant only under the subgroup $U(1)_Q$ of electromagnetic interactions.

The interaction lagrangian of fermion fields with $SU(2)_L \times U(1)_Y$ electro–weak gauge bosons is given by:

$$\begin{aligned} \mathcal{L}_{e-w} &= \sum_{\alpha} \bar{Q}_L^{(\alpha)} \gamma^{\mu} \left(g T_i W_{\mu}^i + \frac{1}{2} g' Y B_{\mu} \right) Q_L^{(\alpha)} \\ &+ \sum_{\alpha} \bar{\Psi}_L^{(\alpha)} \gamma^{\mu} \left(g T_i W_{\mu}^i + \frac{1}{2} g' Y B_{\mu} \right) \Psi_L^{(\alpha)} \\ &+ \sum_{f} \bar{f}_R \gamma^{\mu} \left(\frac{1}{2} g' Y B_{\mu} \right) f_R, \end{aligned} \tag{3.1}$$

where $g, T_i, W_{\mu}^i (i = 1, 3)$ and g', Y, B_{μ} are the coupling constants, the generators and the gauge fields of $SU(2)_L$ and $U(1)_Y$, respectively. The index α can assume 3 values, according to the quark or lepton family to which is referred, whereas f indicates the sum over all right–handed fermions. As anticipated, the $SU(2)_L \times U(1)_Y$ symmetry is spontaneously broken in such a way that the photon field A_{μ} remains massless. Introducing the weak angle θ_W, defined by the relation $\tan\theta_W = g'/g$, and the field Z_{μ} (combination of W_{μ}^3 and

[a] For excellent phenomenological reviews on gauge theories and, in particular, on the Standard Model see Refs.[47–51].

	leptons					
	$l-$ type			$\nu-$ type		
	e	μ	τ	ν_e	ν_μ	ν_τ
m(MeV)	0.511	105.7	1777	$< 7 \times 10^{-6}$	< 0.27	< 31
Q	-1			0		
	quarks					
	$d-$ type			$u-$ type		
	d	s	b	u	c	t
m(MeV)	~ 10	~ 150	~ 4300	~ 5	~ 1200	$\sim 1.7 \times 10^5$
Q	$-1/3$			$+2/3$		

Table 1: Masses and electric charges of quarks and leptons (see Ref. [18] for a discussion about the definition of quark masses).

B_μ orthogonal to A_μ):

$$\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} = \begin{pmatrix} \sin\theta_W & \cos\theta_W \\ \cos\theta_W & -\sin\theta_W \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix}, \tag{3.2}$$

Eq. (3.1) becomes:

$$\begin{aligned} \mathcal{L}_{e-w} &= e\sum_f \bar{f}\gamma^\mu Q f A_\mu \\ &+ \frac{g}{\cos\theta_W}\sum_f \bar{f}\gamma^\mu\left(Q_Z^V - Q_Z^A\gamma_5\right) f Z_\mu \\ &+ \frac{g}{2\sqrt{2}}\left[\sum_\alpha \bar{u}^{(\alpha)}\gamma^\mu(1-\gamma_5)\,d^{(\alpha)}W_\mu^+ + \text{h.c.}\right] \\ &+ \frac{g}{2\sqrt{2}}\left[\sum_\alpha \bar{\nu}^{(\alpha)}\gamma^\mu(1-\gamma_5)\,l^{(\alpha)}W_\mu^+ + \text{h.c.}\right], \end{aligned} \tag{3.3}$$

where $e = g'\cos\theta_W$, $Q_Z^V = T_3 - Q\sin^2\theta_W$ and $Q_Z^A = T_3$.

As in the case of gauge boson masses, quark and lepton masses (§ tab. 1) are dynamically generated by the Yukawa coupling of these fields with the scalars that spontaneously broke $SU(2)_L \times U(1)_Y$ symmetry. The effective lagrangian for quark and lepton masses can be written in the following way

$$\mathcal{L}_m^{eff} = \bar{U}M_U U + \bar{D}M_D D + \bar{L}M_L L, \tag{3.4}$$

where

$$U = \begin{pmatrix} u \\ c \\ t \end{pmatrix}, \qquad D = \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \qquad L = \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}, \tag{3.5}$$

$M_U = \mathrm{diag}\{m_u, m_c, m_t\}$, $M_D = \mathrm{diag}\{m_d, m_s, m_b\}$ and $M_L = \mathrm{diag}\{m_e, m_\mu, m_\tau\}$ (we assume massless neutrinos). The fermion fields which appear in Eq. (3.1) are not necessarily eigenstates of mass matrices. Introducing the unitary matrices V_U, V_D and V_L, so that

$$\begin{pmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{pmatrix} = V_U U, \qquad \begin{pmatrix} d^{(1)} \\ d^{(2)} \\ d^{(3)} \end{pmatrix} = V_D D, \qquad \text{and} \qquad \begin{pmatrix} l^{(1)} \\ l^{(2)} \\ l^{(3)} \end{pmatrix} = V_L L, \tag{3.6}$$

thus writing electro–weak eigenstates in terms of mass eigenstates, Eq. (3.1) becomes

$$\begin{aligned} \mathcal{L}_{e-w} &= e \sum_f \bar{f} \gamma^\mu Q f A_\mu \\ &+ \frac{g}{\cos\theta_W} \sum_f \bar{f} \gamma^\mu \left(Q_Z^V + Q_Z^A \gamma_5 \right) f Z_\mu \\ &+ \frac{g}{2\sqrt{2}} \left[\bar{U} (V_U^\dagger V_D) \gamma^\mu (1 - \gamma_5) D W_\mu^+ + \mathrm{h.c.} \right] \\ &+ \frac{g}{2\sqrt{2}} \left[\sum_\alpha \bar{N} \gamma^\mu (1 - \gamma_5) L W_\mu^+ + \mathrm{h.c.} \right], \end{aligned} \tag{3.7}$$

where

$$N = V_L^\dagger \begin{pmatrix} \nu^{(1)} \\ \nu^{(2)} \\ \nu^{(2)} \end{pmatrix} \doteq \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}. \tag{3.8}$$

Since fermion mass matrices commute with electric charge and with hypercharge (Y), the only difference among Eqs. (3.1) and (3.7) is the presence of the unitary matrix $V_U^\dagger V_D$.

As we have seen in the previous section, a necessary condition to induce CP violation in kaon decays is the presence of weak amplitudes with different weak phases. In the Standard Model framework, this condition is equivalent to the requirement of complex coupling constants in $\mathcal{L}_{e-w}$. These complex couplings must not to be cancelled by a redefinition of field phases and leave $\mathcal{L}_{e-w}$ hermitian. From these two conditions follows that the only coupling constants of $\mathcal{L}_{e-w}$ which can be complex are the matrix elements of $V_U^\dagger V_D$. This matrix, known as the Cabibbo–Kobayashi–Maskawa (CKM) matrix[9,10,17], is the only source of CP violation in the electro–weak sector of the Standard Model.

As anticipated in the introduction, in this review we will not discuss about possible effects due to CP violation in the strong sector.

3.1 The CKM matrix.

In the general case of N_f quark families, $U_{CKM} \doteq V_U^\dagger V_D$ is a $N_f \times N_f$ unitary matrix. The number of independent phases, n_P, and real parameters, n_R, in a $N_f \times N_f$ unitary matrix is given by:

$$n_P = \frac{N_f(N_f+1)}{2} \qquad \text{and} \qquad n_R = \frac{N_f(N_f-1)}{2}. \tag{3.9}$$

Since the $2N_f - 1$ relative phases of the $2N_f$ fermion fields are arbitrary, the non trivial phases in U_{CKM} are

$$n_P^{CP} = \frac{N_f(N_f+1)}{2} - 2N_f + 1 = \frac{(N_f-1)(N_f-2)}{2}. \tag{3.10}$$

In the minimal Standard Model $N_f = 3$, thus the non–trivial phase is unique.

Parametrizing U_{CKM} in terms of 3 angles and one phase, it is possible to write[56–58]:

$$U_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{3.11}$$

$$= \begin{pmatrix} C_\theta C_\sigma & S_\theta C_\sigma & S_\sigma e^{-i\delta} \\ -S_\theta C_\tau - C_\theta S_\sigma S_\tau e^{i\delta} & C_\theta C_\tau - S_\theta S_\sigma S_\tau e^{i\delta} & C_\sigma S_\tau \\ S_\theta S_\tau - C_\theta S_\sigma C_\tau e^{i\delta} & -C_\theta S_\tau - S_\theta S_\sigma C_\tau e^{i\delta} & C_\sigma C_\tau \end{pmatrix}, \tag{3.12}$$

where C_θ and S_θ denote sine and cosine of the Cabibbo angle[9], $C_\sigma = \cos\sigma$, $S_\sigma = \sin\sigma$, etc...

It is important to remark that the CKM phase would not be observable if the mass matrices M_D and M_U, introduced in Eq. (3.4), had degenerate eigenvalues[56,59]. As an example, if m_c was equal to m_t, then V_U would be defined up to the transformation

$$V_U \to \begin{pmatrix} 1 & 0 \\ 0 & U(\bar\theta, \bar\delta_i) \end{pmatrix} V_U, \tag{3.13}$$

where $U(\bar\theta, \bar\delta_i)$ is a unitary 2×2 matrix. Thus adjusting the independent parameters of $U(\bar\theta, \bar\delta_i)$ (1 angle and 3 phases) the phase of U_{CKM} could be rotated away. In other words, the phenomenon of CP violation is intimately related not only to the symmetry breaking mechanism of the electro–weak symmetry (or better of the chiral symmetry, § sect. 4), but also to the breaking mechanism of the flavor symmetry (the global $SU(N_f)$ symmetry that we should have if all the quark masses were equal). The two mechanisms coincide only in the minimal Standard Model.

There is an empirical hierarchy among CKM matrix elements ($S_\sigma \ll S_\tau \ll S_\theta \ll 1$), which let us express them in terms of a single 'scale–parameter' $\lambda = S_\theta \simeq 0.22$ and three coefficients of order 1 (A, ρ ed η)[57]. Defining

$$S_\tau = A\lambda^2, \qquad S_\sigma = A\lambda^3\sigma, \qquad \sigma e^{i\delta} = \rho + i\eta \tag{3.14}$$

and expanding Eq. (3.12) in powers of λ up to $O(\lambda^3)$ terms, one finds:

$$U_{CKM} \simeq \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}. \tag{3.15}$$

An interesting quantity for the study of CP violation is the 'so–called' J_{CP} parameter[59]:

$$J_{CP} = \Im m \left(V_{ai} V_{bj} V_{aj}^* V_{bi}^* \right). \tag{3.16}$$

Process	Parameter	Value
$K \to \pi l \nu$	λ	0.2205 ± 0.0018
$b \to c l \nu$	A	0.83 ± 0.08
$b \to u l \nu$	σ	0.36 ± 0.14
$K_L \to \pi\pi$	$\cos\delta$	0.47 ± 0.32

Table 2: Experimental determination of CKM matrix elements (in the parametrization (3.14)) and relative processes from which are extracted[18] (see sect. 3.3 for the determination of $\cos\delta$).

As can be easily shown, any observable which violates CP must be proportional to this quantity[b]. The unitarity of U_{CKM} insures that J_{CP} is independent of the choice of a, b, i and j ($a \neq b$ and $i \neq j$). In the parametrization (3.14) J_{CP} is given by:

$$J_{CP} = \eta A^2 \lambda^6 + O(\lambda^8) \lesssim 10^{-4}. \tag{3.17}$$

Form Eq. (3.17) we can infer two simple considerations:

- CP violation is naturally suppressed in the Standard Model due to CKM matrix hierarchy.
- Transitions where CP violation should be more easily detected are those where also the CP–conserving amplitude is suppressed by the matrix elements V_{ub} and V_{td}.

3.2 Four–quark hamiltonians.

Differently than weak and electromagnetic interactions, strong interactions are not in a perturbative regime at low energies ($E \lesssim \Lambda_\chi \sim 1$ GeV), i.e. at distances $d \gtrsim 1/\Lambda_\chi \sim 10^{-14}$ cm. This happens because the strong coupling constant has a divergent behaviour in the infrared limit and, presumably, this is also the main reason why quarks and gluons are confined in hadrons. With respect to the scale Λ_χ quarks can be grouped in two categories according to their mass (§ tab. 1): u, d and s are 'light', since $m_{u,d,s} < \Lambda_\chi$, whereas c, b and t are 'heavy', since $\Lambda_\chi < m_{b,c,t}$. As we shall discuss better in the next section, the lightest hadrons with light valence quarks are the pseudoscalar mesons π, K and η (§ tab. 3).

Kaon decays, i.e. $|\Delta S| = 1$ transitions, being processes where initial and final state differ for the number of s quarks ($S|s\rangle = -|s\rangle$), involve the exchange of at least one W boson (see Eq. (3.7) and fig. 1a). In principle one could hope to calculate these decay amplitudes solving separately two problems: i) the perturbative calculation of the weak transition amplitude at the quark level, ii) the non–perturbative calculation of the strong

[b] Let's consider, as an example, the elementary process $u_a d_j \to u_a d_i$. The amplitude is the superposition of at least two processes: $u_a d_j \to W^+ d_i d_j \to u_a d_i$ and $u_a d_j \to u_a W^- u_b \to u_a d_i$, thus $A \sim \alpha V_{ai} V_{aj}^* + \beta V_{bj}^* V_{bi}$. The charge asymmetry, analogously to Eq. (2.57), will be proportional to $\Im m(V_{ai} V_{bj} V_{aj}^* V_{bi}^*)$.

meson	valence quarks	m(MeV)	(I, I_3)	S
π^+	$u\bar{d}$	139.6	$(1,+1)$	0
π^0	$u\bar{u} - d\bar{d}$	135.0	$(1,0)$	0
π^-	$d\bar{u}$	139.6	$(1,-1)$	0
K^+	$u\bar{s}$	493.7	$(1/2,+1/2)$	+1
K^0	$d\bar{s}$	497.7	$(1/2,-1/2)$	+1
$\bar{K}^0$	$s\bar{d}$	497.7	$(1/2,+1/2)$	−1
K^-	$s\bar{u}$	493.7	$(1/2,-1/2)$	−1
η	$u\bar{u} + d\bar{d} - 2s\bar{s}$	547.5	$(0,0)$	0

Table 3: Valence quarks, masses, isospin and strangeness of the pseudoscalar octet (for simplicity we have neglected π^0-η-η' mixing).

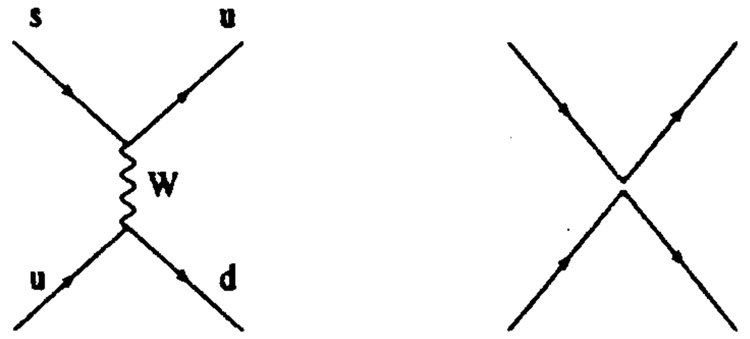

Figure 1: a) Tree–level feynman diagrams for $|\Delta S| = 1$ transitions, at the lowest order in G_F and without strong–interaction corrections. b) The same diagram in the effective theory $M_W \to \infty$.

transition between quark and hadron states. Obviously this separation is not possible, at least in a trivial way, and is necessary to manage strong interaction effects among initial and final state also in weak transitions (§ fig. 2a). Fortunately, both Λ_χ and meson masses are much smaller than the W mass and this helps a lot to simplify the problem.

If we neglect the transferred momenta with respect to M_W, the W–boson propagator becomes point–like

$$\frac{1}{M_W^2 - q^2} \to \frac{1}{M_W^2} \tag{3.18}$$

and the natural scale parameter for the weak amplitudes is given by the Fermi constant

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} \simeq 10^{-5}\text{GeV}. \tag{3.19}$$

In purely non–leptonic $|\Delta S| = 1$ transitions, neglecting strong interaction effects, the amplitudes can be calculated at the lowest order in G_F using the four–quarks effective

hamiltonian

$$\mathcal{H}_{eff}^{|\Delta S|=1} = -\frac{4G_F}{\sqrt{2}}\left[\lambda_u(\bar{s}_L\gamma^\mu u_L)(\bar{u}_L\gamma^\mu d_L) + \lambda_c(\bar{s}_L\gamma^\mu c_L)(\bar{c}_L\gamma^\mu d_L) + \text{h.c.}\right], \tag{3.20}$$

where $\lambda_q = V_{qs}^* V_{qd}$ and $q_L = \frac{1}{2}(1-\gamma_5)q$ (due to their large mass, we neglect, for the moment, the effect of b and t). As it will be clear later, it is convenient to rewrite Eq. (3.20) in the following way:

$$\mathcal{H}_{eff}^{|\Delta S|=1} = -\frac{4G_F}{\sqrt{2}}\sum_{q=u,c}\lambda_q\left(O_q^+ + O_q^-\right) + \text{h.c.}, \tag{3.21}$$

where

$$O_q^\pm = \frac{1}{2}\left[(\bar{s}_L\gamma^\mu q_L)(\bar{q}_L\gamma^\mu d_L) \pm (\bar{s}_L\gamma^\mu d_L)(\bar{q}_L\gamma^\mu q_L)\right]. \tag{3.22}$$

As can be seen from fig. 2, QCD corrections can be calculated more easily in the effective theory, i.e. with a point–like W propagator, than in the full theory. Indeed, in the first case, feynman diagrams with four 'full propagators' are reduced to diagrams with only three 'full propagators', simplifying the calculation. However, the presence of point–like propagators induces new ultraviolet divergences, not present in the full theory, that must be eliminated by an appropriate renormalization of the four–quarks operators[60,61,c]:

$$\begin{aligned}\langle F|\mathcal{H}_w^{|\Delta S|=1}|I\rangle &= \frac{g^2}{8}\int \mathrm{d}^4x D_W^{\mu\nu}(x, M_W)\langle F|T\left(J_\mu(x)J_\nu^\dagger(0)\right)|I\rangle \\ &\to \frac{G_F}{\sqrt{2}}\sum_i C_i(\mu)\langle F|O_i(\mu)|I\rangle.\end{aligned} \tag{3.23}$$

This procedure is nothing but an application of the Wilson's 'Operator Product Expansion'[62], the technique which let us expand a non–local product of operators, in this case the weak currents, in a series of local terms. The scale μ, which appears in Eq. (3.23), is a consequence of the renormalization procedure which eliminates the 'artificial' divergences of the effective operators. The requirement that the product $C_i(\mu)O_i(\mu)$, and thus all physical observables, be independent of μ, fix unambiguously the evolution of the coefficients C_i as a function of μ:

$$\left[\delta_{ij}\left(\mu\frac{\partial}{\partial\mu} + \beta_s(g_s)\frac{\partial}{\partial g_s}\right) - \gamma_{ij}^T(g_s)\right]C_j(\mu) = 0. \tag{3.24}$$

Eq. (3.24), known as the Callan–Symanzik equation for the coefficients C_i, takes this simple form only in a 'mass–independent' regularization scheme[63,64]. The functions $\beta(g_s)$ and $\gamma_{ij}(g_s)$ are defined by

$$\beta_s = \mu\frac{\partial}{\partial\mu}g_s(\mu) \qquad \text{and} \qquad \gamma_{ij} = \hat{\gamma}_{ij} - 2\gamma_j\delta_{ij}, \tag{3.25}$$

---

[c] The first identity in Eq. (3.23) follows from Eq. (3.7), defining $J_\mu(x) = \bar{U}(x)(V_U^\dagger V_D)\,\gamma^\mu\,(1-\gamma_5)\,D(x)$ and denoting by $D_W^{\mu\nu}(x, M_W)$ the W propagator in spatial coordinates.

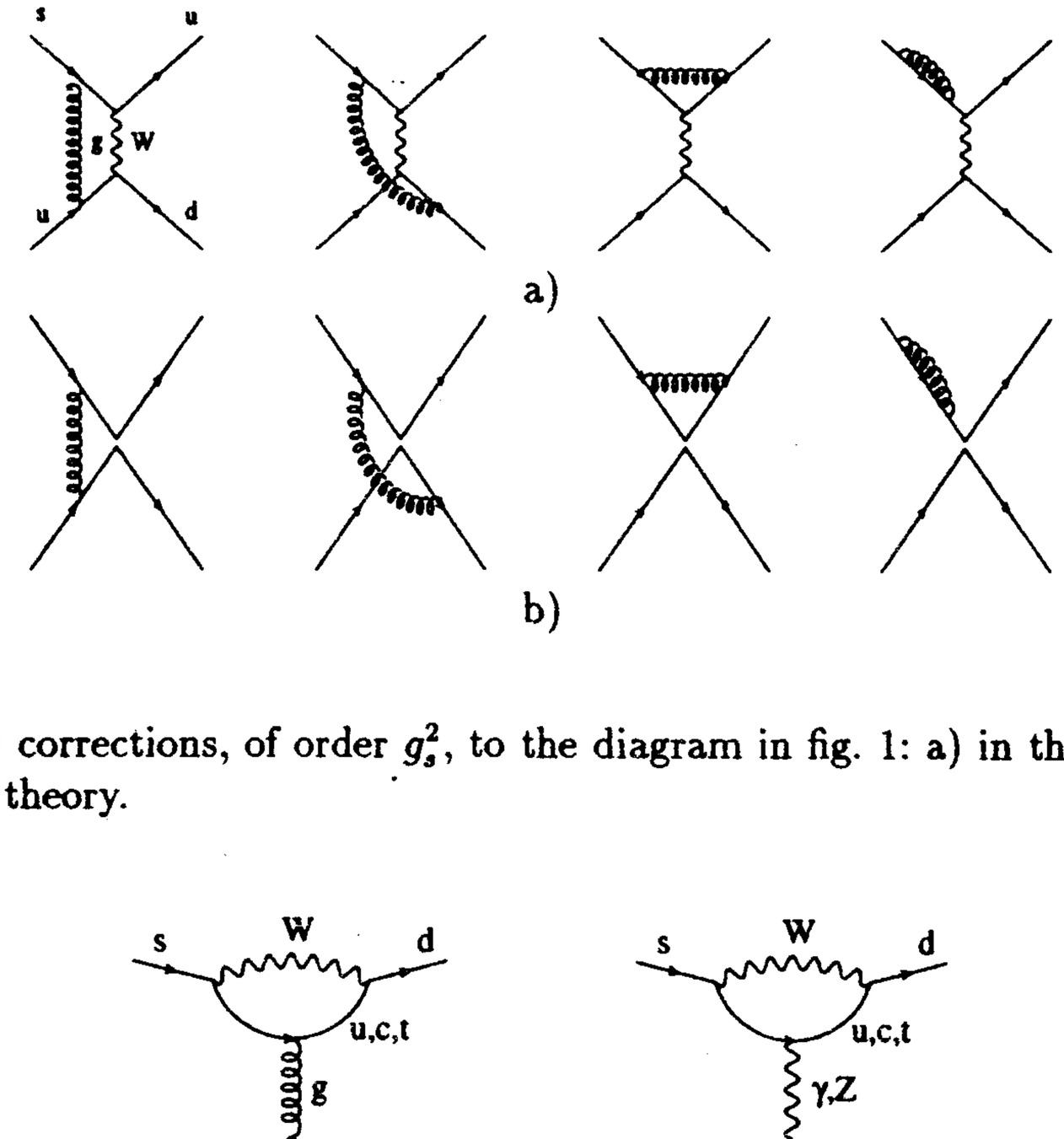

Figure 2: QCD corrections, of order g_s^2, to the diagram in fig. 1: a) in the full theory; b) in the effective theory.

Figure 3: Penguin diagrams: a) 'gluon penguin'; b) 'electromagnetic penguin'.

where $\hat{\gamma}_{ij}$ is the anomalous dimension matrix of the effective operators[d] and γ_J is the anomalous dimension of the weak current. In addition to Eq. (3.24), which rules the evolution of the C_i as a function of μ, in order to use Eq. (3.23) is necessary to fix the values of the C_i at a given scale, imposing the identity in the last term ('matching' procedure). This scale is typically chosen to be of the order of the W mass, where the perturbative calculations are much simpler ($g_s(M_W) \ll 1$).[e]

If we consider only the diagrams in fig. 2 and we neglect the effects due to non-vanishing light quark masses, the operator O_i which appear in Eq. (3.23) are just the

[d] Given a set of operators O_i, which mix each other through strong interactions, calling Z_{ij} the matrix of renormalization constants of such operators ($O_i^{ren} = Z_{ij}^{-1} O_j$), the anomalous dimension matrix is defined by $\gamma_{ij} = Z_{ik}^{-1} \mu \frac{\partial}{\partial \mu} Z_{kj}$.

[e] For a wider discussion about matching conditions and about the integration of Eq. (3.24) see Ref.[22,65,66].

$O_q^{\pm}$ of Eq. (3.22). These operators are renormalized only in a multiplicative way, as a consequence γ_{ij} is diagonal:

$$\gamma_{ij} = \frac{g_s^2}{4\pi^2} \begin{bmatrix} \gamma_+ & 0 \\ 0 & \gamma_- \end{bmatrix} = \frac{g_s^2}{4\pi^2} \begin{bmatrix} +1 & 0 \\ 0 & -2 \end{bmatrix}. \tag{3.26}$$

Since $C_q^{\pm}(M_W) = \lambda_q[1 + O(g_s(M_W))]$, solving Eq. (3.23) one finds[60,61 f]

$$\begin{aligned} \mathcal{H}_{eff}^{|\Delta S|=1} &= -\frac{4G_F}{\sqrt{2}} \sum_{q=u,c} \lambda_q \left\{ \left[\frac{g_s(M_W)}{g_s(\mu)}\right]^{\frac{\gamma_+}{2\beta_0}} O_q^+(\mu) \left[\frac{g_s(M_W)}{g_s(\mu)}\right]^{\frac{\gamma_-}{2\beta_0}} O_q^-(\mu) \right\} + \text{h.c.} \\ &\simeq -\frac{4G_F}{\sqrt{2}} \sum_{q=u,c} \lambda_q \left\{ \left[1 - \frac{g_s^2(\mu)}{4\pi^2} \log\left(\frac{M_W}{\mu}\right)\right] O_q^+(\mu) \right. \\ &\qquad \left. + \left[1 + 2\frac{g_s^2(\mu)}{4\pi^2} \log\left(\frac{M_W}{\mu}\right)\right] O_q^-(\mu) \right\} + \text{h.c.}, \end{aligned} \tag{3.27}$$

where $\beta_0 = \frac{1}{12}(33 - 2N_f)$ is defined by

$$\beta_s(g_s) = -\frac{g_s^2}{4\pi^2} \left(\beta_0 g_s + O(g_s^3)\right). \tag{3.28}$$

Eq. (3.27) is a good approximation to the weak hamiltonian for $m_c < \mu < m_b$. The coefficients $C_q^{\pm}(\mu)$ keep track, indeed, of all the leading QCD corrections; i.e. of all the terms of order $g_s(\mu)^{2n} \log(M_W/\mu)^n$. However, Eq. (3.27) is not sufficient to study CP-violating effects: in this case is necessary to consider also other diagrams. As it is well known, an important role is played by the so–called 'penguin diagrams'[67–72] (§ fig. 3) which, though suppressed with respect to those in fig. 2, give rise to new four–quark operators with different weak phases. The suppression of the diagrams in fig. 3 is nothing but a particular case of the famous GIM mechanism[10]: due to the unitarity of CKM matrix, their contribution vanishes in the limit $m_u = m_c = m_t$.

The complete set of operators relevant to non–leptonic $|\Delta S| = 1$ transitions, is given by the following 12 dimension–six terms[24,g]:

$$\begin{aligned} O_1 &= (\bar{s}_L^\alpha \gamma^\mu d_{L\alpha})(\bar{u}_L^\beta \gamma^\mu u_{L\beta}), \\ O_2 &= (\bar{s}_L^\alpha \gamma^\mu d_{L\beta})(\bar{u}_L^\beta \gamma^\mu u_{L\alpha}), \\ O_{3,5} &= (\bar{s}_L^\alpha \gamma^\mu d_{L\alpha}) \sum_{q=u,d,s,c} (\bar{q}_{L,R}^\beta \gamma^\mu q_{L,R\beta}), \\ O_{4,6} &= (\bar{s}_L^\alpha \gamma^\mu d_{L\beta}) \sum_{q=u,d,s,c} (\bar{q}_{L,R}^\beta \gamma^\mu q_{L,R\alpha}), \\ O_{7,9} &= (\bar{s}_L^\alpha \gamma^\mu d_{L\alpha}) \sum_{q=u,d,s,c} e_q (\bar{q}_{L,R}^\beta \gamma^\mu q_{L,R\beta}), \end{aligned}$$

---

f For simplicity we neglect the effect due to the b threshold in the integration of Eq. (3.24).

g The number of independent operators decrease to 11 and 7, for $\mu < m_b$ and $\mu < m_c$, respectively[24].

$$
\begin{aligned}
O_{8,10} &= (\bar{s}_L^\alpha \gamma^\mu d_{L\beta}) \sum_{q=u,d,s,c} e_q (\bar{q}_{L,R}^\beta \gamma^\mu q_{L,R\alpha}), \\
O_1^c &= (\bar{s}_L^\alpha \gamma^\mu d_{L\alpha})(\bar{c}_L^\beta \gamma^\mu c_{L\beta}), \\
O_2^c &= (\bar{s}_L^\alpha \gamma^\mu d_{L\beta})(\bar{c}_L^\beta \gamma^\mu c_{L\alpha}),
\end{aligned} \tag{3.29}
$$

where α and β are the color indices and e_q is the electric charge of the quark q. Using the relation $\lambda_u + \lambda_c + \lambda_t = 0$, in the basis (3.29) the weak hamiltonian assumes the following form:

$$
\begin{aligned}
\mathcal{H}_{eff}^{|\Delta S|=1} &= -\frac{4G_F}{\sqrt{2}} \Big\{ \lambda_u \left[C_1(\mu) O_1(\mu) + C_2(\mu) O_2(\mu) \right] \\
&\quad - \lambda_u \left[C_1(\mu) O_1^c(\mu) + C_2(\mu) O_2^c(\mu) \right] - \lambda_t \sum_{i=3}^{10} C_i(\mu) O_i(\mu) \Big\} + \text{h.c.}
\end{aligned} \tag{3.30}
$$

In Refs.[21,22] the 10×10 anomalous dimension matrix of the coefficients $C_i(\mu)$ has been calculated at two loops, including corrections of order α_s^2, $\alpha_s \alpha_{em}$ and α_{em}^2. Correspondingly the initial condition for the $C_i(\mu)$, at $\mu = M_W$, have been calculated including terms of order $\alpha_s(M_W)$ and $\alpha_{em}(M_W)$. Using these results is possible to calculate all the next–to–leading–order corrections to the Wilson coefficients of the effective hamiltonian.

After the work of Refs.[21,22] is useless to push further the perturbative calculation of the Wilson coefficients. At this point, the main source of error in estimating CP violation in kaon decays, is represented by the non–perturbative evaluation of the hadronic matrix elements of Eq. (3.23). In the next subsection, following Ciuchini et al.[24], we shall see how this problem has been solved in $K \to 2\pi$ using lattice results.

3.3 $K \to 2\pi$ parameters ϵ and ϵ'.

3.3.1 $\mathcal{H}_{eff}^{|\Delta S|=2}$ and the estimate of ϵ.

In the Standard Model the value of ϵ cannot be predicted but is an important constraint on the CKM phase. From Eqs. (2.19) and (2.33), assuming $\arg(\epsilon) = \pi/4$ and neglecting terms of order $|\epsilon|^2$, follows

$$
\epsilon \simeq \frac{e^{i\pi/4}}{2\sqrt{2}} \left(\frac{\Im m M_{12}}{\Re e M_{12}} \right) = -\frac{e^{i\pi/4}}{\sqrt{2}} \left(\frac{\Im m M_{12}}{\Delta M_K} \right). \tag{3.31}
$$

In order to calculate M_{12} is necessary to determine the effective hamiltonian responsible of $|\Delta S| = 2$ transitions. In this case the situation is much simpler than in the $|\Delta S| = 1$ case, previously discussed, since there is only one relevant operator, the one created by the box diagram of fig. 4. Thus the effective hamiltonian for $|\Delta S| = 2$ transitions can be written as

$$
\begin{aligned}
\mathcal{H}_{eff}^{|\Delta S|=2} &= \frac{G_F^2}{4\pi^2} M_W^2 (\bar{s}_L \gamma^\mu d_L)^2 [\lambda_c^2 \eta_1 F(x_c) + \lambda_t^2 \eta_2 F(x_t) \\
&\quad + 2\lambda_c \lambda_t \eta_3 F(x_c, x_t)] + \text{h.c.},
\end{aligned} \tag{3.32}
$$

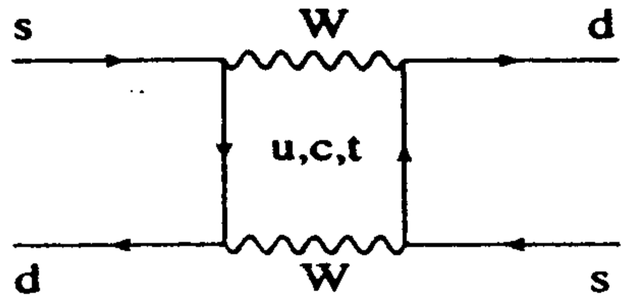

Figure 4: a) Box diagram for $|\Delta S| = 2$ transitions, without QCD corrections.

where $F(x_q)$ and $F(x_q, x_j)$ are the Inami–Lim functions[73], $x_q = m_q^2/M_W^2$, and $\eta_i = 1+O(g_s^2)$ are the QCD corrections, calculated at the next–to–leading order in Refs.[74,75]. Since

$$\Im m(M_{12}) = \frac{1}{2M_K} \Im m \left[\langle \bar{K}^0 | \mathcal{H}_{eff}^{|\Delta S|=2} | K^0 \rangle \right], \tag{3.33}$$

from the previous equations (3.31–3.33) follows

$$\begin{aligned} |\epsilon| \;=\; & \frac{G_F^2 M_W^2}{4\sqrt{2}\pi^2 M_K \Delta M_K} A^2 \lambda^6 \sigma \sin\delta \langle \bar{K}^0 | (\bar{s}_L \gamma^\mu d_L)^2 | K^0 \rangle \\ & \times \left[\eta_3 F(x_c, x_t) - \eta_1 F(x_c) + A^2 \lambda^4 (1 - \sigma \cos\delta) \eta_2 F(x_t) \right] \end{aligned} \tag{3.34}$$

where A, λ, δ and σ are the CKM parameters defined in sect. 3.1.

Few comments about Eq. (3.34) before going on:

- Both the η_i and the matrix element $\langle \bar{K}^0 | (\bar{s}_L \gamma^\mu d_L)^2 | K^0 \rangle$ depend on μ, but their product is scale independent.
- Due to the large value of the top mass, the term proportional to $F(x_t)$ is relevant even if it is suppressed by the factor $A^2\lambda^4$.
- In order to avoid the calculation of the matrix element $\langle \bar{K}^0 | (\bar{s}_L \gamma^\mu d_L)^2 | K^0 \rangle$, one could try to evaluate ΔM_K using $\mathcal{H}_{eff}^{|\Delta S|=2}$. However, this is not convenient since $\Re e(M_{12})$ receive also a long distance contribution that is difficult to evaluate with high accuracy[76].

In Ref.[24] the matrix element has been parametrized in the following way:

$$\langle \bar{K}^0 | (\bar{s} \gamma^\mu (1 - \gamma_5) d)^2 | K^0 \rangle = \frac{8}{3} f_K^2 M_K^2 B_K \alpha_s(\mu)^{6/25}, \tag{3.35}$$

where $f_K = \sqrt{2} F_K = 160$ MeV is the K–meson decay constant (§ sect. 4) and B_K is a μ–independent parameter.[h] Lattice estimates of this matrix element at scales $\mu \sim$

[h] If one considers also the next–to–leading terms in the η_i, then Eq. (3.35) must be modified, in order to preserve the μ invariance of B_K.

$2-3$ GeV imply[77] $B_K = 0.75 \pm 0.15$. With this result and the experimental value of ϵ, Eq. (3.34) imposes two possible solutions for $\cos\delta$, with different signs (§ fig. 5). The negative solution can be eliminated imposing additional conditions (coming from lattice estimates the $\bar{B}_d - B_d$ mixing) and the final result of Ref.[24] is:

$$\cos\delta = 0.47 \pm 0.32. \tag{3.36}$$

3.3.2 The estimate of ϵ'.

As shown in sect. 2.2, assuming the isospin decomposition of $K \to 2\pi$ amplitudes, it follows

$$\epsilon' = i\frac{e^{i(\delta_2-\delta_0)}}{\sqrt{2}}\frac{\omega}{\Re e A_0}\left[\omega^{-1}\Im m(A_2) - \Im m(A_0)\right]. \tag{3.37}$$

Actually the decomposition (2.28) is not exactly true, it receives small corrections due to the mass difference $m_u - m_d \neq 0$, which breaks isospin symmetry.

The main effect generated by the mass difference among u and d quarks, is to induce a mixing between the π^0 ($I = 1$) and the two isospin–singlet η and η'. As a consequence the transition $K^0 \to \pi^0\pi^0$ can occur also through the intermediate state $\pi^0\eta(\eta')$ ($K^0 \to \pi^0\eta(\eta') \to \pi^0\pi^0$). Due to the hierarchy of weak amplitudes ($\Re e A_0 \gg \Re e A_2$), we can safely neglect isospin–breaking terms in A_0. Furthermore, we know that $K^0 \to \pi^0\eta(\eta')$ amplitudes are $\Delta I = 1/2$ transitions, thus the global effect of isospin breaking in ϵ' can be simply reduced to a correction of $\Im m A_2$ proportional to $\Im m A_0$[78]. Following Refs.[78,79] we define

$$\Im m A_2 = \Im m A_2' + \Omega_{IB}\omega\Im m A_0, \tag{3.38}$$

where A_2' is the 'pure' $\Delta I = 3/2$ amplitude (without isospin–breaking terms). In order to estimate Ω_{IB} is necessary to evaluate $\pi^0 - \eta - \eta'$ mixing and the imaginary parts of $K^0 \to \pi^0\eta(\eta')$ amplitudes. The first problem is connected with the relation among quark and meson masses, and can be partially solved in the framework of Chiral Perturbation Theory (§ sect. 4.2.1). On the other hand, the second problem requires the non–perturbative knowledge of weak matrix elements. Evaluating these elements in the large N_c limit, Buras and Gerard[79] estimated $\Omega_{IB} \simeq 0.25$. Due to the large uncertainties which affect this estimate, in the following we shall assume[80] $\Omega_{IB} = 0.25 \pm 0.10$.

Using Eq. (3.38), the expression of ϵ' becomes

$$\epsilon' = i\frac{e^{i(\delta_2-\delta_0)}}{\sqrt{2}}\frac{\omega}{\Re e A_0}\left[\omega^{-1}\Im m(A_2') - (1-\Omega_{IB})\Im m(A_0)\right], \tag{3.39}$$

where A_0 and A_2' can be calculated using the effective hamiltonian (3.30) in Eq. (2.25).[i] Analogously to the case of ϵ, is convenient to introduce opportune B–factors to parametrize

[i] Bertolini et al.[81] pointed out that a non–negligible contribution to ϵ' could be generated by the dimension–5 gluonic–dipole operator, not included in the basis (3.29). The matrix element of this operator is however suppressed in the chiral expansion (see the discussion about the electric–dipole operator in sect. 6.2) and was overestimated in Ref. [81].

	$\sqrt{\frac{1}{24}}\langle 2\pi,0\|O_i\|K^0\rangle_{VIA}$	$\sqrt{\frac{1}{12}}\langle 2\pi,2\|O_i\|K^0\rangle_{VIA}$
O_3	$+X/3$	0
O_4	$+X$	0
O_5	$-Z/3$	0
O_6	$-Z$	0
O_7	$+2Y/3+Z/6+X/2$	$+Y/3-X/2$
O_8	$+2Y+Z/2+X/6$	$+Y-X/6$
O_9	$-X/3$	$+2X/3$
O_1^c	$+X/3$	0
O_2^c	$+X$	0

Table 4: Matrix elements of the four–quarks operators of $\mathcal{H}_{eff}^{|\Delta S|=1}$ in the vacuum insertion approximation; $X = f_\pi(M_K^2 - M_\pi^2)$, $Y = f_\pi M_K^4/(m_s+m_d)^2$ and $Z = 4Y(f_K - f_\pi)/f_\pi$.

$\mathcal{H}_{eff}^{|\Delta S|=1}$ matrix elements. Following again Ref.[24] we define:

$$\langle 2\pi, I|O_i(\mu)|K^0\rangle = B_i^{\frac{1+I}{2}}(\mu)\langle 2\pi, I|O_i|K^0\rangle_{VIA}, \tag{3.40}$$

where $\langle 2\pi, I|O_i|K^0\rangle_{VIA}$ indicates the matrix element calculated in the vacuum insertion approximation[32].

VIA results for matrix elements which contribute[j] to $\Im m(A_0)$ and $\Im m(A_2')$ are reported in table 4, in tables 5 and 6 we report Wilson coefficients and corresponding B–factors at $\mu = 2$ GeV. The two column of table 5 correspond to different regularization schemes: the 't Hooft–Veltman scheme (HV) and the naive–dimensional–regularization scheme (NDR). The differences among the C_i values in the two tables give an estimate of the next–to–next–to–leading–order corrections which has been neglected.

The dominant contribution to the real parts of A_0 and A_2 is generated by O_1 and O_2. The lattice estimates of the corresponding B–factors, which must be substantially different from one in order to reproduce the observed $\Delta I = 1/2$ enhancement, are affected by large uncertainties and are not reported in table 5. Fortunately this uncertainty does not affect the imaginary parts, which in the basis (3.29) are dominated by O_6 and O_8:

$$\begin{aligned}\Im m(A_0) &\simeq \frac{G_F}{\sqrt{2}}\lambda^5 A^2\sigma\sin\delta(C_6 B_6^{1/2} Z),\\ \Im m(A_2) &\simeq -G_F\lambda^5 A^2\sigma\sin\delta(C_8 B_8^{3/2} Y).\end{aligned} \tag{3.41}$$

Using these equations we can derive a simple and interesting phenomenological expression (similar to the one proposed in Ref.[82]) for $\Re e(\epsilon'/\epsilon)$:

$$\Re e\left(\frac{\epsilon'}{\epsilon}\right) \simeq \left(3.0\times 10^{-3}\right)\left[B_6^{1/2} - \tilde{r}B_8^{3/2}\right]A^2\sigma\sin\delta = (2.6\pm 2.3)\times 10^{-4}, \tag{3.42}$$

[j] O_1 and O_2 do not contribute since $\Im m(\lambda_u) = 0$; O_{10} has been eliminated through the relation $O_{10} = O_9 + O_4 - O_3$ which holds for $\mu < m_b$.

	HV	NDR
C_1	$(-3.29 \pm 0.37 \pm 0.00) \times 10^{-1}$	$(-3.13 \pm 0.39 \pm 0.00) \times 10^{-1}$
C_2	$(104.13 \pm 0.54 \pm 0.00) \times 10^{-2}$	$(11.54 \pm 0.23 \pm 0.00) \times 10^{-1}$
C_3	$(1.73 \pm 0.26 \pm 0.00) \times 10^{-2}$	$(2.07 \pm 0.33 \pm 0.00) \times 10^{-2}$
C_4	$(-3.82 \pm 0.44 \pm 0.01) \times 10^{-2}$	$(-5.19 \pm 0.71 \pm 0.01) \times 10^{-2}$
C_5	$(1.20 \pm 0.11 \pm 0.00) \times 10^{-2}$	$(10.54 \pm 0.16 \pm 0.02) \times 10^{-3}$
C_6	$(-5.08 \pm 0.72 \pm 0.03) \times 10^{-2}$	$(-0.72 \pm 0.13 \pm 0.00) \times 10^{-1}$
C_7	$(0.01 \pm 0.00 \pm 0.18) \times 10^{-3}$	$(0.01 \pm 0.04 \pm 0.20) \times 10^{-3}$
C_8	$(0.77 \pm 0.12 \pm 0.12) \times 10^{-3}$	$(0.81 \pm 0.16 \pm 0.14) \times 10^{-3}$
C_9	$(-6.71 \pm 0.27 \pm 0.68) \times 10^{-3}$	$(-7.49 \pm 0.15 \pm 0.75) \times 10^{-3}$

Table 5: Wilson coefficients of $\mathcal{H}_{eff}^{|\Delta S|=1}$ operators, at $\mu = 2$ GeV, calculated including next–to–leading–order corrections in two different regularization schemes[24]. The first error is due to the uncertainty on $\alpha_s(\mu)$, the second to the uncertainty on m_t.

$B_{1c,2c}^{1/2}$	$B_{3,4}^{1/2}$	$B_{5,6}^{1/2}$	$B_{7,8,9}^{1/2}$	$B_{7,8}^{3/2}$	$B_9^{3/2}$
$0 - 0.15^{(*)}$	$1 - 6^{(*)}$	1.0 ± 0.2	$1^{(*)}$	1.0 ± 0.2	0.62 ± 0.10

Table 6: B–factors, defined in Eq. (3.40) at $\mu = 2$ GeV. The entries marked with '*' are pure 'theoretical guesses', whereas the others are obtained by lattice simulations[24].

where

$$\tilde{r} = \frac{\sqrt{2}}{(1 - \Omega_{IB})\omega} \left| \frac{C_8}{C_6} \right| \simeq (0.6 \pm 0.2). \tag{3.43}$$

From Eq. (3.42) it is clear that the weak phases of A_0 and A_2' accidentally tends to cancel each other. As anticipated in the previous section, this cancellation is due to the large value m_t, which enhance C_8 (for $m_t \sim 200$ GeV we found $\tilde{r} \sim 1$, whereas for $m_t \sim 100$ GeV, $\tilde{r} \sim 10^{-1}$). Nevertheless, the other essential ingredient of this cancellation is the '$\Delta I = 1/2$ rule', i.e. the dynamical suppression of $\Delta I = 3/2$ amplitudes with respect to $\Delta I = 1/2$ ones (the ω^{-1} factor in Eq. (3.43) is essential for the enhancement of $\tilde{r}$).

An accurate statistical analysis of the theoretical estimate of $\Re e(\epsilon'/\epsilon)$ has been recently carried out by Ciuchini et al.[24]. In fig. 5 we report some results of this analysis. Histograms have been obtained by varying, according to their errors, all quantities involved in the calculation of ϵ and ϵ': Wilson coefficients, B–factors, experimental values of α_s and m_t, CKM parameters, Ω_{IB} and m_s (the latter is extracted by lattice simulations). The final estimate of $\Re e(\epsilon'/\epsilon)$ thus obtained is[24]:

$$\Re e \left(\frac{\epsilon'}{\epsilon} \right) = (3.1 \pm 2.5) \times 10^{-4}, \tag{3.44}$$

in agreement with previous and more recent analyses[80,23,66,83].

Actually, in Ref.[66] the final error on $\Re e(\epsilon'/\epsilon)$ is larger since the various uncertainties have been combined linearly and not in a gaussian way, like in Ref.[24]. A large uncertainty has been obtained also in Ref.[83], where the matrix elements have been estimated in a completely different approach. Nevertheless, all analyses agree on excluding a value of $\Re e(\epsilon'/\epsilon)$ substantially larger than 1×10^{-3}.

3.4 B decays.

CP violation in kaon decays is strongly suppressed by CKM–matrix hierarchy: ϵ, as an example, is of order $O(\lambda^4) \sim O(10^{-3})$. On the other hand, in the decays of B_d and B_s mesons this suppression is avoidable in several cases[84,85] (see the discussion at the end of sect. 3.1) and the study of CP violation turns out to be more various and promising.

With respect to the $K^0 - \bar{K}^0$ system, $B_d - \bar{B}_d$ and $B_s - \bar{B}_s$ systems have the following interesting differences:

- Due to the large number of initial and final states, both even and odd under CP, the difference of the decay widths is much lower than the mass difference, i.e. $|\Gamma_{12}| \ll |M_{12}|$. As a consequence, according to Eq. (2.18), the mixing parameters ϵ_{B_d} and ϵ_{B_s} have very small real parts.

- Mass differences are originated by box diagrams (similar to the one of fig. 4) which, differently from the $K^0 - \bar{K}^0$ case, are dominated by top–quark exchange not only in the imaginary part but also in the real part. In the CKM phase convention of Eq. (3.12) one finds:

$$\frac{1 - \epsilon_{B_d}}{1 + \epsilon_{B_d}} \simeq \frac{V_{td}}{V_{td}^*} \doteq e^{-2i\beta}, \tag{3.45}$$

$$\frac{1 - \epsilon_{B_s}}{1 + \epsilon_{B_s}} \simeq \frac{V_{ts}}{V_{ts}^*} \simeq 1. \tag{3.46}$$

Since real parts of ϵ_{B_d} and ϵ_{B_s} are small, the study of CP violation via charge asymmetries of semileptonic decays (§ sect. 2.3) is not convenient in neutral–B–meson systems. The best way to observe a CP violation in these channels[84,85] is to compare the time evolution of the states $|B_q(t)\rangle$ and $|\bar{B}_q(t)\rangle$ (states which represent, at $t = 0$, B_q and $\bar{B}_q$ mesons) in a final CP–eigenstate $|f\rangle$ ($CP|f\rangle = \eta_f|f\rangle$):

$$\Gamma(B_q(t) \to f) \propto e^{-\Gamma_{B_q} t} \left[1 - \eta_f \lambda_f \sin(\Delta M_{B_q} t)\right], \tag{3.47}$$

$$\Gamma(\bar{B}_q(t) \to f) \propto e^{-\Gamma_{B_q} t} \left[1 + \eta_f \lambda_f \sin(\Delta M_{B_q} t)\right]. \tag{3.48}$$

An evidence of $\lambda_f \neq 0$ necessarily implies CP violation, either in the mixing or in the decay amplitudes. As we shall see in the following, for particular states $|f\rangle$, the CKM mechanism predicts $\lambda_f = O(1)$.

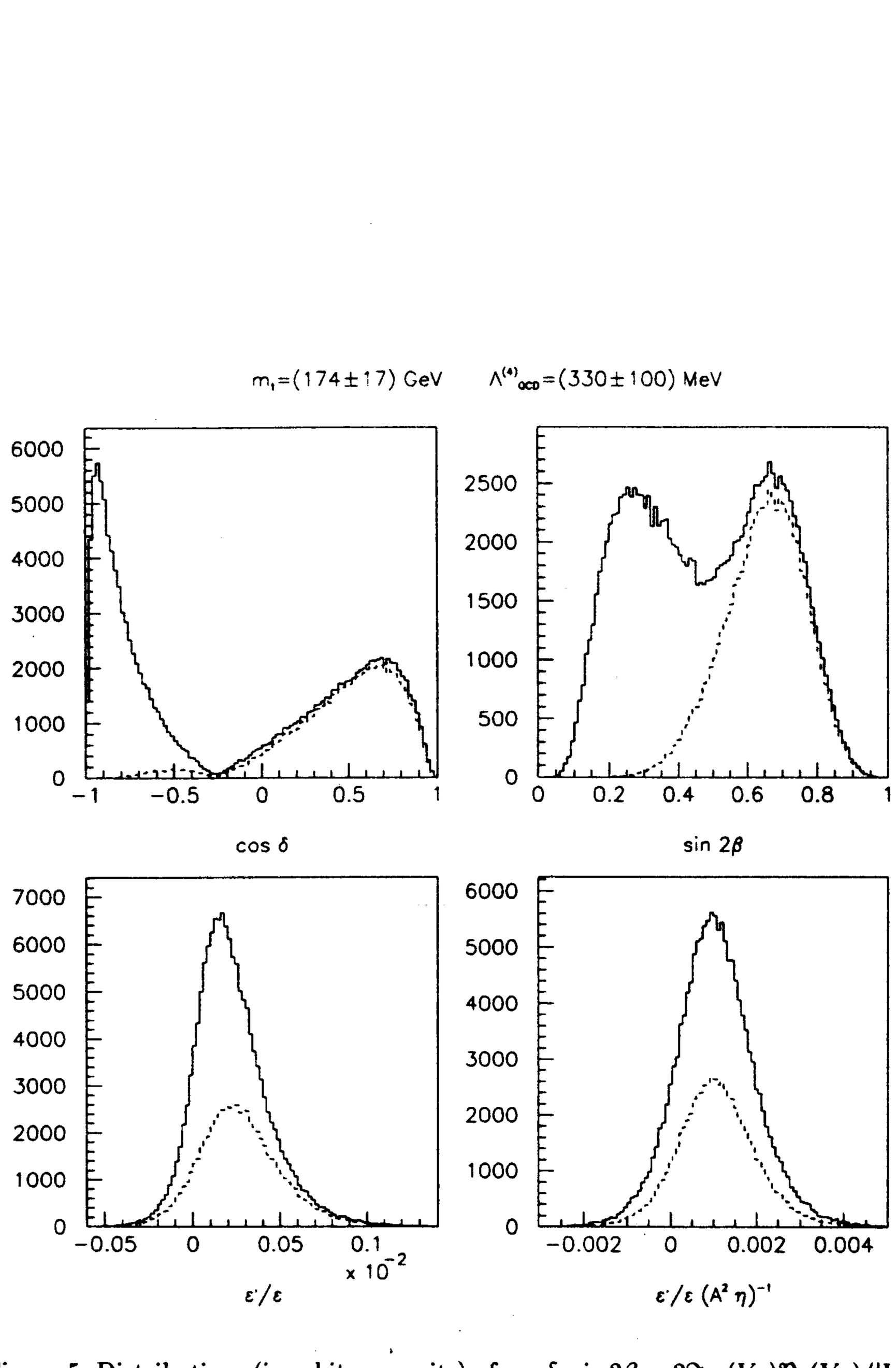

Figure 5: Distributions (in arbitrary units) of $\cos\delta$, $\sin 2\beta = 2\Im m(V_{td})\Re e(V_{td})/|V_{td}|^2$, ϵ'/ϵ and $\epsilon'/\epsilon(A^2\eta)^{-1}$ as obtained by Ciuchini et al.[24]. Dotted histograms have been obtained adding the additional constraint coming from $\bar{B}_d - B_d$ mixing.

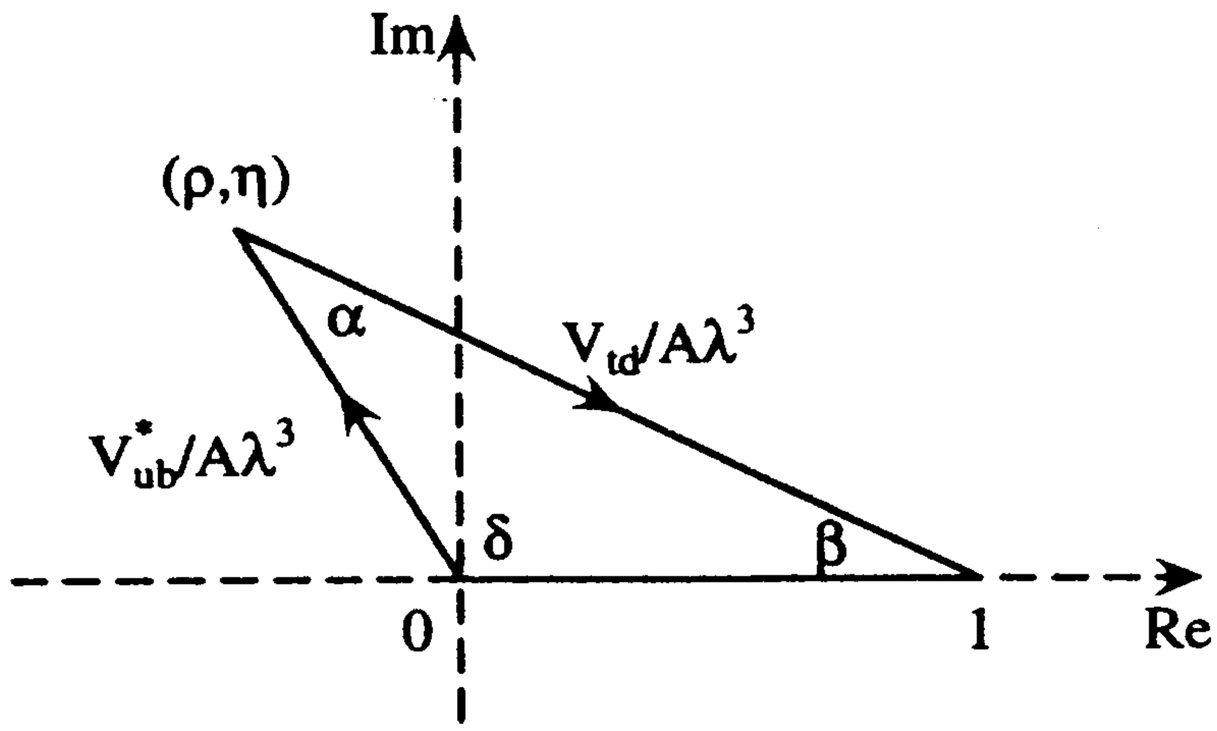

Figure 6: Unitarity triangle in the complex plane.

If CP violation was originated only at the level of $B_q - \bar{B}_q$ mixing, λ_f would not depend on the decay channel. On the contrary, in the Standard Model λ_f assumes different values according to the channel. In all transitions where only one weak amplitude is dominant, is possible to factorize strong interaction effects and to extract CKM matrix–elements *independently from the knowledge of hadronic matrix elements.* According to the dominant process at the quark level, λ_f assumes the following values[86]:

$$\begin{array}{ll} \lambda_f = \sin 2\beta & B_d,\ b \to c \\ \lambda_f = \sin 2(\beta+\delta) \doteq \sin 2\alpha & B_d,\ b \to u \\ \lambda_f = 0 & B_s,\ b \to c \\ \lambda_f = \sin 2\delta & B_s,\ b \to u \end{array} \tag{3.49}$$

where δ is the CKM phase in the parametrization (3.12) and β, already introduced in Eq. (3.45), is given by:

$$\tan\beta = \frac{\sigma \sin\delta}{1 - \sigma\cos\delta} = \frac{\eta}{1-\rho}. \tag{3.50}$$

The phases α, β and δ, which appears in Eq. (3.49), have an interesting phenomenological interpretation: are the angles of the so–called 'unitarity triangle' (§ fig. 6). Indeed, from CKM–matrix unitarity, which implies

$$V^*_{ub}V_{ud} + V^*_{cb}V_{cd} + V^*_{tb}V_{td} = 0, \tag{3.51}$$

i.e., at the leading order in λ,

$$V^*_{ub} + V_{td} = A\lambda^3, \tag{3.52}$$

follows the relation

$$\alpha + \beta + \delta = \pi. \tag{3.53}$$

Obviously, CKM–matrix unitarity impose also other constraints, in addition to Eq. (3.51), which can be re–formulated in terms of different triangles. However, the one in fig. 6 is the most interesting one since the three sizes are of the same order in λ and thus the triangle is not degenerate.

Limits on CKM–parameter ρ and η (§ tab. 2), coming both from K and B physics, put some constraints on the angles α, β and δ. Recent correlated analyses of such limits[24,87] leads to the conclusion that, whereas $\sin 2\delta$ and $\sin 2\alpha$ can vanish, $\sin 2\beta$ is necessarily different from zero and possibly quite large:

$$0.23 \leq \sin 2\beta \leq 0.84. \tag{3.54}$$

Fortunately, the measurement of $\sin 2\beta$ is also the most accessible from the experimental point of view. Indeed, the decay $B_d \to \Psi K_{S,L}$ is dominated[k] by the tree–level process $\bar{b} \to c\bar{c}s$, thus, according to Eq. (3.49), from this decay is possible to extract in a clean way $\sin 2\beta = \lambda_{\Psi K}$. The measurement of $\lambda_{\Psi K}$ is one of the main goal of next–generation high–precision experiments on B decays and will represent a fundamental test for the CKM mechanism of CP violation.

The measurements of the other two phases (α and δ), very interesting from the theoretical point of view, both to exclude new 'super–weak' models (models where CP is generated only by $|\Delta S| = 2$ and $|\Delta B| = 2$ interactions) and to test CKM–matrix unitarity (limiting the presence of new quark families), are much more difficult. In the case of α, for instance, the most promising channel is the decay $B_d \to \pi\pi$, dominated by the tree–level transition $\bar{b} \to u\bar{u}d$, but the very small branching ratio and the contamination of penguin diagrams with different weak phases[88], makes the measurement of $\lambda_{\pi\pi}$ quite difficult and the successive extraction of $\sin 2\alpha$ not very clean. A detailed analysis of all the processes that can be studied in order to measure these phases is beyond the purpose of this article, we refer the reader to the numerous works on this subject which are present in the literature (see e.g. Refs.[35,86,88–90] and references cited therein).

4 Chiral Perturbation Theory.

As already stated in the previous section, color interactions between quarks and gluons are non perturbative at low energies, and the confinement phenomenon is probably the most evident consequence of this behaviour. Nevertheless, from the experimental point of view is known that at very low energies pseudoscalar–octet mesons (§ tab. 3) interact weakly, both among themselves and with nucleons. Therefore it is reasonable to expect that with a suitable choice of degrees of freedom QCD can be treated perturbatively even at low energies. Chiral Perturbation Theory[25–27] (CHPT), using the pseudoscalar–octet mesons as degrees of freedom, has exactly this goal.[a]

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[k] Actually there is also a small contribution coming from the penguin diagram $\bar{b} \to sq\bar{q}$, however this term has the same weak phase (zero in the standard CKM parametrization) of the dominant one.

[a] For excellent reviews on CHPT and, in particular, for its applications in kaon dynamics see Refs.[38,39,51,91–95].

Neglecting light quark masses, the QCD lagrangian

$$\tilde{\mathcal{L}}_{QCD} = \sum_{q=u,d,s} \bar{q}\gamma^{\mu}\left(i\partial_{\mu} - g_s\frac{\lambda_a}{2}G^a_{\mu}\right)q - \frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + O(\text{heavy quarks}), \qquad (4.1)$$

a part from the local invariance under $SU(3)_C$, posses a global invariance under $SU(N_{ql})_L \times SU(N_{ql})_R \times U(1)_V \times U(1)_A$, where $N_{ql} = 3$ is the number of massless quarks. The $U(1)_V$ symmetry, which survives also in the case of non–vanishing quark masses, is exactly conserved and its generator is the barionic number. On the other hand, the $U(1)_A$ symmetry is explicitly broken at the quantum level by the abelian anomaly[96,14]. $G = SU(3)_L \times SU(3)_R$ is the group of chiral transformations:

$$\psi_{L,R} \xrightarrow{G} g_{L,R}\psi_{L,R}, \qquad \text{where} \qquad \psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \qquad \text{and} \qquad g_{L,R} \in G, \qquad (4.2)$$

spontaneously broken by the quark condensate ($\langle 0|\bar{\psi}\psi|0\rangle \neq 0$). The subgroup which remains unbroken after the breaking of G is $H = SU(3)_V \equiv SU(3)_{L+R}$ (the famous $SU(3)$ of the 'eightfold way'[6]), as a consequence also the coset G/H is isomorphic to $SU(3)$.

The fundamental idea of CHPT is that, in the limit $m_u = m_d = m_s = 0$ (chiral limit), pseudoscalar–octet mesons are Goldstone bosons generated by the spontaneous breaking of G into H. Since Goldstone fields can be always re-defined in such a way that interact only through derivative couplings[28], this hypothesis justify the weak behaviour of pseudoscalar interactions at low energies. If these mesons were effectively Goldstone bosons, they should be massless, actually this is not the case due to the light–quark–mass terms which explicitly break G. Nevertheless, since $m_{u,d,s} < \Lambda_\chi$, is natural to expect that these breaking terms can be treated as small perturbations. The fact that pseudoscalar–meson masses are much smaller than the typical hadronic scale ($M_\pi^2/\Lambda_\chi^2 \ll 1$) indicates that also this hypothesis is reasonable. Summarizing, the two basic assumptions of CHPT are[95]:

[1] *In the chiral limit, pseudoscalar–octet mesons are Goldstone bosons originated by the spontaneous breaking of G into H.*

[2] *The mass terms of light quarks can be treated as perturbations.*

According to these hypotheses, to describe QCD interactions of pseudoscalar mesons is necessary to consider the most general lagrangian invariant under G, written in terms of Goldstone–boson fields, and add to it the breaking terms, which transform linearly under G.[25] The problem of this approach is that the lagrangian built in this way is non renormalizable and thus contains an infinite number of operators. Nevertheless, as we shall see in the following, in the case of low energy processes ($E < \Lambda_\chi$), the error done by considering only a finite number of such operators is under control (of order $(E/\Lambda_\chi)^n$) .

4.1 Non–linear realization of G.

Goldstone–boson fields parametrize the coset space G/H and thus do not transform linearly under G. The general formalism to construct invariant operators, or operators which

transform linearly, in terms of the Goldstone–boson fields of a spontaneously broken (compact, connected and semisimple) symmetry, has been analyzed in detail by Callan, Coleman, Wess and Zumino[97,98]. In this subsection we shall only illustrate the application of this formalism to the chiral symmetry case.

First of all is possible to define a unitary matrix u (3×3), which depends on the Goldstone–boson fields (ϕ_i), and which transforms in the following way:

$$\begin{aligned} u(\phi_i) &\xrightarrow{G} g_R u(\phi_i) h^{-1}(g,\phi_i) = h(g,\phi_i) u(\phi_i) g_L^{-1} \\ u(\phi_i)^\dagger &\xrightarrow{G} g_L u(\phi_i)^\dagger h^{-1}(g,\phi_i) = h(g,\phi_i) u(\phi_i)^\dagger g_R^{-1}, \end{aligned} \tag{4.3}$$

where $h(g,\phi_i)$, the so–called 'compensator–field', is an element of the subgroup H. If $g \in H$, h is a unitary matrix, independent of the ϕ_i, which furnishes a linear representation of H: if Ψ is a matrix which transforms linearly under H, then

$$\Psi_i \xrightarrow{G} h(g,\phi_i) \Psi_i h^{-1}(g,\phi_i). \tag{4.4}$$

There are different parametrizations of u in terms of the fields ϕ_i, which correspond to different choices of coordinates in the coset space G/H. A convenient parametrization is the exponential parametrization:[b]

$$\begin{aligned} u^2 = U &= e^{i\sqrt{2}\Phi/F}, \\ \Phi &= \frac{1}{\sqrt{2}} \sum_i \lambda_i \phi^i = \begin{bmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}} \end{bmatrix}, \end{aligned} \tag{4.5}$$

where F is a dimensional constant ($\dim[F]=\dim[\Phi]$) that, as we shall see, can be related to the pseudoscalar–meson decay constant. Note that $U \xrightarrow{G} g_R U g_L^{-1}$.

Successively, it is convenient to introduce the following derivative operators:

$$u_\mu \doteq i(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger) = i u^\dagger \partial_\mu U u^\dagger = u_\mu^\dagger \qquad u_\mu \xrightarrow{G} h u_\mu h^\dagger, \tag{4.6}$$

which transforms like Ψ, and

$$\Gamma_\mu \doteq \frac{1}{2}(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger) = -\Gamma_\mu^\dagger \qquad \Gamma_\mu \xrightarrow{G} h \Gamma_\mu h^\dagger + h \partial_\mu h^\dagger, \tag{4.7}$$

which let us build the covariant derivative of Ψ:

$$\nabla_\mu \Psi = \partial_\mu \Psi - [\Gamma_\mu, \Psi]. \tag{4.8}$$

---

[b] We denote by η_8 the octet component of the η meson.

With these definitions is very simple to construct the operators we are interested in: if A is any operator which transforms linearly under H (like Ψ, u_μ and their covariant derivatives), then $uAu^\dagger$ and $u^\dagger Au$ transforms linearly under G, whereas their trace is invariant:

$$\begin{aligned} uAu^\dagger &\xrightarrow{G} g_R(uAu^\dagger)g_R^{-1}, \\ u^\dagger Au &\xrightarrow{G} g_L(u^\dagger Au)g_L^{-1}. \end{aligned} \tag{4.9}$$

4.2 Lowest–order lagrangians.

In absence of external fields, the invariant operator which contains the lowest number of derivatives is unique:[c] $\langle u_\mu u^\mu \rangle = \langle \partial_\mu U \partial^\mu U^\dagger \rangle$. Fixing the coupling constant of this operator in order to get the kinetic term of spin–less fields, leads to:

$$\tilde{\mathcal{L}}_S^{(2)} = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger \rangle = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + O(\Phi^4). \tag{4.10}$$

This lagrangian is the chiral realization, at the lowest order in the derivative expansion, of $\tilde{\mathcal{L}}_{QCD}$.

To include explicitly breaking terms, and to generate in a systematic way Green functions of quark currents, is convenient to modify $\tilde{\mathcal{L}}_{QCD}$, i.e. the QCD lagrangian in the chiral limit, coupling external sources to quark currents. Following the work of Gasser and Leutwyler[26,27], we introduce the sources v_μ, a_μ, $\hat{s}$ and $\hat{p}$, so that

$$\begin{aligned} r_\mu = v_\mu + a_\mu &\xrightarrow{G} g_R r_\mu g_R^{-1}, \\ l_\mu = v_\mu - a_\mu &\xrightarrow{G} g_L l_\mu g_L^{-1}, \\ \hat{s} + i\hat{p} &\xrightarrow{G} g_R(\hat{s} + i\hat{p})g_L^{-1}, \\ \hat{s} - i\hat{p} &\xrightarrow{G} g_L(\hat{s} - i\hat{p})g_R^{-1}, \end{aligned} \tag{4.11}$$

and we consider the lagrangian

$$\mathcal{L}_{QCD}(v,a,\hat{s},\hat{p}) = \tilde{\mathcal{L}}_{QCD} + \bar{\psi}\gamma^\mu(v_\mu + a_\mu\gamma_5)\psi - \bar{\psi}(\hat{s} - i\hat{p}\gamma_5)\psi. \tag{4.12}$$

By this way we achieve two interesting results[95]:

- The generating functional

$$e^{iZ(v,a,\hat{s},\hat{p})} = \int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}G\; e^{i\int d^4x \mathcal{L}_{QCD}(v,a,\hat{s},\hat{p})} \tag{4.13}$$

  is explicitly invariant under chiral transformations, but the explicit breaking of G can be nonetheless obtained by calculating the Green functions, i.e. the functional derivatives of $Z(v,a,\hat{s},\hat{p})$, at

$$v_\mu = a_\mu = \hat{p} = 0 \qquad\qquad \hat{s} = M_q = \mathrm{diag}(m_u, m_d, m_s). \tag{4.14}$$

[c] We denote by $\langle A \rangle$ the trace of A.

to the dominant terms. Thus the chiral lagrangian for $|\Delta S| = 1$ non–leptonic transitions, at order $(G_F p^2 e^0) + (G_F p^0 e^2)$, is given by:

$$\mathcal{L}_W^{(2)} = \frac{G_F}{\sqrt{2}} \lambda_u F^4 \left[\sum_{i=8,27} g_i W_i^{(2)} + g_{\underline{8}} W_{\underline{8}}^{(0)} \right] + \text{h.c.} \tag{4.32}$$

The three constants g_i which appear in $\mathcal{L}_W^{(2)}$ are not fixed by chiral symmetry but is natural to expect them to be of the order of the Wilson coefficients of table 5. The g_i are real in the limit where CP is an exact symmetry.

In principle, the g_i could be determined either by comparison with experimental data, on $K \to 2\pi$ or $K \to 3\pi$, or by by comparison with theoretical estimates, coming from lattice QCD or other non–perturbative approaches[76,102–104]. In practice, the choice is reduced because: i) there are no experimental information on the imaginary parts of the weak amplitudes; ii) lattice calculations for the real parts of $K \to 2\pi$ amplitudes are still not reliable (the estimates are dominated by the large errors on the B-factors of O_1 and O_2); iii) all the other non–perturbative approaches are affected from large theoretical uncertainties. As a consequence, in our opinion the best choice to determine the g_i is to fix the real parts by comparison with experimental data (those on $K \to 2\pi$ for simplicity), and to fix the imaginary parts by comparison with lattice calculations[105].

Using the lagrangian (4.32) at tree level, from Eq. (2.28) follows:[f]

$$A_0 = \sqrt{2} F \left[\left(M_K^2 - M_\pi^2 \right) \left(G_8 + \frac{1}{9} G_{27} \right) - \frac{2}{3} F^2 G_{\underline{8}} \right], \tag{4.33}$$

$$A_2' = F \left[\frac{10}{9} G_{27} \left(M_K^2 - M_\pi^2 \right) - \frac{2}{3} F^2 G_{\underline{8}} \right], \tag{4.34}$$

where, for simplicity, we have introduced the dimensional couplings $G_i = G_F \lambda_u g_i / \sqrt{2}$. Neglecting the contribution of the $(8_L, 8_R)$ operator,[g] the comparison with the experimental data (2.35–2.36) leads to:

$$|G_8| = 9.1 \times 10^{-6} \text{ GeV}^{-2}, \tag{4.35}$$

$$|g_{27}/g_8| = \frac{9\sqrt{2}\omega}{10} = 5.7 \times 10^{-2}. \tag{4.36}$$

For what concerns imaginary parts, the comparison with the results shown in sect. 3.3 leads to[105]:

$$\begin{aligned} \Im m g_8 = \Im m \left(\frac{\lambda_t}{\lambda_u} \right) \Bigg[& \frac{C_1}{3} B_{1c}^{1/2} + C_2 B_{2c}^{1/2} + \frac{C_3}{3} B_3^{1/2} + C_4 B_4^{1/2} \\ & - \left(\frac{C_5}{3} B_5^{1/2} + C_6 B_6^{1/2} \right) \frac{Z}{X} - \frac{C_9}{3} B_9^{1/2} \Bigg], \end{aligned}$$

[f] In the following we will neglect isospin–breaking effects but in ϵ'/ϵ (§ sect. 3.3), since there are not sufficient data to systematically analyze isospin breaking beyond $K \to 2\pi$[91,106].

[g] As can be seen from Eq. (3.30), the contribution of $(8_L, 8_R)$ operators in the real parts is completely negligible.

amplitude	exper. fit	$O(p^2)$ – prediction
a_c	-95.4 ± 0.4	-76.0 ± 0.3
a_n	$+84.4 \pm 0.6$	$+69.9 \pm 0.6$
b_c	-26.9 ± 0.3	-17.5 ± 0.1
b_n	-28.1 ± 0.5	-18.0 ± 0.2
b_2	-3.9 ± 0.4	-3.1 ± 0.4

Table 7: Comparison between experimental data and lowest order CHPT predictions for the dominant $K \to 3\pi$ amplitudes[106] (§ sect. 5).

$$\begin{aligned} \Im m g_{27} &= \Im m \left(\frac{\lambda_t}{\lambda_u}\right) \left[\frac{6C_9}{9} B_9^{3/2}\right], \\ \Im m g_8 &= \Im m \left(\frac{\lambda_t}{\lambda_u}\right) \left[-C_7 B_7^{3/2} - 3C_8 B_8^{3/2}\right] \left(\frac{Y}{\sqrt{2}F_\pi^3}\right). \end{aligned} \tag{4.37}$$

Once fixed the g_i, by comparison with $K \to 2\pi$ amplitudes, the theory is absolutely predictive in all other non–leptonic channels: the comparison between these predictions and the experimental data leads to useful indications about the convergence of the derivative (or chiral) expansion. In table 7 we report the results of a fit[106] on the experimental data, together with the predictions of $\mathcal{L}_W^{(2)}$, for the dominant $K \to 3\pi$ amplitudes (§ sect. 5). As can be noticed, the discrepancy between lowest order (order p^2) chiral predictions and data is about 30%. To obtain a better agreement is necessary to consider next–order (order p^4) corrections[106]. As we shall see in sect. 6, the need of considering $O(p^4)$ terms is even more evident in the case of radiative decays, where the lowest order predictions vanish except for the bremsstrahlung amplitudes.

4.3 Generating functional at order p^4.

In the previous subsection we have seen how to build the chiral realization of $\mathcal{L}_{QCD}$ and of the non–leptonic effective hamiltonian at the lowest order in the chiral expansion. At this order Green functions can be calculated using the above lagrangians at tree level. On the other hand, at the next order, is necessary to calculate the whole generating functional to obtain Green functions in terms of meson fields.

In the case of strong interactions we can rewrite the generating functional (4.13) in the following way:

$$e^{iZ(v,a,\hat{s},\hat{p})} = \int \mathcal{D}U(\Phi)\; e^{i\int d^4x \mathcal{L}_S(U,v,a,\hat{s},\hat{p})} \tag{4.38}$$

where $\mathcal{L}_S(U,v,a,\hat{s},\hat{p})$ is a local function of meson fields and external sources. Since $Z(v,a,\hat{s},\hat{p})$ is locally invariant for chiral transformations, except for the anomalous term[107,108], it is natural to expect that also $\mathcal{L}_S(U,v,a,\hat{s},\hat{p})$ be locally invariant[25]. Indeed, it has been shown by Leutwyler[109] that the freedom in the definition of U and $\mathcal{L}_S$ let us

always to put the latter in a locally–chiral–invariant form. Only the anomalous part of the functional cannot be written in terms of locally–invariant operators[110,111].

The expansion of $\mathcal{L}_S$ in powers of p, by means of the power counting rules (4.20–4.21),

$$\mathcal{L}_S = \mathcal{L}_S^{(2)} + \mathcal{L}_S^{(4)} + ..., \tag{4.39}$$

induces a corresponding expansion of the generating functional. At the lowest order we have

$$Z^{(2)}(v, a, \hat{s}, \hat{p}) = \int \mathrm{d}^4 x \mathcal{L}_S^{(2)}(U, v, a, \hat{s}, \hat{p}). \tag{4.40}$$

At the next order is necessary to consider both one–loop amplitudes generated by $\mathcal{L}_S^{(2)}$ and local terms of $\mathcal{L}_S^{(4)}$. As anticipated at the beginning of this section, $\mathcal{L}_S^{(2)}$ is non renormalizable, however, by symmetry arguments, all one–loop divergences generated by $\mathcal{L}_S^{(2)}$ which cannot be re–absorbed in a re–definition of $\mathcal{L}_S^{(2)}$ coefficients have exactly the same structure of $\mathcal{L}_S^{(4)}$ local terms. The same happens at order p^6: non–re–absorbed divergences generated at two loop by $\mathcal{L}_S^{(2)}$ and at one loop by $\mathcal{L}_S^{(4)}$ have the structure of $\mathcal{L}_S^{(6)}$ local terms. Thus the theory is renormalizable order by order in the chiral expansion.

It is important to remark that loops play a fundamental role: generating the imaginary parts of the amplitudes let us to implement the unitarity of the theory in a perturbative way.

Furthermore, the loop expansion suggest a natural scale for the expansion in powers of p, i.e. for the scale Λ_χ which rules the suppression (p^2/Λ_χ^2) of $O(p^{n+2})$ terms with respect to $O(p^n)$ ones. Since any loop carries a factor $1/(4\pi F)^2$ ($1/F^2$ comes from the expansion of U and $1/16\pi^2$ from the integration on loop variables), the naive expectation is[112]

$$\Lambda_\chi \sim 4\pi F_\pi = 1.2 \text{ GeV}. \tag{4.41}$$

Obviously Eq. (4.41) is just and indicative estimate of Λ_χ, more refined analysis suggest that is lightly in excess (see the discussion in Ref.[95]), but is sufficient to understand that in kaon decays, where $|p| \leq M_K$, the convergence might be slow, as shown in the previous subsection.

At order p^4, beyond loops and local terms of $\mathcal{L}_S^{(4)}$ there is also the anomalous term, the so–called Wess–Zumino–Witten functional[110,111] (Z_{WZW}). Thus the complete expression of $Z^{(4)}$ is:

$$Z^{(4)}(v, a, \hat{s}, \hat{p}) = \int \mathrm{d}^4 x \mathcal{L}_S^{(4)}(U, v, a, \hat{s}, \hat{p}) + Z_{1-loop}^{(4)} + Z_{WZW}. \tag{4.42}$$

Whereas Z_{WZW} is finite and is not renormalized (§ sect. 4.3.2), $Z_{1-loop}^{(4)}$ is divergent and is necessary to regularize it. Using dimensional regularization, chiral power counting insures that the divergent part of $Z_{1-loop}^{(4)}$ is of order p^4 and, as already stated, has the structure of $\mathcal{L}_S^{(4)}$ local terms. In d dimension we can write

$$Z_{1-loop}^{(4)} = -\Lambda(\mu) \sum_i \gamma_i O_i^{(4)} + Z_{1-loop}^{(4)fin}(\mu), \tag{4.43}$$

where

$$\Lambda(\mu) = \frac{\mu^{d-4}}{(4\pi)^2}\left\{\frac{1}{d-4} - \frac{1}{2}\left[\ln(4\pi) + 1 + \Gamma'(1)\right]\right\}, \tag{4.44}$$

γ_i are appropriate coefficients, independent of d, and $Z_{1-loop}^{(4)fin}(\mu)$ is finite in the limit $d \to 4$. By this way, calling L_i the coefficients of $\mathcal{L}_S^{(4)}$ operators:

$$\mathcal{L}_S^{(4)} = \sum L_i O_i^{(4)}, \tag{4.45}$$

and defining

$$L_i = L_i^r(\mu) + \gamma_i \Lambda(\mu), \tag{4.46}$$

the sum of the first two terms in Eq. (4.42) is renormalized:

$$\int \mathrm{d}^4x \mathcal{L}_S^{(4)}(L_i) + Z_{1-loop}^{(4)} = \int \mathrm{d}^4x \mathcal{L}_S^{(4)}(L_i^r(\mu)) + Z_{1-loop}^{(4)fin}(\mu). \tag{4.47}$$

4.3.1 $O(p^4)$ Strong counterterms.

The most general lagrangian of order p^4, invariant under local–chiral transformations, Lorentz transformations, P, C and T, consists of 12 operators[26]:

$$\begin{aligned}
\mathcal{L}_S^{(4)} &= L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle \\
&+ L_3\langle D_\mu U^\dagger D_\mu U D^\nu U^\dagger D^\nu U\rangle + L_4\langle D_\mu U^\dagger D^\mu U\rangle\langle \chi^\dagger U + U^\dagger\chi\rangle \\
&+ L_5\langle D_\mu U^\dagger D^\mu U(\chi^\dagger U + U^\dagger\chi)\rangle + L_6\langle \chi^\dagger U + U^\dagger\chi\rangle^2 + L_7\langle \chi^\dagger U - U^\dagger\chi\rangle^2 \\
&+ L_8\langle \chi^\dagger U\chi^\dagger U + U^\dagger\chi U^\dagger\chi\rangle - iL_9\langle F_R^{\mu\nu} D_\mu U D^\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D^\nu U\rangle \\
&+ L_{10}\langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle + L_{11}\langle F_R^{\mu\nu} F_{R\mu\nu} + F_L^{\mu\nu} F_{L\mu\nu}\rangle + L_{12}\langle \chi\chi^\dagger\rangle.
\end{aligned} \tag{4.48}$$

Since at order p^4 this lagrangian operate only at the tree level, the equation of motion of $L_S^{(2)}$,

$$\Box U U^\dagger - U\Box U^\dagger = \chi U^\dagger - U\chi^\dagger - \frac{1}{3}\langle \chi U^\dagger - U\chi^\dagger\rangle, \tag{4.49}$$

has been used to reduce the number of independent operators[26].

The constants $L_1 \div L_{10}$ of Eq. (4.48) are not determined by the theory alone and must be fixed by experimental data. The value of the renormalized constants, defined by Eq. (4.46), together with the corresponding scale factor γ_i and the processes used to fix them are reported in table 8 at $\mu = M_\rho \simeq 770$ GeV. To obtain the $L_i^r(\mu)$ at different scales, using Eq. (4.46) we find

$$L_i^r(\mu_1) = L_i^r(\mu_2) + \frac{\gamma_i}{(4\pi)^2}\ln\frac{\mu_2}{\mu_1}. \tag{4.50}$$

It is important to remark that the processes where the $L_i^r(\mu)$ appear are more than those used to fix them, thus the theory is predictive (see e.g. Refs.[95,39] for a discussion on CHPT tests in the strong–interaction sector).

i	$L_i^r(M_\rho) \times 10^3$	process	γ_i
1	0.4 ± 0.3	K_{e4}, $\pi\pi \to \pi\pi$	3/32
2	1.35 ± 0.3	K_{e4}, $\pi\pi \to \pi\pi$	3/16
3	-3.5 ± 1.1	K_{e4}, $\pi\pi \to \pi\pi$	0
4	-3.5 ± 0.5	Zweig rule	1/8
5	1.4 ± 0.5	F_K/F_π	3/8
6	-0.2 ± 0.3	Zweig rule	11/144
7	-0.4 ± 0.2	Gell-Mann-Okubo, L_5, L_8	0
8	0.9 ± 0.3	$M_{K^0} - M_{K^+}$, L_5	5/48
9	6.9 ± 0.7	$\langle r^2 \rangle_V^\pi$	1/4
10	-5.5 ± 0.7	$\pi \to e\nu\gamma$	$-1/4$
11			$-1/8$
12			5/24

Table 8: Values of the $L_i^r(M_\rho)$, processes used to fix them, and relative scale factors[113].

The constants L_{11} and L_{12} are not measurable because the corresponding operators are contact terms of the external–field, necessary to renormalize the theory but without any physical meaning.

Finally, using the $L_i^r(M_\rho)$ fixed by data, we can verify the reliability of the naive estimate of Λ_χ (4.41). Using, as an example, $L_9^r(M_\rho)$ (the largest value in table 8), from the tree–level calculation of the electromagnetic pion form factor, follows

$$f_\pi^{e.m.}(t) \doteq 1 + \langle r^2 \rangle_V^\pi t + O(t^2) = 1 + \frac{2L_9^r(M_\rho)}{F_\pi^2} t + O(t^2), \tag{4.51}$$

which implies

$$\Lambda_\chi^2 \gtrsim \frac{F_\pi^2}{2L_9^r(M_\rho)} \simeq M_\rho^2. \tag{4.52}$$

4.3.2 The WZW functional.

The generating functional which reproduces the QCD chiral anomaly in terms of meson fields was originally built by Wess and Zumino[110], successively has been re–formulated by Witten[111] in the following way:

$$\begin{aligned} Z_{WZW}(l,r) &= -\frac{iN_c}{240\pi^2} \int_{M^5} \mathrm{d}^5 x \epsilon^{ijklm} \langle U^\dagger \partial_i U \partial_j U^\dagger \partial_k U \partial_l U^\dagger \partial_m U \rangle \\ &\quad -\frac{iN_c}{48\pi^2} \int \mathrm{d}^4 x \epsilon^{\mu\nu\rho\sigma} \left[W(U,l,r)_{\mu\nu\rho\sigma} - W(1,l,r)_{\mu\nu\rho\sigma} \right], \end{aligned} \tag{4.53}$$

where

$$W(U,l,r)_{\mu\nu\rho\sigma} = \langle U l_\mu l_\nu l_\rho U^\dagger r_\sigma + \frac{1}{4} U l_\mu U^\dagger r_\nu U l_\rho U^\dagger r_\sigma + i U \partial_\mu l_\nu l_\rho U^\dagger r_\sigma$$

$$
\begin{aligned}
&+i\partial_\mu r_\nu U l_\rho U^\dagger r_\sigma - iU^\dagger\partial_\mu U l_\nu U^\dagger r_\rho U l_\sigma - \partial_\mu U^\dagger \partial_\nu r_\rho U l_\sigma \\
&+\partial_\mu U^\dagger \partial_\nu U U^\dagger r_\rho U l_\sigma + U^\dagger \partial_\mu U l_\nu \partial_\rho l_\sigma + U^\dagger \partial_\mu U \partial_\nu l_\rho l_\sigma \\
&-iU^\dagger \partial_\mu U l_\nu l_\rho l_\sigma + \frac{1}{2} U^\dagger \partial_\mu U l_\nu U^\dagger \partial_\rho U l_\sigma + iU^\dagger \partial_\mu U \partial_\nu U^\dagger \partial_\rho U l_\sigma \\
&-(U \leftrightarrow U^\dagger, l_\mu \leftrightarrow r_\mu)\rangle. \qquad (4.54)
\end{aligned}
$$

The Z_{WZW} functional let us to compute all the contributions generated by the chiral anomaly to electromagnetic and semileptonic decays of pseudoscalar mesons. This does not mean that there are no other contributions to these decays. However, since Z_{WZW} satisfies the anomalous Ward identities, contributions not generated by Z_{WZW} must be locally invariant under chiral transformations.

Chiral power counting insures that Z_{WZW} coefficients are not renormalized by next-order contributions (for a detailed discussion about the odd-intrinsic-parity sector at $O(p^6)$ see Refs.[114,95]).

4.3.3 $O(p^4)$ Weak counterterms.

Also in the case of non-leptonic transitions, in order to calculate the Green functions at order p^4 is convenient to introduce an appropriate generating functional. Since we are interested only in contributions of order G_F, we proceed analogously to the strong interaction case (§ sect. 4.3) with the simple substitution

$$
\begin{aligned}
\mathcal{L}_S^{(2)} &\rightarrow \mathcal{L}_S^{(2)} + \mathcal{L}_W^{(2)}, \\
\mathcal{L}_S^{(4)} &\rightarrow \mathcal{L}_S^{(4)} + \mathcal{L}_W^{(4)}, \qquad (4.55)
\end{aligned}
$$

where $\mathcal{L}_W^{(4)}$ is an $O(p^4)$ lagrangian that transforms linearly under G like $\mathcal{L}_S^{(2)}$ and consequently absorbs all one-loop divergences generated by $\mathcal{L}_S^{(2)} \times \mathcal{L}_W^{(2)}$.

The operators of order p^4 which transforms like $(8_L, 1_R)$ and $(27_L, 1_R)$ under G, have been classified for the first time by Kambor, Missimer and Wyler[115]: the situation is worse than in the strong case because the number of independent operators is much larger. For this reason, since $\Delta I = 3/2$ amplitudes are experimentally very suppressed, following Ecker, Kambor and Wyler[116] we shall limit to consider only $(8_L, 1_R)$ operators.

In Ref.[116] the number of independent $(8_L, 1_R)$ operators has been reduced to 37 using the lowest order equation of motion for U and the Cayley-Hamilton theorem. Successively, terms that contribute only to processes with external W bosons (i.e. terms which generate $O(G_F^2)$ corrections to semileptonic decays) and contact terms have been isolated. By this way, the number of independent operators relevant to non-leptonic kaon decays at $O(G_F p^4)$ turns out to be only 22.

In the basis of Ref.[116] the $(8_L, 1_R)$ component of the $O(p^4)$ weak lagrangian is written in the following way

$$
\mathcal{L}_W^{(4)} = G_8 F^2 \sum_{i=1}^{37} N_i W_i^{(4)} + \text{h.c.}, \qquad (4.56)
$$

i	$W_i^{(4)}$	decay channel	γ_i
1	$\langle \lambda u^\dagger u_\mu u^\mu u_\nu u^\nu u \rangle$	$\geq 3\pi$	2
2	$\langle \lambda u^\dagger u_\mu u_\nu u^\nu u^\mu u \rangle$	$\geq 3\pi$	$-1/2$
3	$\langle \lambda u^\dagger u_\mu u_\nu u \rangle \langle u^\mu u^\nu \rangle$	$\geq 3\pi$	0
4	$\langle \lambda u^\dagger u_\mu u \rangle \langle u^\mu u_\nu u^\nu \rangle$	$> 3\pi$	1
5	$\langle \lambda u^\dagger \{\chi_+, u_\mu u^\mu\} u \rangle$	$\geq 2\pi$	3/2
6	$\langle \lambda u^\dagger u_\mu u \rangle \langle \chi_+ u^\mu \rangle$	$\geq 2\pi$	$-1/4$
7	$\langle \lambda u^\dagger \chi_+ u \rangle \langle u_\mu u^\mu \rangle$	$\geq 2\pi$	$-9/8$
8	$\langle \lambda u^\dagger u_\mu u^\mu u \rangle \langle \chi_+ \rangle$	$\geq 2\pi$	$-1/2$
9	$\langle \lambda u^\dagger [\chi_-, u_\mu u^\mu] u \rangle$	$\geq 2\pi$	3/4
10	$\langle \lambda u^\dagger \chi_+^2 u \rangle$	$\geq 2\pi$	2/3
11	$\langle \lambda u^\dagger \chi_+ u \rangle \langle \chi_+ \rangle$	$\geq 2\pi$	$-13/18$
12	$\langle \lambda u^\dagger \chi_-^2 u \rangle$	$\geq 2\pi$	$-5/12$
13	$\langle \lambda u^\dagger \chi_- u \rangle \langle \chi_- \rangle$	$\geq 2\pi$	0
14	$i\langle \lambda u^\dagger \{f_+^{\mu\nu}, u_\mu u_\nu\} u \rangle$	$\geq \pi\gamma\ (E)$	1/4
15	$i\langle \lambda u^\dagger u_\mu f_+^{\mu\nu} u_\nu u \rangle$	$\geq \pi\gamma\ (E)$	1/2
16	$i\langle \lambda u^\dagger \{f_-^{\mu\nu}, u_\mu u_\nu\} u \rangle$	$\geq 2\pi\gamma\ (E)$	$-1/4$
17	$i\langle \lambda u^\dagger u_\mu f_-^{\mu\nu} u_\nu u \rangle$	$\geq 2\pi\gamma\ (E)$	0
18	$\langle \lambda u^\dagger \left(f_{+\mu\nu}^2 - f_{-\mu\nu}^2\right) u \rangle$	$\geq \pi\gamma\gamma\ (E)$	$-1/8$
28	$i\epsilon_{\mu\nu\rho\sigma} \langle \lambda u^\dagger u^\mu u \rangle \langle u^\nu u^\rho u^\sigma \rangle$	$\geq 3\pi\gamma\ (M)$	0
29	$\langle \lambda u^\dagger \left[\tilde{f}_{+\mu\nu} - \tilde{f}_{-\mu\nu}, u^\mu u^\nu\right] u \rangle$	$\geq 2\pi\gamma\ (M)$	0
30	$\langle \lambda u^\dagger u^\mu u \rangle \langle u^\nu \tilde{f}_{+\mu\nu} \rangle$	$\geq \pi\gamma\ (M)$	0
31	$\langle \lambda u^\dagger u^\mu u \rangle \langle u^\nu \tilde{f}_{-\mu\nu} \rangle$	$\geq 2\pi\gamma\ (M)$	0

Table 9: $W_i^{(4)}$ operators, in the basis of Ref.[116], relevant to non–leptonic kaon decays at $O(G_F)$, with relative scale factors. In the third column are indicated the processes which the operators can contribute to: the symbol $>$ indicates that π or γ can be added, whereas (E) and (M) indicate electric and magnetic transitions, respectively; no distinction is made for real or virtual photons.

where the N_i are adimensional constants. The 22 relevant–operator $W_i^{(4)}$ are reported in table 9, where, for simplicity, has been introduced the fields

$$\begin{aligned} f_\pm^{\mu\nu} &= uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_R^{\mu\nu}u, \qquad \tilde{f}_{\pm\mu\nu} = \epsilon_{\mu\nu\rho\sigma}f_\pm^{\rho\sigma}, \\ \chi_\pm &= u^\dagger\chi u^\dagger \pm u\chi u. \end{aligned} \tag{4.57}$$

Analyzing the effects of the $W_i^{(4)}$ in $K \to 2\pi$, $K \to 3\pi$, $K \to \pi\gamma^*$, $K \to \pi\gamma\gamma$, $K \to 2\pi\gamma$ and $K \to 3\pi\gamma$ decays, some interesting consequences (which we shall discuss more in detail in the next sections) can be deduced:

- It is not possible to fix the coefficients $N_5 \div N_{13}$: their effect is just to renormalize the value of G_8 fixed at $O(p^2)$ (in principle, some combinations could be fixed by off–shell processes, like $K \to \pi\pi^*$).

- Two combinations of $N_1 \div N_3$ can be fixed by widths and linear slopes of $K \to 3\pi$, then is possible to make predictions for the quadratic slopes of these decays[117] (§ sect. 5). As shown in Ref.[118], radiative non–leptonic processes do not add further information about $N_1 \div N_{13}$.

- The coefficients $N_{14} \div N_{18}$ and three independent combinations of $N_{28} \div N_{31}$ could in principle fixed by the analysis of radiative non–leptonic decays (unfortunately present data are too poor). Then, also in this case several predictions could be made[119] (§ sect. 6).

Obviously, the above statements are valid only for the real parts of the coefficients N_i. For what concerns the imaginary parts, related to CP violation, to date there are neither useful experimental information nor lattice results. In order to make definite predictions is necessary to implement an hadronization model. Nevertheless, as we shall see in the following, chiral symmetry alone is still very useful to relate each other different CP–violating observables.

4.4 Models for counterterms.

Due to the large number of $O(p^4)$ counterterms, both in the strong and expecially in the non–leptonic weak sector, it is interesting to consider theoretical models which let us to predict the value of counterterms at a given scale. By construction these models have nothing to do with the chiral constraints, already implemented, but are based on additional less–rigorous assumptions dictated by the phenomenology of strong interactions at low energy.

There are different classes of such models (for an extensive discussion see Ref.[38,39]); one of the most interesting hypothesis is the idea that counterterms are saturated, around $\mu = M_\rho$, by the contributions coming from low–energy–resonance (ρ, ω, η', etc...) exchanges[120,121]. In the framework of this hypothesis (known as 'chiral duality') it is assumed that the dominant contribution is generated by spin–1 mesons, in agreement with the old idea of 'vector meson dominance'.

In order to calculate the resonance contribution to counterterms, is necessary: i) to consider the most general chiral–invariant lagrangian containing both resonance and pseudoscalar meson fields; ii) to integrate over the resonance degrees of freedom, in order to obtain a non–local effective action for pseudoscalar mesons only; iii) to expand this action in terms of local operators. Since strong and electromagnetic coupling constants of resonance fields are experimentally known, in the case of $\mathcal{L}_S^{(4)}$ this procedure leads to interesting unambiguous predictions[121].

As an example, to calculate spin–1 resonance effects, we can introduce two antisymmetric tensors $V^{\mu\nu}$ and $A^{\mu\nu}$, which describe the $SU(3)_{L+R}$ octets of 1^{--} and 1^{++} resonances, and which under G transform in the following way:

$$R^{\mu\nu} \xrightarrow{G} h(g,\phi_i) R^{\mu\nu} h^{-1}(g,\phi_i) \qquad R^{\mu\nu} = V^{\mu\nu},\ A^{\mu\nu}. \tag{4.58}$$

The lowest–order chiral lagrangian describing $V^{\mu\nu}$ and $A^{\mu\nu}$ interactions with pseudoscalar mesons and gauge fields is:[h]

$$\begin{aligned} \mathcal{L}_{V,A}^{(2)} &= \mathcal{L}_{kin}(V) + \mathcal{L}_{kin}(A) + \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu}\rangle \\ &\quad + \frac{iG_V}{\sqrt{2}}\langle V_{\mu\nu} u^\mu u^\nu\rangle + \frac{F_A}{2\sqrt{2}}\langle A_{\mu\nu} f_-^{\mu\nu}\rangle, \end{aligned} \tag{4.59}$$

where

$$\mathcal{L}_{kin}(R) = -\frac{1}{2}\langle \nabla^\mu R_{\mu\nu}\ \nabla_\sigma\ R^{\sigma\nu}\rangle + \frac{1}{4}M_R^2\langle R_{\mu\nu}R^{\mu\nu}\rangle. \tag{4.60}$$

Integrating over resonance degrees of freedom and expanding up to $O(p^4)$, leads to identify the following contribution to the L_i:

$$\begin{aligned} &L_1^V = \frac{G_V^2}{8M_V^2}, \qquad L_2^V = -L_3^V/3 = 2L_1^V, \qquad L_9^V = \frac{F_V G_V}{2M_V^2}, \\ &L_{10}^V = \frac{F_A^2}{4M_A^2} - \frac{F_V^2}{4M_V^2}, \qquad L_4^V = L_5^V = L_6^V = L_7^V = L_8^V = 0. \end{aligned} \tag{4.61}$$

The constants G_V, F_V and F_A can be experimentally fixed by the measurements of $\Gamma(V \to \pi\pi)$, $\Gamma(V \to e^+e^-)$ and $\Gamma(A \to \pi\gamma)$, respectively. The results obtained by this procedure for the non–vanishing L_i^V are reported in the second column of table 10: as can be noticed the agreement with the fitted L_i is very good. For the constants L_{4-8}, which do not receive any contribution from spin–1 resonances, is necessary to calculate the contribution of scalar resonances. At any rate, as can be noticed from table 8, these constants have smaller values respect to the dominant ones (L_3, L_9 and L_{10}) to which spin–1 resonance contribute. We finally note that imposing on the lagrangian (4.59) Weinberg sum rules[123] and the so–called KSFR relations[124,125] (which are in good agreement with experimental data) then F_V, G_V, F_A and M_A satisfy the following identities:

$$F_V = 2G_V = \sqrt{2}F_A = \sqrt{2}F_\pi, \qquad M_A = \sqrt{2}M_V. \tag{4.62}$$

[h] Actually, the choice of Eqs. (4.58–4.60) to describe spin–1 resonances couplings is not unique, there exist different formulations which however lead to equivalent results[122].

i	$L_i^r(M_\rho) \times 10^3$	$L_i^V \times 10^3$	$L_i^V \times 10^3$ $(*)$
1	0.4 ± 0.3	0.6 ± 0.2	0.9
2	1.35 ± 0.3	1.2 ± 0.3	1.8
3	-3.5 ± 1.1	-3.0 ± 0.7	-4.9
9	6.9 ± 0.7	7.0 ± 0.4	7.3
10	-5.5 ± 0.7	-6.0 ± 0.8	-5.5

Table 10: Comparison between the fitted $L_i^r(M_\rho)$ (first column) and the vector–meson–dominance predictions (4.61). The values in the second column have been obtained with F_V, G_V, F_A, M_V and M_A fixed by experimental data, the corresponding errors are related to the different possibilities to fix these constants (F_V, as an example, can be fixed either from $\Gamma(\rho \to e^+e^-)$ or from $\Gamma(\omega \to e^+e^-)$). The values reported in the last column have been obtained using the relations (4.62) and fixing $M_V = M_\rho$.

As a consequence, in this case the L_i^V can be expressed in term of a single parameter: M_V. The values of the L_i^V thus obtained are reported in the third column of table 10: in spite of the simplicity of the model, even in this case the agreement is remarkable.

4.4.1 The factorization hypothesis of $\mathcal{L}_W$.

Clearly, in the sector of non–leptonic weak interactions the situation is more complicated since there are no experimental information about weak resonance couplings. To make predictions is necessary to add further assumptions, one of these is the factorization hypothesis[103,126,127]. Since the dominant terms of the four–quarks hamiltonian are factorizable as the product of two left–handed currents, we assume that also the chiral weak lagrangian has the same structure.

As we have seen in sect.4.2, the lowest–order chiral realization of the left–handed current $\bar{q}_L \gamma^\mu q_L$ is given by the functional derivative of $Z^{(2)}$ with respect to the external source l_μ:

$$J_\mu^{(1)} = \frac{\delta Z^{(2)}(l, r, \hat{s}, \hat{p})}{\delta l_\mu} = -\frac{1}{2} F^2 L_\mu. \tag{4.63}$$

Furthermore, since the lowest–order weak lagrangian can be written as

$$\mathcal{L}_W^{(2)} = 4G_8 \langle \lambda J_\mu^{(1)} J^{\mu(1)} \rangle + \text{h.c.}, \tag{4.64}$$

the factorization hypothesis of $\mathcal{L}_W^{(4)}$ consists of assuming the following structure:

$$\mathcal{L}_{W-fact}^{(4)} = 4k_f G_8 \langle \lambda \left\{ J_\mu^{(1)}, J^{\mu(3)} \right\} \rangle + \text{h.c.}, \tag{4.65}$$

where k_f is a positive parameter of order 1 and $J^{\mu(3)}$ is the chiral realization of the left–handed current at order p^3. In general $J^{\mu(3)}$ can be expressed as functional derivative of

$\mathcal{L}_S^{(4)}$, with respect to the source l_μ, and in this case depends on the value of the $L_i^r(\mu)$. In order to make the model more predictive, relating it to the vector–meson–dominance hypothesis previously discussed, one can assume $L_i^r(M_\rho) = L_i^V$.

To date, experimental data on weak $O(p^4)$ counterterms are very poor, not sufficient to draw quantitative conclusions about the validity of the factorization hypothesis. At any rate, in the only channels where there are useful and reliable experimental information, i.e. $K \to 3\pi$ and $K^+ \to \pi^+e^+e^-$ decays, the estimates of the sign and of the order of magnitude of counterterms, effectuated within this model, are more or less correct[116]. Only in the next years, when new high–statistics data on both neutral and charged kaon decays will be available, it will be possible to make an accurate analysis of the non–leptonic sector. With the expected new data it will be possible not only to test the factorization model, but also to study in general the convergence of the chiral expansion at $O(p^4)$ in the non–leptonic sector.

5 $K \to 3\pi$ decays.

5.1 Amplitude decomposition.

There are four distinct channels for $K \to 3\pi$ decays:

$$\begin{array}{rll} K^\pm & \to \pi^\pm\pi^\pm\pi^\mp & I = 1,2, \\ K^\pm & \to \pi^0\pi^0\pi^\pm & I = 1,2, \\ K^0(\bar{K}^0) & \to \pi^\pm\pi^\mp\pi^0 & I = 0,1,2, \\ K^0(\bar{K}^0) & \to \pi^0\pi^0\pi^0 & I = 1. \end{array} \tag{5.1}$$

Near each channel we have indicated the final–state isospin assuming $\Delta I \leq 3/2$.

In order to write the transition amplitudes, it is convenient to introduce the following kinematical variables:

$$s_i \doteq (p_K - p_i)^2, \qquad \text{and} \qquad s_0 \doteq \frac{1}{3}(s_1 + s_2 + s_3) = \frac{1}{3}M_K^2 + \frac{1}{3}\sum_{i=1}^{3} M_{\pi_i}^2, \tag{5.2}$$

where p_K and p_i denote kaon and π_i momenta (π_3 indicates the odd pion in the first three channels). With these definition, the isospin decomposition of $K \to 3\pi$ amplitudes is given by[128–130]:

$$\begin{aligned} A_{++-} &= A(K^+ \to \pi^+\pi^+\pi^-) = 2A_c(s_1,s_2,s_3) + B_c(s_1,s_2,s_3) + B_2(s_1,s_2,s_3), \\ A_{+00} &= A(K^+ \to \pi^0\pi^0\pi^+) = A_c(s_1,s_2,s_3) - B_c(s_1,s_2,s_3) + B_2(s_1,s_2,s_3), \\ A_{+-0} &= \sqrt{2}A(K^0 \to \pi^+\pi^-\pi^0) = A_n(s_1,s_2,s_3) - B_n(s_1,s_2,s_3) + C_0(s_1,s_2,s_3) \\ &\qquad +2[B_2(s_3,s_2,s_1) - B_2(s_1,s_3,s_2)]/3, \\ A_{000} &= \sqrt{2}A(K^0 \to \pi^0\pi^0\pi^0) = 3A_n(s_1,s_2,s_3). \end{aligned} \tag{5.3}$$

The amplitudes A_i, B_i ($i = c,n,2$) and C_0 transform in the following way under s_i permutations: the A_i are completely symmetric, C_0 is antisymmetric for any exchange $s_i \leftrightarrow s_j$,

finally the B_i are symmetric in the exchange $s_1 \leftrightarrow s_2$ and obey to the relation

$$B_i(s_1, s_2, s_3) + B_i(s_3, s_2, s_1) + B_i(s_1, s_3, s_2) = 0. \tag{5.4}$$

For what concerns isospin, $A_{c,n}$ and $B_{c,n}$ belong to transitions in $I = 1$, whereas B_2 and C_0 belong to $I = 2$ and $I = 0$, respectively.

Differently than in $K \to 2\pi$, in the first three channels of Eq. (5.3) there are two amplitudes, which differ for the transformation property under s_i–permutations, that lead to the same final state ($I = 1$). For this reason is convenient to introduce the two matrices

$$T_c = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \qquad T_n = \begin{pmatrix} 1 & -1 \\ 3 & 0 \end{pmatrix}, \tag{5.5}$$

which project the symmetric and the antisymmetric components of the $I = 1$ state in the physical channels for charged and neutral kaon decays:

$$\begin{pmatrix} A^{(1)}_{++-} \\ A^{(1)}_{+00} \end{pmatrix} = T_c \begin{pmatrix} A_c(s_i) \\ B_c(s_i) \end{pmatrix} \qquad \begin{pmatrix} A^{(1)}_{+-0} \\ A^{(1)}_{000} \end{pmatrix} = T_n \begin{pmatrix} A_n(s_i) \\ B_n(s_i) \end{pmatrix}. \tag{5.6}$$

Experimentally, the event distributions in $K \to 3\pi$ transitions are analyzed in terms of two adimensional and independent variables:

$$X = \frac{s_2 - s_1}{M_\pi^2} \qquad \text{and} \qquad Y = \frac{s_3 - s_0}{M_\pi^2}, \tag{5.7}$$

the so–called Dalitz variables. Since the three–pion phase space is quite small ($M_K - 3M_\pi < 100$ MeV), terms with elevate powers of X and Y, corresponding to states with high angular momenta, are very suppressed (see Ref.[131] and references cited therein). Until now the distributions have been analyzed including up to quadratic terms in X and Y

$$|A(K \to 3\pi)|^2 \propto 1 + gY + jX + hY^2 + kX^2. \tag{5.8}$$

The parameters $g \div k$ are the 'Dalitz Plot slopes'. In table (11) we report the experimental data for the different channels.

To relate the decomposition (5.3) with experimental data is necessary to expand A_i, B_i and C_0 in terms of X and Y. According to the transformation properties under s_i–permutations follows:

$$\begin{aligned} A_i &= a_i + c_i(Y^2 + X^2/3) + ..., \\ B_i &= b_i Y + d_i(Y^2 - X^2/3) + e_i Y(Y^2 + X^2/3) + ..., \\ C_0 &= f_0 X(Y^2 - X^2/9) + ..., \end{aligned} \tag{5.9}$$

where dots indicate terms at least quartic in X and Y. The parameters $a_i, b_i, ...f_0$ are real if strong re–scattering is neglected and CP is conserved.

Since we are interested only in CP violating effects, we shall limit to consider only the dominant terms in each amplitude and we will neglect completely the C_0 amplitude that

channel	$\Gamma(s^{-1}) \times 10^{-6}$	$g \times 10^1$	$j \times 10^3$	$h \times 10^2$	$k \times 10^2$
$\pi^+\pi^+\pi^-$	4.52 ± 0.04	-2.15 ± 0.03	/	1.2 ± 0.8	-1.0 ± 0.3
$\pi^-\pi^-\pi^+$	4.52 ± 0.04	-2.17 ± 0.07	/	1.0 ± 0.6	-0.8 ± 0.2
$\pi^0\pi^0\pi^\pm$	1.40 ± 0.03	5.94 ± 0.19	/	3.5 ± 1.5	–
$(\pi^+\pi^-\pi^0)_L$	2.39 ± 0.04	6.70 ± 0.14	1.1 ± 0.8	7.9 ± 0.7	1.0 ± 0.2
$(\pi^+\pi^-\pi^0)_S$	$4.5^{+2.6}_{-1.5} \times 10^{-3}$	–	–	–	–
$(\pi^0\pi^0\pi^0)_L$	4.19 ± 0.16	/	/	-0.33 ± 0.13	–
$(\pi^0\pi^0\pi^0)_S$	< 0.4	/	/	–	–

Table 11: Experimental data for widths and slopes in $K \to 3\pi$ decays[18,132,133]. The symbol / indicates terms forbidden by Bose symmetry.

is very suppressed. With this assumption, the decomposition (5.3) contains at most linear terms in $X$and Y:

$$\begin{aligned} A_{++-} &= 2a_c + (b_c + b_2)Y, \\ A_{00+} &= a_c - (b_c - b_2)Y, \\ A_{+-0} &= a_n - b_n Y + \frac{2}{3} b_2 X, \\ A_{000} &= 3a_n. \end{aligned} \tag{5.10}$$

5.2 Strong re–scattering.

As we have seen in sect. 2, to estimate CP violation in charged–kaon decays is fundamental to know strong re–scattering phases of the final state.

Differently than in $K \to 2\pi$, $K \to 3\pi$ re–scattering phases are not constants but depend on the kinematical variables X and Y. Furthermore, in the $I = 1$ final state, the two amplitudes with different symmetry are mixed by re–scattering[130]. Projecting, by means of T_c and T_n, $I = 1$ physical amplitudes in the basis of amplitudes with definite symmetry, is possible to introduce a unique re–scattering matrix R, relative to the $I = 1$ final state, so that

$$\begin{pmatrix} A^{(1)}_{++-} \\ A^{(1)}_{+00} \end{pmatrix}_R = T_c R \begin{pmatrix} A_c \\ B_c \end{pmatrix} = T_c R T_c^{-1} \begin{pmatrix} A^{(1)}_{++-} \\ A^{(1)}_{+00} \end{pmatrix}, \tag{5.11}$$

$$\begin{pmatrix} A^{(1)}_{+-0} \\ A^{(1)}_{000} \end{pmatrix}_R = T_n R \begin{pmatrix} A_c \\ B_c \end{pmatrix} = T_n R T_n^{-1} \begin{pmatrix} A^{(1)}_{+-0} \\ A^{(1)}_{000} \end{pmatrix}. \tag{5.12}$$

The matrix R has diagonal elements which preserve the symmetry under s_i–permutations as well as off–diagonal elements which transform symmetric amplitudes into antisymmetric ones (and vice versa). Since the phase space is limited, we expect re–scattering phases to

be small, i.e. that R can be expanded in the following way:

$$R = 1 + i \begin{pmatrix} \alpha(s_i) & \beta'(s_i) \\ \alpha'(s_i) & \beta(s_i) \end{pmatrix}, \tag{5.13}$$

with $\alpha(s_i), \beta(s_i), \alpha'(s_i), \beta'(s_i) \ll 1$. Analogously, for the re–scattering in $I = 2$ we can introduce a phase $\delta(s_i) \ll 1$, so that

$$B_2(s_i)_R = B_2(s_i)\left[1 + i\delta(s_i)\right]. \tag{5.14}$$

Moreover, from the transformation properties of the amplitudes follows[130]

$$\begin{aligned} \alpha(s_i) &= \alpha_0 + O(X^2, Y^2), \\ \alpha'(s_i) &= \alpha'_0 Y + O(X^2, Y^2), \\ \beta(s_i) &= \beta_0 + O(X, Y), \\ \beta'(s_i) &= \beta'_0(Y^2 + X^2/3)/Y + O(X^2, Y^2), \\ \delta(s_i) &= \delta_0 + O(X, Y). \end{aligned} \tag{5.15}$$

With these definitions, the complete re–scattering of Eq. (5.10), including up to linear terms in X and Y, is given by:

$$\begin{aligned} (A_{++-})_R &= 2a_c[1 + i\alpha_0 + i\alpha'_0 Y/2] + b_c Y[1 + i\beta_0] + b_2 Y[1 + i\delta_0] \\ &= 2a_c[1 + i\alpha_0] + b_c Y\left[1 + i\left(\beta_0 + \frac{a_c}{b_c}\alpha'_0\right)\right] + b_2 Y[1 + i\delta_0], \\ (A_{00+})_R &= a_c[1 + i\alpha_0 - i\alpha'_0 Y] - b_c Y[1 + i\beta_0] + b_2 Y[1 + i\delta_0] \\ &= a_c[1 + i\alpha_0] - b_c Y\left[1 + i\left(\beta_0 + \frac{a_c}{b_c}\alpha'_0\right)\right] + b_2 Y[1 + i\delta_0], \\ (A_{+-0})_R &= a_n[1 + i\alpha_0 - i\alpha'_0 Y] - b_n Y[1 + i\beta_0] + \frac{2}{3} b_2 X[1 + i\delta_0] \\ &= a_n[1 + i\alpha_0] - b_n Y\left[1 + i\left(\beta_0 + \frac{a_n}{b_n}\alpha'_0\right)\right] + \frac{2}{3} b_2 X[1 + i\delta_0], \\ (A_{000})_R &= 3a_n[1 + i\alpha_0]. \end{aligned} \tag{5.16}$$

The first three amplitudes have been expressed in two different ways to stress that Y–dependent imaginary parts receive contributions from the re–scattering of both symmetric amplitudes ($a_{c,n}$) and antisymmetric ones ($b_{c,n}$).

5.3 CP–violating observables.

Considering only widths and linear slopes (as can be noticed from table 11, quadratic slopes have large errors), we can define the following CP–violating observables in $K \to 3\pi$ transitions:

$$\eta_{+-0} \doteq \left.\frac{A^S_{+-0}}{A^L_{+-0}}\right|_{X=Y=0} \doteq \epsilon + \epsilon'_{+-0}, \tag{5.17}$$

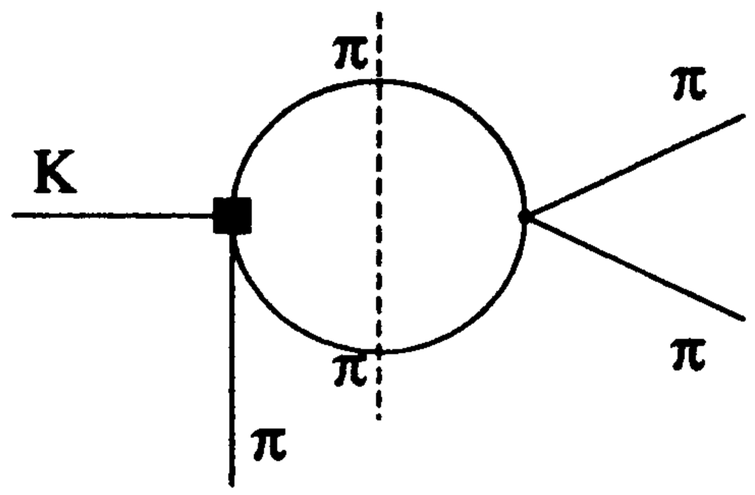

Figure 7: One–loop diagram for $K \to 3\pi$ re–scattering phases.

$$\eta_{000} \doteq \left.\frac{A^S_{000}}{A^L_{000}}\right|_{X=Y=0} \doteq \epsilon + \epsilon'_{000}, \tag{5.18}$$

$$\eta^X_{+-0} \doteq \left(\frac{\partial A^L_{+-0}/\partial X}{\partial A^S_{+-0}/\partial X}\right)_{X=Y=0} \doteq \epsilon + \epsilon^X_{+-0}, \tag{5.19}$$

$$(\delta_g)_\tau \doteq \frac{g^{++-} - g^{--+}}{g^{++-} + g^{--+}}, \tag{5.20}$$

$$(\delta_g)_{\tau'} \doteq \frac{g^{+00} - g^{-00}}{g^{+00} + g^{-00}}. \tag{5.21}$$

The first three observables belong to neutral kaons and, as explicitly shown, have an indirect CP–violating component. On the other hand $(\delta_g)_\tau$ and $(\delta_g)_{\tau'}$ are pure indices of direct CP violation. In principle, analogously to Eqs. (5.20–5.21), also the asymmetries of charged–kaon widths can be considered. However, since the integral over the Dalitz Plot of the terms linear in Y is zero, the width asymmetries are very suppressed respect to the slope asymmetries[105] and we will not consider them.

Using the definitions of K_S and K_L, and applying CPT to the decomposition (5.16), leads to

$$\epsilon'_{+-0} = \epsilon'_{000} = i\left(\frac{\Im m a_n}{\Re e a_n} - \frac{\Im m A_0}{\Re e A_0}\right), \tag{5.22}$$

$$\epsilon^X_{+-0} = i\left(\frac{\Im m b_2}{\Re e b_2} - \frac{\Im m A_0}{\Re e A_0}\right), \tag{5.23}$$

$$(\delta_g)_\tau = \frac{\Im m(a_c^* b_c)(\alpha_0 - \beta_0) + \Im m(a_c^* b_2)(\alpha_0 - \delta_0)}{\Re e(a_c^* b_c) + \Re e(a_c^* b_2)}, \tag{5.24}$$

$$(\delta_g)_{\tau'} = \frac{\Im m(a_c^* b_c)(\alpha_0 - \beta_0) - \Im m(a_c^* b_2)(\alpha_0 - \delta_0)}{\Re e(a_c^* b_c) - \Re e(a_c^* b_2)}. \tag{5.25}$$

where A_0 is the $K \to 2\pi$ decay amplitude in $I = 0$.

5.4 Estimates of CP violation.

The lowest order (p^2) CHPT results for the weak amplitudes of Eqs. (5.22–5.25) are:

$$a_c = -\frac{M_K^2}{3}\left[G_8 + \frac{2}{3}G_{27} + \frac{3F^2}{M_K^2}G_{\underline{8}}\right], \tag{5.26}$$

$$a_n = +\frac{M_K^2}{3}\left[G_8 - G_{27}\right], \tag{5.27}$$

$$b_c = +M_\pi^2\left[G_8 - \frac{7}{12}G_{27}\left(1 - \frac{15}{7}\rho_\pi\right) + \frac{3F^2}{4M_K^2}G_{\underline{8}}(1+\rho_\pi)\right], \tag{5.28}$$

$$b_2 = -M_\pi^2\left[\frac{15}{4}G_{27}\left(1 + \frac{1}{3}\rho_\pi\right) + \frac{3F^2}{4M_K^2}G_{\underline{8}}(1+\rho_\pi)\right], \tag{5.29}$$

where $\rho_\pi = M_\pi^2/(M_K^2 - M_\pi^2) \simeq 1/12$.

To estimate re–scattering phases at the lowest non–vanishing order in CHPT, is necessary to calculate the imaginary part of one–loop diagrams of fig. 7. The complete analytical results for the phases introduced in sect. 5.2 can be found in Refs.[105,130], for what concerns the parameters which enter in Eqs. (5.22–5.25) we have:

$$\alpha_0 = \frac{\sqrt{1 - 4M_\pi^2/s_0}}{32\pi F^2}(2s_0 + M_\pi^2) \simeq 0.13, \tag{5.30}$$

$$\beta_0 = -\delta_0 = \frac{\sqrt{1 - 4M_\pi^2/s_0}}{32\pi F^2}(s_0 - M_\pi^2) \simeq 0.05. \tag{5.31}$$

Using Eqs. (5.26–5.31) and the estimates of the imaginary parts of $L_W^{(2)}$ coefficients (§ sect. 4.2.2), we can finally predict the value of the observables (5.22–5.25) within the Standard Model[134].

5.4.1 Charge asymmetries.

In figs. 8 and 9 we show the results of a statistical analysis of $(\delta_g)_\tau$ and ϵ', obtained implementing in the program of Ref.[24] (§ fig. 5) the calculation of $(\delta_g)_\tau$.[a] The most interesting aspect of this analysis, as already stressed in Ref.[105], is that in $(\delta_g)_\tau$, differently than in ϵ', the interference between weak phases of $(8_L, 1_R)$ and $(8_L, 8_R)$ operators is constructive. Thus, within the Standard Model, charge asymmetries in $K^\pm \to (3\pi)^\pm$ decays could be more interesting than ϵ' in order to observe direct CP violation. Unfortunately, the theoretical estimates of these asymmetries are far from the expected sensitivities of next–future experiments (at KLOE[135] $\sigma[(\delta_g)_\tau]$ is expected to be[39] $\sim 10^{-4}$).

The results for $(\delta_g)_{\tau'}$ are very similar to those of $(\delta_g)_\tau$, a part from the sign which is opposite[105], we will not show them in detail since $(\delta_g)_\tau$ is more interesting from the experimental point of view. The mean value of $(\delta_g)_\tau$ and $(\delta_g)_{\tau'}$, together with the corresponding width asymmetries, are reported in table 12. As anticipated, the width asymmetries are

[a] For the real parts of the amplitudes a_c, b_c and b_2 we have used the experimental data.

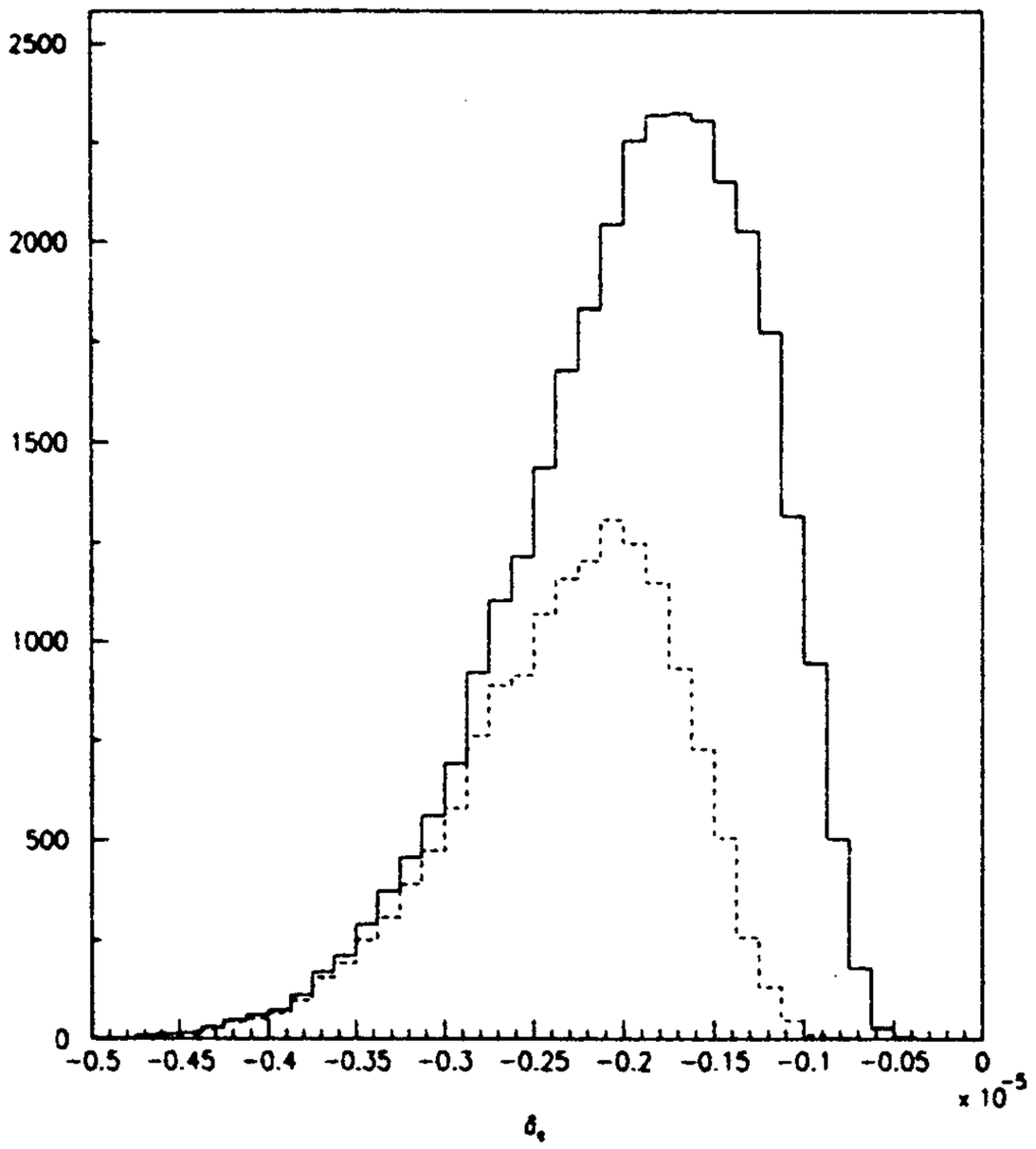

Figure 8: Predictions for the charge asymmetry $(\delta_g)_\tau$ at the lowest non–vanishing order in CHPT. The two histograms represent the probability distribution in arbitrary units, like in fig. 5; the dashed one has been obtained adding the supplementary conditions coming from $B_d - \bar{B}_d$ mixing .

definitively suppressed with respect to the slope asymmetries. Analogous results to those reported in table 12 have been obtained also by other authors[136,137,b].

It is important to remark that the previous analysis has been obtained using the lowest–order CHPT results for the weak amplitudes and, differently than in $K \to 2\pi$, could be sensibly modified by next–order corrections. The difference between $K \to 2\pi$ and $K \to 3\pi$ is that in former the CP–violating interference is necessarily between a $\Delta I = 1/2$ and a $\Delta I = 3/2$ amplitude, whereas in the latter the interference is between two $\Delta I = 1/2$ amplitudes (a_c and b_c). At the lowest order there is only one dominant $(8_L, 1_R)$ operator, thus the phase difference between a_c and b_c is determined by the suppressed $(27_L, 1_R)$ operator. At the next order, $O(p^4)$, there are different $(8_L, 1_R)$ operators and we can expect that the interference is no more suppressed by the ω factor[78]. Unfortunately in this case is not easy to make definite statements, since there are no reliable information about $O(p^4)$–operator weak phases. Nevertheless, according to general considerations, is still possible to put an interesting limit[138] on $(\delta_g)_\tau$.

[b] Actually, in Ref.[136], as well as in Ref.[105], also some isospin breaking effects have been included. We prefer to neglect these effects for two reasons: i) there are not sufficient data to analyze systematically isospin breaking in all $K \to 3\pi$ channels; ii) as we will discuss in the following, these effects are completely negligible with respect to possible next–order CHPT corrections.

asimmetry	exp. limit	th. estimate
$(\delta_g)_\tau$	$-(0.70 \pm 0.53) \times 10^{-2}$	$-(2.3 \pm 0.6) \times 10^{-6}$
$(\delta_\Gamma)_\tau$	$(0.04 \pm 0.06) \times 10^{-2}$	$-(6.0 \pm 2.0) \times 10^{-8}$
$(\delta_g)_{\tau'}$	-	$(1.3 \pm 0.4) \times 10^{-6}$
$(\delta_\Gamma)\|_{\tau'}$	$(0.0 \pm 0.3) \times 10^{-2}$	$(2.4 \pm 0.8) \times 10^{-7}$

Table 12: Experimental limits and theoretical estimates for the charge asymmetries in $K^\pm \to \pi^\pm\pi^\pm\pi^\mp(\tau)$ and $K^\pm \to \pi^\pm\pi^0\pi^0(\tau')$ decays, calculated at the lowest non–vanishing order in CHPT.

From Eq. (5.24), neglecting $\Delta I = 3/2$ amplitudes, follows

$$(\delta_g)_\tau = \frac{(\alpha_0 - \beta_0)}{\Re e a_c}\left[\Im m b_c \frac{\Re e a_c}{\Re e b_c} - \Im m a_c\right]; \tag{5.32}$$

since

$$\Im m a_c^{(2)} - \frac{\Re e a_c^{(2)}}{\Re e b_c^{(2)}} \Im m b_c^{(2)} \simeq \omega \Im m a_c^{(2)}, \tag{5.33}$$

expanding imaginary parts at order p^2 we obtain, as anticipated, a result proportional to ω. On the other hand, expanding the imaginary parts up to $O(p^4)$, and neglecting $O(\omega)$ terms, leads to

$$(\delta_g)_\tau = (\alpha_0 - \beta_0)\left[\frac{\Im m b_c^{(4)}}{\Re e b_c^{(2)}} - \frac{\Im m a_c^{(4)}}{\Re e a_c^{(2)}}\right]. \tag{5.34}$$

In the more optimistic case we can expect that the two $O(p^4)$ phases are of the same order of $\Im m A_0/\Re e A_0$ and that their interference is constructive, thus[138]

$$|(\delta_g)_\tau| < 2(\alpha_0 - \beta_0)\left|\frac{\Im m A_0}{\Re e A_0}\right| < 10^{-5}. \tag{5.35}$$

A final comment on the value of $(\delta_g)_\tau$ before going on. The limit (5.35) is proportional to the phase difference $(\alpha_0 - \beta_0) \simeq 0.08$ which *is not* equal to the difference between constants and Y–dependent re–scattering terms, as explicitly shown in Eq. (5.16). In the literature this subtle difference has been sometimes ignored (probably due to numerical analysis of the re–scattering) and, as a consequence, overestimates of $(\delta_g)_\tau$ have been obtained[139].

5.4.2 The parameters ϵ'_{+-0} and ϵ^X_{+-0}.

Defining the 'weak phases'

$$\phi_8 = \frac{\Im m G_8}{\Re e G_8}, \qquad \phi_{27} = \frac{\Im m G_{27}}{\Re e G_{27}} \qquad \text{and} \qquad \phi_{\underline{8}} = \frac{F^2}{M_K^2}\frac{\Im m G_{\underline{8}}}{\Re e G_{27}}, \tag{5.36}$$

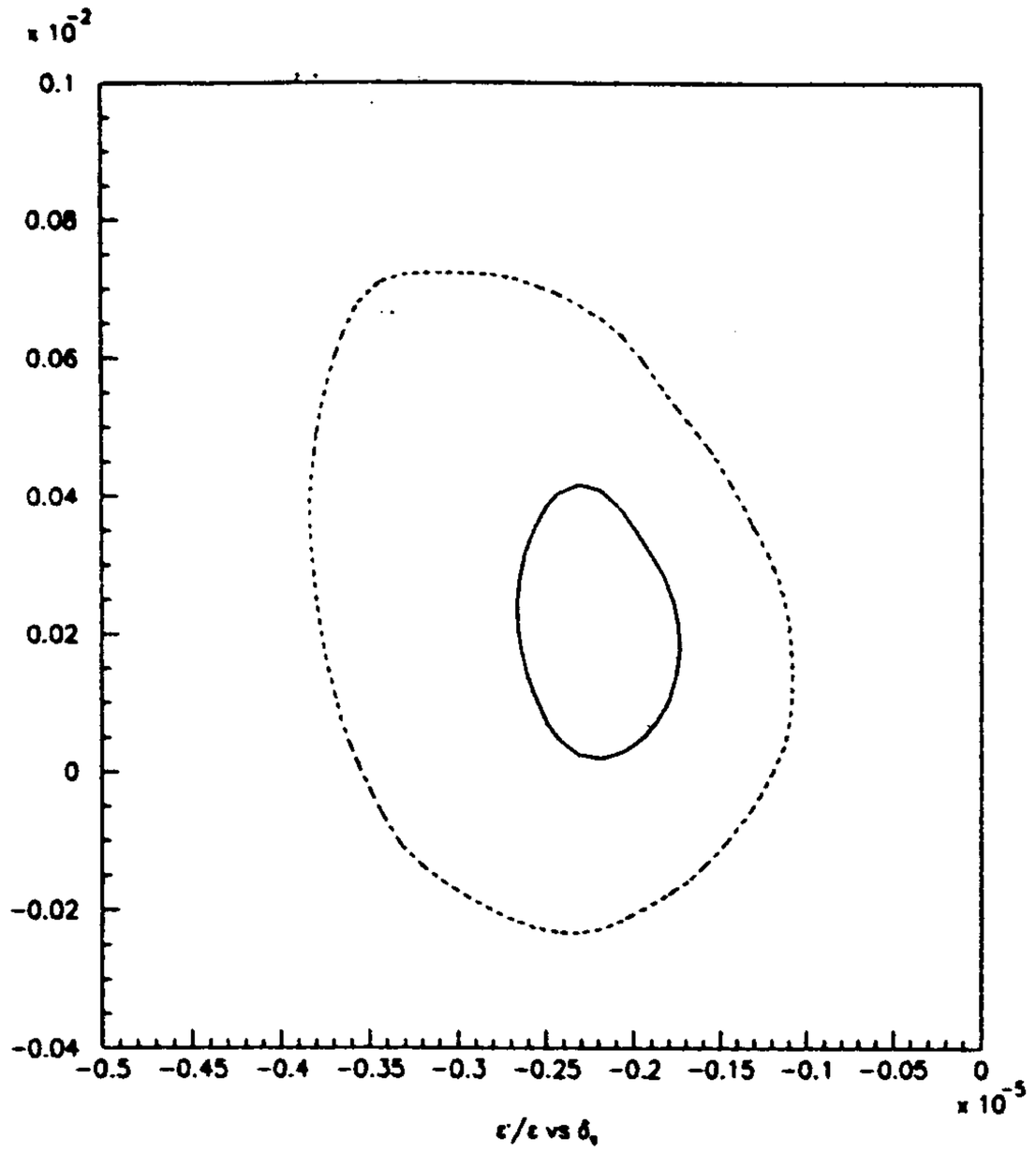

Figure 9: Comparison between $(\delta_g)_\tau$ and ϵ'/ϵ. Full and dotted lines indicate 5% and 68% contours around the central value, respectively.

we can write

$$\begin{aligned}\epsilon'_{+-0} &= i\omega\sqrt{2}\left[\phi_8 - \phi_{27} + \frac{3}{5}\phi_{\underline{8}} + O(\omega,\rho_\pi)\right], \\ \epsilon^X_{+-0} &= i\left[\phi_{27} - \phi_8 + \frac{1}{30}\phi_{\underline{8}} + O(\omega,\rho_\pi)\right]. \end{aligned} \tag{5.37}$$

For ϵ'_{+-0} the situation is exactly the same as for $(\delta_g)_\tau$, i.e. the ω–suppression could be removed by next–order CHPT corrections[78]. On the other hand for ϵ^X_{+-0}, which is necessarily proportional to the phase difference between a $\Delta I = 3/2$ (b_2) and a $\Delta I = 1/2$ amplitude, the lowest–order prediction is definitively more stable with respect to next–order corrections.

At the leading order in CHPT there is a simple relation between ϵ'_{+-0} and ϵ':

$$\epsilon'_{+-0} = -2i|\epsilon'|[1 + O(\Omega_{IB},\omega,\rho_\pi)], \tag{5.38}$$

It is interesting to note that this relation, obtained many years ago by Li and Wolfenstein[140], who considered only $(8_L,1_R)$ and $(27_L,1_R)$ operators, is still valid in presence of the lowest–order $(8_L,8_R)$ operator.

For what concerns next–order corrections, analogously to the case of $(\delta_g)_\tau$, we can estimate the upper limit for the enhancement of ϵ^X_{+-0} and ϵ'_{+-0} with respect to ϵ'. In the more optimistic case, we can assume to avoid the ω–suppression and the accidental cancellation between B_6 and B_8 in (3.42), without 'paying' anything for having considered

next–to–leading–order terms in CHPT. According to this hypothesis, from Eq. (3.42) follows

$$|\epsilon'_{+-0}|, |\epsilon^X_{+-0}| \lesssim 3 \times 10^{-3}\omega^{-1}|\epsilon|A^2\sigma \sin\delta \sim 5 \times 10^{-5}. \tag{5.39}$$

5.5 Interference measurements for $\eta_{3\pi}$ parameters.

The parameters η_{000}, η_{+-0} and η^X_{+-0}, being defined as the ratio of two amplitudes (analogously to η_{+-} and η_{00} of $K \to 2\pi$), can be directly measured only by the analysis of the interference term in the time evolution of neutral kaons. This kind of measurement, achievable by several experimental apparata[141,142], assumes a particular relevance in the case of the Φ–factory[131,143,39]. Since this method is very general and is useful for instance also in $K_{L,S} \to 2\pi\gamma$ decays, we will briefly discuss it (see Ref.[39] for a more detailed discussion).

The antisymmetric $K^0 - \bar{K}^0$ state, produced by the Φ decay, can be written as

$$\phi \to \frac{N}{\sqrt{2}}\left[K_S^{(\vec{q})}K_L^{(-\vec{q})} - K_L^{(\vec{q})}K_S^{(-\vec{q})}\right], \tag{5.40}$$

where $\vec{q}$ denotes the spatial momenta of one of the two kaons and N is a normalization factor. The decay amplitude in the final state $|a^{(\vec{q})}(t_1), b^{(-\vec{q})}(t_2)\rangle$ is thus given by:

$$\begin{aligned} A\left(a^{(\vec{q})}(t_1), b^{(-\vec{q})}(t_2)\right) \;=\; & \frac{N}{\sqrt{2}}\big[A(K_S \to a)e^{-i\lambda_S t_1}A(K_L \to b)e^{-i\lambda_L t_2} \\ & -A(K_L \to a)e^{-i\lambda_L t_1}A(K_S \to b)e^{-i\lambda_S t_2}\big]. \end{aligned} \tag{5.41}$$

Integrating the modulus square of this amplitude with respect to t_1 and t_2, keeping fixed the difference $t = t_1 - t_2$, and integrating with respect to all possible directions of $\vec{q}$, leads to

$$\begin{aligned} I(a,b;t) &= \int d\Omega_q dt_1 dt_2 |A(a(t_1), b(t_2))|^2 \delta(t_1 - t_2 - t) \\ &\propto \frac{e^{-\Gamma|t|}}{2\Gamma}\Big\{|A_S^a|^2|A_L^b|^2 e^{-\frac{\Delta\Gamma}{2}t} \\ &\quad + |A_L^a|^2|A_S^b|^2 e^{+\frac{\Delta\Gamma}{2}t} - 2\Re\left[A_S^a A_L^{a*} A_L^b A_S^{b*} e^{+i\Delta m t}\right]\Big\}, \end{aligned} \tag{5.42}$$

where

$$\Gamma = \frac{\Gamma_S + \Gamma_L}{2}, \qquad \Delta\Gamma = \Gamma_S - \Gamma_L \qquad \text{and} \qquad \Delta m = m_L - m_S. \tag{5.43}$$

$I(a,b;t)$ represents the probability to have in the final state $K_{S,L} \to a$ and $K_{L,S} \to b$ decays separated by a time interval t.

By choosing appropriately $|a\rangle$ and $|b\rangle$, is possible to construct interesting asymmetries. As an example, a convenient choice to study $K \to 3\pi$ amplitudes is given by $|a\rangle = |3\pi\rangle$

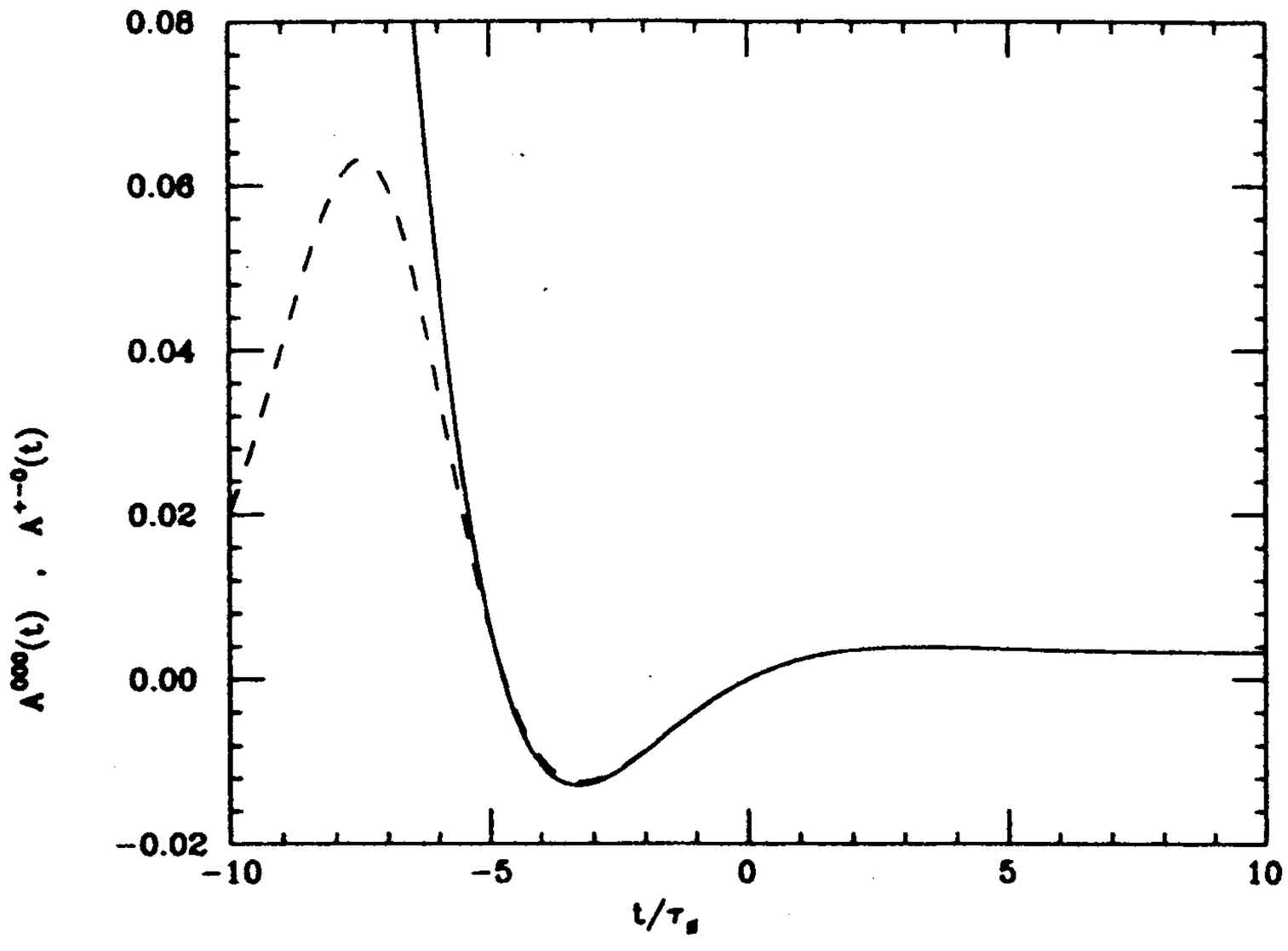

Figure 10: The asymmetries $A^{000}(t)$ (full line) and $A^{+-0}(t)$ (dotted line). The plots have been obtained fixing $\eta^{000} = \eta^{+-0} = |\epsilon|e^{i\pi/4}$.

and $|b\rangle = |\pi l\nu\rangle$ (as shown in sect. 2.3, $|A(K_S \to \pi l\nu)| = |A(K_L \to \pi l\nu)|$), which let us consider the following asymmetry[39]

$$\begin{aligned} A^{123}(t) &= \frac{\int [I(\pi^1\pi^2\pi^3, l^+\pi^-\nu; t) - I(\pi^1\pi^2\pi^3, l^-\pi^+\nu; t)]\, d\phi_{3\pi}\, d\phi_{l\pi\nu}}{\int [I(\pi^1\pi^2\pi^3, l^+\pi^-\nu; t) + I(\pi^1\pi^2\pi^3, l^-\pi^+\nu; t)]\, d\phi_{3\pi}\, d\phi_{l\pi\nu}}, \\ &= \frac{2(\Re\epsilon)e^{+\frac{\Delta\Gamma}{2}t} - 2\Re\left(\eta^{123}e^{+i\Delta mt}\right)}{e^{+\frac{\Delta\Gamma}{2}t} + \frac{\Gamma_S^{123}}{\Gamma_L^{123}}e^{-\frac{\Delta\Gamma}{2}t}}, \end{aligned} \tag{5.44}$$

where $d\phi_{3\pi}$ and $d\phi_{l\pi\nu}$ indicate final-state phase–space elements.

The peculiarity[144] of $A^{123}(t)$, with respect to analogous distributions measurable in different experimental set up, like fixed–target experiments, is the fact that $A^{123}(t)$ can be studied for $t < 0$. Events with $t < 0$ are those where the semileptonic decay occurs after the three–pion one, thus, as can be seen from Eq. (5.44) and fig. 10, are much more sensible to the CP–violating $K_S \to 3\pi$ amplitude. Obviously the statistics of these events is very low, and tends to zero for $t \ll 0$, but for small times ($|t| \lesssim 5\tau_S$) the decrease of statistics does not compensate the increase of sensibility.

The asymmetry $A^{123}(t)$ is very useful to measure both η_{000} and η_{+-0}. The measurement of η^X_{+-0} is more difficult since it requires an X-odd integration over the Dalitz Plot[39] which drastically reduces the statistics.

At any rate, the sensitivity which should be reached on η_{000} and η_{+-0} at KLOE is of the order of 10^{-3}, still far form direct–CP–violating effects expected in the Standard

Model. Present bounds on η_{+-0} are of the order of 10^{-2}.[141,142]

6 $K \to \pi\pi\gamma$ decays.

6.1 Amplitude decomposition.

The channels of $K \to \pi\pi\gamma$ transitions are three:

$$\begin{aligned} K^{\pm} &\to \pi^{\pm}\pi^{0}\gamma, \\ K^{0}(\bar{K}^{0}) &\to \pi^{+}\pi^{-}\gamma, \\ K^{0}(\bar{K}^{0}) &\to \pi^{0}\pi^{0}\gamma, \end{aligned} \tag{6.1}$$

in any channel is possible to distinguish an electric (E) and a magnetic (M) amplitude. The most general form, dictated by gauge and Lorentz invariance, for the transition amplitude $K(p_K) \to \pi_1(p_1)\pi_2(p_2)\gamma(\epsilon, q)$ is given by:

$$A(K \to \pi\pi\gamma) = \epsilon_\mu \left[E(z_i)(qp_1p_2^\mu - qp_2p_1^\mu) + M(z_i)\epsilon^{\mu\nu\rho\sigma}p_{1\nu}p_{2\rho}q_\sigma\right]/M_K^3, \tag{6.2}$$

where

$$z_i \doteq \frac{p_i q}{M_K^2} \quad (i = 1, 2) \qquad \text{and} \qquad z_3 \doteq z_1 + z_2 = \frac{p_K q}{M_K^2}. \tag{6.3}$$

E and M thus defined are adimensional. Summing over the photon–helicity states, the differential width of the decay is given by:

$$\begin{aligned} \frac{\mathrm{d}\Gamma}{\mathrm{d}z_1\mathrm{d}z_2} &= \frac{M_K}{4(4\pi)^3}\left(|E(z_i)|^2 + |M(z_i)|^2\right) \\ &\quad \times \left[z_1z_2(1 - 2z_3 - r_1^2 - r_2^2) - r_1^2z_2^2 - r_2^2z_1^2\right], \end{aligned} \tag{6.4}$$

where $r_i = M_{\pi_i}/M_K$. Thus there is no interference among E and M if the photon helicity is not measured.

In the limit where the photon energy goes to zero, the electric amplitude is completely determined by the Low theorem[145], which relates $A(K \to \pi\pi\gamma)$ to $A(K \to \pi\pi)$. For this reason is convenient to re–write E in two parts: the 'bremsstrahlung' E_{IB} and the 'direct–emission' E_{DE}. The bremsstrahlung amplitude contains the term fixed by the Low theorem, which diverges for $E_\gamma \to 0$ and that corresponds in the classical limit to the external–charged–particle radiation. If eQ_i is the electric charge of the pion π_i, we have

$$E_{IB}(z_i) \doteq \frac{eA(K \to \pi_1\pi_2)}{M_K z_3}\left(\frac{Q_2}{z_2} - \frac{Q_1}{z_1}\right). \tag{6.5}$$

The electric direct emission amplitude is by definition $E_{DE} \doteq E - E_{IB}$ and, according to the low theorem, we know that $E_{DE} = \text{cost.} + O(E_\gamma)$. The magnetic term by construction does not receive bremsstrahlung contributions (is a pure direct–emission term) thus, analogously to the previous case, $M = \text{cost.} + O(E_\gamma)$. As can be noticed by table 13, in the

decay	BR(bremsstrahlung)	BR(direct emission)
$K^{\pm} \to \pi^{\pm}\pi^0\gamma$ $(T_c^*=(55-90)MeV)$	$(2.57 \pm 0.16) \times 10^{-4}$	$(1.8 \pm 0.4) \times 10^{-5}$
$K_S \to \pi^+\pi^-\gamma$ $(E_\gamma^*>50MeV)$	$(1.78 \pm 0.05) \times 10^{-3}$	$< 9 \times 10^{-5}$
$K_L \to \pi^+\pi^-\gamma$ $(E_\gamma^*>20MeV)$	$(1.49 \pm 0.08) \times 10^{-5}$	$(3.19 \pm 0.16) \times 10^{-5}$
$K_S \to \pi^0\pi^0\gamma$	/	–
$K_L \to \pi^0\pi^0\gamma$	/	$< 5.6 \times 10^{-6}$

Table 13: Experimental values of $K \to \pi\pi\gamma$ branching ratios[18] (E_γ^* and T_c^* are the photon energy and the π^+ kinetic energy in the kaon rest frame, respectively).

process	E_{IB}	E_1	M_1	E_2	M_2
$K^{\pm} \to \pi^{\pm}\pi^0\gamma$	ω	·	·	·	·
$K_S \to \pi^+\pi^-\gamma$	·	·	CP	CP	·
$K_L \to \pi^+\pi^-\gamma$	CP	CP	·	·	CP
$K_S \to \pi^0\pi^0\gamma$	/	/	/	CP	·
$K_L \to \pi^0\pi^0\gamma$	/	/	/	·	CP

Table 14: Suppression factors for $K \to \pi\pi\gamma$ amplitudes: · = allowed transitions, CP = CP–violating transitions, ω = amplitudes suppressed by the $\Delta I = 1/2$ rule, / = completely forbidden amplitudes (by $Q_1 = Q_2 = 0$ or by Bose symmetry).

cases where the corresponding $K \to \pi\pi$ amplitude is not suppressed the pole for $E_\gamma \to 0$ naturally enhances the bremsstrahlung contribution respect to the direct emission one.

The last decomposition which is convenient to introduce is the so–called multipole expansion for the direct–emission amplitudes E_{DE} ed M:

$$E_{DE}(z_i) = E_1 + E_2(z_1 - z_2) + O\left[(z_1 - z_2)^2\right], \tag{6.6}$$

$$M(z_i) = M_1 + M_2(z_1 - z_2) + O\left[(z_1 - z_2)^2\right]. \tag{6.7}$$

This decomposition is useful essentially for two reasons: i) since the phase space is limited ($|z_1 - z_2| < 0.2$) high–order multipoles are suppressed; ii) in the neutral channels even and odd multipoles have different CP–transformation properties: $CP(E_J) = (-1)^{J+1}$, $CP(M_J) = (-1)^J$.

6.1.1 CP–violating observables.

As can be noticed by tables 13 and 14, $K_S \to \pi^+\pi^-\gamma$ and $K_{S,L} \to \pi^0\pi^0\gamma$ decays are not very interesting for the study of CP violation. The first is dominated by the bremsstrahlung, which 'hides' other contributions, whereas neutral channels are too suppressed to observe any kind of interference. The theoretical branching ratios for the latter[146–148] are below 10^{-8}.

Interesting channels for CP violation are $K^{\pm} \to \pi^{\pm}\pi^0\gamma$ and $K_L \to \pi^+\pi^-\gamma$, where the bremsstrahlung is suppressed and consequently it is easier to measure interference between the latter and other amplitudes. If the photon polarization is not measured and the multipoles E_2 and M_2 are neglected, we can define only two observables which violates CP:

$$\eta_{+-\gamma} = \frac{A(K_L \to \pi^+\pi^-\gamma)_{E_{IB}+E_1}}{A(K_S \to \pi^+\pi^-\gamma)_{E_{IB}+E_1}}, \tag{6.8}$$

$$\delta\Gamma = \frac{\Gamma(K^+ \to \pi^+\pi^0\gamma) - \Gamma(K^- \to \pi^-\pi^0\gamma)}{\Gamma(K^+ \to \pi^+\pi^0\gamma) + \Gamma(K^- \to \pi^-\pi^0\gamma)}. \tag{6.9}$$

In the case where also E_2 and the photon polarization are considered, is possible to add other two $K_L \to \pi^+\pi^-\gamma$ observables, proportional to the interference of $(E_{IB} + E_1)$ with E_2 and M_1. The first is the Dalitz Plot asymmetry in the $\pi^+ \leftrightarrow \pi^-$ exchange, the second is the $\phi \to -\phi$ asymmetry, where ϕ is the angle between the γ–polarization plane and the $\pi^+ - \pi^-$ plane. However, these observables are less interesting than those of Eqs. (6.8–6.9), because are not pure signals of direct CP violation and are suppressed by the interference with higher order multipoles[149,148]. In the following we will not consider them.

By the definition of $\eta_{+-\gamma}$, using the identities

$$E_{IB}(K_L) = \eta_{+-}E_{IB}(K_S), \tag{6.10}$$

$$E_1(K_L) = \tilde{\epsilon}E_1(K_1) + E_1(K_2), \tag{6.11}$$

follows

$$\begin{aligned}\eta_{+-\gamma} &= \frac{E_{IB}(K_L) + E_1(K_L)}{E_{IB}(K_S) + E_1(K_S)} \\ &= \eta_{+-} + \frac{(\tilde{\epsilon} - \eta_{+-})E_1(K_1) + E_1(K_2)}{E_{IB}(K_S)}\left[1 + O\left(\frac{E_1(K_S)}{E_{IB}(K_S)}\right)\right].\end{aligned} \tag{6.12}$$

From the previous equation we deduce that, contrary to the statement of Cheng[150], the difference

$$\epsilon'_{+-\gamma} \doteq \eta_{+-\gamma} - \eta_{+-} \tag{6.13}$$

is an index of direct CP violation. Identifying in Eq. (6.2) the (p_1, p_2) pair with (p_+, p_-) and factorizing strong phases, we can write

$$E_1(K_1) = e^{i\delta_n}\Re e E_n, \tag{6.14}$$

$$E_1(K_2) = ie^{i\delta_n}\Im m E_n, \tag{6.15}$$

$$E_{IB}(K_S) = -e^{i\delta_0}\left(\frac{e\sqrt{2}\Re e A_0}{M_K z_+ z_-}\right)[1 + O(\omega, \epsilon)], \tag{6.16}$$

where E_n is a complex amplitude which becomes real in the limit of CP conservation. Using this decomposition we find

$$\epsilon'_{+-\gamma} = \frac{e^{i(\delta_n - \delta_0)}M_K z_+ z_- \Re e E_n}{e\sqrt{2}\Re e A_0}\left[\epsilon' + i\left(\frac{\Im m A_0}{\Re e A_0} - \frac{\Im m E_n}{\Re e E_n}\right)\right](1 + O(\omega, \epsilon))$$

$$\simeq \frac{ie^{i(\delta_n-\delta_0)}M_K z_+ z_- \Re e E_n}{e\sqrt{2}\Re e A_0}\left(\frac{\Im m A_0}{\Re e A_0}-\frac{\Im m E_n}{\Re e E_n}\right), \tag{6.17}$$

where the second identity follows from the fact that the weak–phase difference between A_0 and E_n is not suppressed by ω.

The observable $\delta\Gamma$ is a pure index of direct CP violation. Actually, analogously to the case of $K^\pm \to (3\pi)^\pm$ decays, instead of the width asymmetry is more convenient to consider the asymmetry of quantities which are directly proportional to interference terms (like the $g^\pm$ slopes in $K^\pm \to (3\pi)^\pm$). For this purpose is useful to consider the quantities $\Gamma^\pm_{DE}(E^*_\gamma)$, defined by

$$\Gamma^\pm_{DE}(E^*_\gamma) = \int_0^{E^*_\gamma} \mathrm{d}E_\gamma \left[\frac{\partial\Gamma(K^\pm \to \pi^\pm\pi^0\gamma)}{\partial E_\gamma} - \frac{\partial\Gamma(K^\pm \to \pi^\pm\pi^0\gamma)_{IB}}{\partial E_\gamma}\right], \tag{6.18}$$

where $\Gamma(K^\pm \to \pi^\pm\pi^0\gamma)_{IB}$ is obtained by Eq. (6.4) setting $E = E_{IB}$ and $M = 0$. In the limit where the magnetic term in Eq. (6.4) is negligible, the expression of

$$\delta\Gamma_{DE} = \frac{\Gamma^+_{DE} - \Gamma^-_{DE}}{\Gamma^+_{DE} + \Gamma^-_{DE}} \tag{6.19}$$

is very simple: setting $(p_1, p_2) \equiv (p_\pm, p_0)$ and factorizing strong phases analogously to Eqs. (6.14–6.16)

$$E_1(K^\pm) = \pm e^{i\delta_c} E_c, \tag{6.20}$$

$$E_{IB}(K^\pm) = \mp e^{i\delta_2}\left(\frac{3e\Re e A_2}{2M_K z_\pm z_0}\right), \tag{6.21}$$

we obtain:

$$\delta\Gamma_{DE} = \frac{\Im m(A_2 E_c^*)\sin(\delta_2-\delta_c)}{\Re e(A_2 E_c^*)\cos(\delta_2-\delta_c)} \doteq \epsilon'_{+0\gamma}\tan(\delta_c-\delta_2). \tag{6.22}$$

If the magnetic term is not negligible, Eq. (6.22) is modified in

$$\delta\Gamma_{DE} = \frac{\epsilon'_{+0\gamma}}{1+R}\tan(\delta_c-\delta_2), \tag{6.23}$$

where

$$R = \left\{\int \mathrm{d}z_+\mathrm{d}z_0\left[z_+z_0(1-2z_3-r_+^2-r_0^2)-r_+^2z_0^2-r_0^2z_+^2\right]|M(z_i)|^2\right\}\times$$
$$\times\left\{2\mathrm{d}z_+\mathrm{d}z_0\left[z_+z_0(1-2z_3-r_+^2-r_0^2)-r_+^2z_0^2-r_0^2z_+^2\right]\Re e(E_1^*(z_i)E_{IB}(z_i))\right\}^{-1}. \tag{6.24}$$

Analogously to $\epsilon'_{+-\gamma}$, also

$$\epsilon'_{+0\gamma} = \left(\frac{\Im m E_c}{\Re e E_c}-\frac{\Im m A_2}{\Re e A_2}\right) = \left(\frac{\Im m E_c}{\Re e E_c}-\frac{\Im m A_0}{\Re e A_0}\right) - \frac{\sqrt{2}|\epsilon'|}{\omega} \tag{6.25}$$

is not suppressed by the $\Delta I = 1/2$ rule.

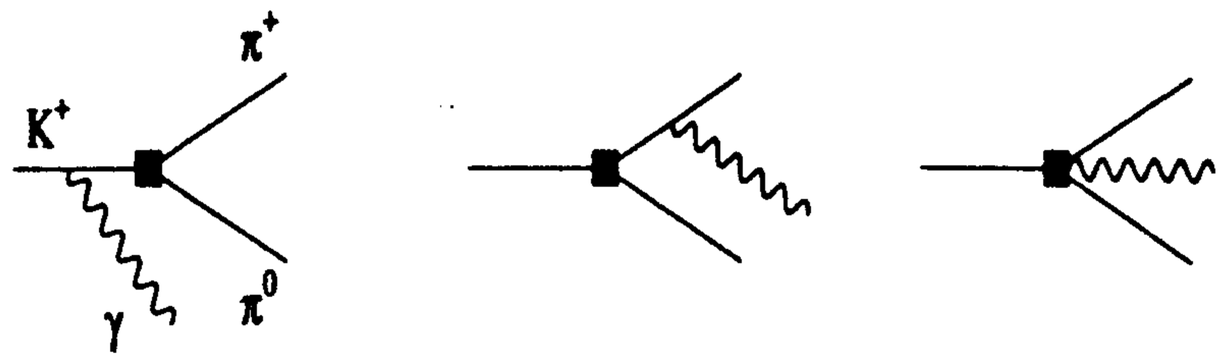

Figure 11: Tree–level diagrams for the transition $K^+ \to \pi^+\pi^0\gamma$. The black box indicates the weak vertex.

6.1.2 $K \to \pi\pi\gamma$ amplitudes in CHPT.

The lowest order CHPT diagrams which contribute $K \to \pi\pi\gamma$ transitions are shown in fig. 11. At this order only the bremsstrahlung amplitude is different from zero. As can be easily deduced from Eq. (6.2), is necessary to go beyond the lowest order to obtain non–vanishing contributions to direct emission amplitudes. At order p^4 electric amplitudes receive contributions from both loops (§ fig. 12) and counterterms, whereas the magnetic amplitudes receive contributions only by local operators[146].[a]

The complete $O(p^4)$ calculation of electric direct–emission amplitudes, carried out in Refs.[146,151,148], give rise to two interesting results:

- The loop contribution is finite both in $\pi^+\pi^0\gamma$ and $\pi^+\pi^-\gamma$.
- In both channels the counterterm combination is the same.

Neglecting the small contribution of $\pi - K$ and $K - \eta$ loops, the explicit $O(p^4)$ expression of the weak amplitudes E_n and E_c is:

$$\begin{aligned} E_n &= \frac{eG_8M_K^3}{4\pi^2F_\pi}\left[\frac{64\pi^2M_K^2}{1+\rho_\pi}\Re e\widetilde{C_{20}}(M_K^2,0) - N_{E_1}^{(4)}\right] \\ &\simeq \frac{eG_8M_K^3}{4\pi^2F_\pi}[1.3 - N_{E_1}^{(4)}], \qquad (6.26) \\ E_c &= \frac{eG_8M_K^3}{8\pi^2F_\pi}N_{E_1}^{(4)}, \qquad (6.27) \end{aligned}$$

where the function $\widetilde{C_{20}}(x,y)$ is defined in the appendix, $\rho_\pi = M_\pi^2/(M_K^2 - M_\pi^2)$ and

$$N_{E_1}^{(4)} = (4\pi)^2\left[N_{14} - N_{15} - N_{16} - N_{17}\right] \qquad (6.28)$$

is a μ–independent combination of $\mathcal{L}_W^{(4)}$ coefficients (§ sect. 4.3.3). For what concerns strong

[a] In $\pi^0\pi^0\gamma$ channels there is no contribution even at $O(p^4)$.

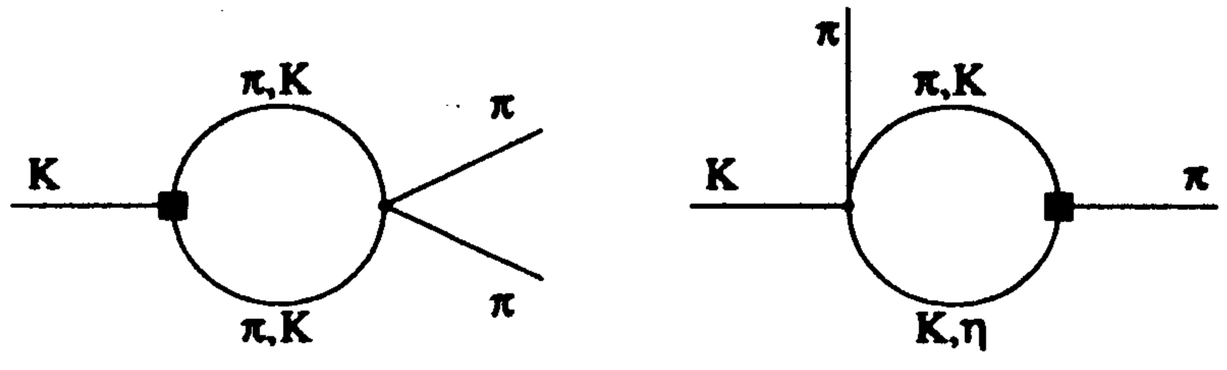

Figure 12: One–loop diagrams relevant to the direct emission amplitudes in $K \to \pi\pi\gamma$ decays; for simplicity we have omitted the photon line, which has to be attached to any charged line and to any vertex.

phases, always at $O(p^4)$, we find:

$$\begin{aligned} \delta_n &= \arctan\left(\frac{64\pi^2 M_K^2 \Im m\widetilde{C_{20}}(M_K^2,0)}{64\pi^2 M_K^2 \Re e\widetilde{C_{20}}(M_K^2,0) - (1+\rho_\pi)\Re e N_{E_1}^{(4)}}\right) \\ &\simeq -\arctan\left(\frac{0.5}{1.3 - \Re e N_{E_1}^{(4)}}\right), \qquad (6.29) \\ \delta_c &= 0. \qquad (6.30) \end{aligned}$$

Up to date it is impossible to determine the value of $\Re e N_{E_1}^{(4)}$ using experimental data, since available information on $K^+ \to \pi^+\pi^0\gamma$ are not accurate enough to distinguish between electric and magnetic amplitudes. To estimate $\Re e N_{E_1}^{(4)}$ is necessary to assume some theoretical model. In the framework of the factorization model discussed in sect. 4.4, which we expect give correct indications about sign and order of magnitude of counterterms, the result is

$$\Re e N_{E_1}^{(4)} = -k_f \frac{8\pi^2 F_\pi^2}{M_V^2} = -(0.5 \div 1), \qquad (6.31)$$

thus:

- In $K^+ \to \pi^+\pi^0\gamma$ the interference between E_1 and E_{IB} is positive (the loop contribution is negligible).
- In $K_S \to \pi^+\pi^-\gamma$ loop and counterterm contributions are of the same order of the same sign and interfere destructively with the bremsstrahlung.

For what concerns higher order electric multipoles, local $O(p^4)$ contributions to E_2 are forbidden by power–counting, but the kinematical dependence of the loop amplitudes generate a non–vanishing contribution of this kind in $K^+ \to \pi^+\pi^0\gamma$ and $K^0 \to \pi^+\pi^-\gamma$. However, the $O(p^4)$ prediction for this higher order electric multipole is very small:[146,148]

$$E_2^{(4)}(K_2) \simeq \frac{eG_8 M_K^3}{8\pi^2 F_\pi}[0.005(z_+ - z_-)] \qquad (6.32)$$

and we belive that the dominant contribution is generated only at $O(p^6)$, where counterterms are not forbidden. Indeed, following Ref.[148] we can write

$$E_2^{(6)}(K_2) = \frac{eG_8 M_K^5}{48\pi^4 F_\pi^3} N_{E_2}^{(6)}(z_+ - z_-), \tag{6.33}$$

and by power counting we expect $N_{E_2}^{(6)} \sim O(1)$.

For a detailed discussion about magnetic multipoles we refer the reader the analysis of Ecker, Neufeld and Pich[146].

6.2 Estimates of CP violation.

Using Eqs. (6.26–6.27) and the $O(p^2)$ expression of A_0, is possible to relate each other the direct–CP–violating parameters of $K \to \pi\pi\gamma$:

$$\epsilon'_{+-\gamma} = \frac{ie^{i(\delta_n-\delta_0)} M_K^2 \Re e N_{E_1}^{(4)} z_+ z_-}{F_\pi^2} \frac{\Im m N_{E_1}^{(4)}}{\Re e N_{E_1}^{(4)}} [1 + (\Omega_{IB}, \omega, \rho_\pi)], \tag{6.34}$$

$$\epsilon'_{+0\gamma} = \frac{\Im m N_{E_1}^{(4)}}{\Re e N_{E_1}^{(4)}} - \frac{\sqrt{2}|\epsilon'|}{\omega} = -\frac{ie^{i(\delta_0-\delta_n)} F_\pi^2}{\Re e N_{E_1}^{(4)} M_K^2 z_+ z_-} \epsilon'_{+-\gamma} - \frac{\sqrt{2}|\epsilon'|}{\omega} \tag{6.35}$$

Eq. (6.35) is the analogous of Eq. (5.38), which relates direct–CP–violating parameters of $K \to 2\pi$ and $K \to 3\pi$. However, since it does not imply $O(\omega)$ cancellation among $\Delta I = 1/2$ amplitudes, Eq. (6.35) is definitively more stable than Eq. (5.38) with respect to next–order CHPT corrections.

For what concerns numerical estimates of $\epsilon'_{+0\gamma}$ and $\epsilon'_{+-\gamma}$, proceeding similar to the $K \to 3\pi$ case, i.e. assuming that all weak phases are of the order of $\Im m A_0 / \Re e A_0$ and that interfere constructively, we find

$$|\epsilon'_{+-\gamma}| \lesssim (3 \times 10^{-5}) z_+ z_-, \qquad \text{and} \qquad |\epsilon'_{+0\gamma}| \lesssim 10^{-4}. \tag{6.36}$$

Since the parameter R introduced in Eq. (6.23) is positive (due to the constructive interference between E_{IB} and E_1 in $K^+ \to \pi^+\pi^0\gamma$) the limit on $|\epsilon'_{+0\gamma}|$ imply

$$|\delta\Gamma_{DE}| \lesssim 10^{-4} \qquad \text{and} \qquad |\delta\Gamma| \lesssim 10^{-5} \tag{6.37}$$

in agreement with the estimates of Refs.[152,153,154].

Actually, the four–quark–operator basis used for $K \to 2\pi$ and $K \to 3\pi$ decays is not complete for $K \to \pi\pi\gamma$ transitions. In this case we should add to $\mathcal{H}_{eff}^{|\Delta S|=1}$ the dimension–five electric–dipole operator[73]:

$$\mathcal{H}_{eff}^{|\Delta S|=1;\gamma} = \mathcal{H}_{eff}^{|\Delta S|=1} - \frac{4G_F}{\sqrt{2}} [\lambda_t C_{11}(\mu) O_{11}(\mu) + \text{h.c.}], \tag{6.38}$$

$$O_{11} = i(m_s \bar{s}_R \sigma_{\mu\nu} d_L + m_d \bar{s}_L \sigma_{\mu\nu} d_R) F^{\mu\nu}. \tag{6.39}$$

This operator generates a new short–distance contribution to the weak phases of $\Delta I = 1/2$ amplitudes. However, the matrix elements of O_{11} are suppressed with respect to those of O_6 (the dominant operator in the imaginary part of A_0), because are different from zero only at $O(p^6)$ in CHPT. Indeed, according to the chiral power counting exposed in sect. 4, we have $m_q \sim O(p^2)$, $F^{\mu\nu} \sim O(p^2)$ and $\bar{q}\sigma_{\mu\nu}q \sim \partial_\mu\phi\partial_\nu\phi \sim O(p^2)$ (for an explicit chiral realization of O_{11} see Ref.[148]). Furthermore, since the Wilson coefficient of this operator is quite small[73,155] $|C_{11}| < 2 \times 10^{-2}$, it is reasonable to expect that limits (6.36) are still valid.[b]

Also in the $K \to \pi\pi\gamma$ case the situation is not very promising from the experimental point of view:

- The parameter $\eta_{+-\gamma}$ has been recently measured at Fermilab[156], with an error $\sigma(\eta_{+-\gamma}) \sim 3 \times 10^{-4}$. In the next years new high–statistics fixed–target experiments should reach $\sigma(\eta_{+-\gamma}) \gtrsim 10^{-5}$.

- The asymmetry in the widths will be measured at KLOE[135] with an error[39] $\sigma(\delta\Gamma_{DE}) \gtrsim 10^{-3}$.

7 Decays with two photons in the final state.

7.1 $K \to \gamma\gamma$.

According to the photon polarizations, which can be parallel ($\sim F^{\mu\nu}F_{\mu\nu}$) or perpendicular ($\sim \epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$), we can distinguish two channels in $K^0(\bar{K}^0) \to \gamma\gamma$ transitions. The two channels transform under CP in such a way that the parameters

$$\eta_{||} = \frac{A(K_L \to 2\gamma_{||})}{A(K_S \to 2\gamma_{||})} = \epsilon + \epsilon'_{||}, \tag{7.1}$$

$$\eta_{\perp} = \frac{A(K_S \to 2\gamma_{\perp})}{A(K_L \to 2\gamma_{\perp})} = \epsilon + \epsilon'_{\perp}, \tag{7.2}$$

measurable in interference experiments,[a] would be zero if CP was not violated[159,160,35].

It is useful to separate the amplitude contributions into two classes: the long– and the short–distance ones. The first are generated by a non–leptonic transition ($K \to \pi$ or $K \to 2\pi$), ruled by $\mathcal{H}_{eff}^{|\Delta S|=1}$, followed by an electromagnetic process ($\pi \to \gamma\gamma$ or $\pi\pi \to \gamma\gamma$) which produces the two photons. The latter are determined by new operators, bilinear in the quark fields, like the electric–dipole operator (§ sect. 6.2) and the operator generated by the box diagram of fig. 13. By construction short–distance contributions, recently analyzed by Herrlich and Kalinowski[161], are either suppressed by the GIM mechanism or forbidden by

[b] The value of $\epsilon'_{+0\gamma}$ obtained by Dib and Peccei[155], that overcame the limit (6.37), is overestimated, as recently confirmed by one of the authors[86].

[a] The need of interference experiments would drop if photon polarizations were directly measurable.

decay	branching ratio
$K_L \to \gamma\gamma$	$(5.73 \pm 0.27) \times 10^{-4}$
$K_S \to \gamma\gamma$	$(2.4 \pm 1.2) \times 10^{-6}$
$K_L \to \pi^0\gamma\gamma$	$(1.70 \pm 0.28) \times 10^{-6}$
$K^\pm \to \pi^\pm\gamma\gamma$	$\sim 10^{-6}$

Table 15: Experimental data on $K_{L,S} \to \gamma\gamma$ and $K \to \pi\gamma\gamma$ decays[18,157,158]

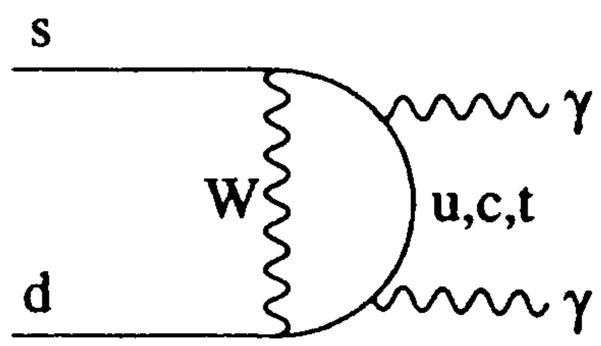

Figure 13: Short–distance contribution to $K \to \gamma\gamma$ transitions.

the Furry theorem[60]. By comparing the short–distance calculation[161] with the experimental widths, we find:

$$\left|\frac{A_{short-d}(K \to \gamma\gamma)}{A_{long-d}(K \to \gamma\gamma)}\right| < 10^{-4}. \tag{7.3}$$

In CHPT the first non–vanishing contribution to $K_S \to \gamma\gamma$ starts at $O(p^4)$ and is generated only by loop diagrams (§ fig. 14). The absence of counterterms, which implies the finiteness of the loop calculation, leads to the unambiguous prediction[162,163]:

$$BR(K_S \to \gamma\gamma)^{O(p^4)} = 2.1 \times 10^{-6}. \tag{7.4}$$

This result is in good agreement with the experimental data (§ tab. 15). Indeed, we expect that $O(p^6)$ contributions in this channel are small because: i) are not enhanced by near–by resonance exchanges, ii) unitarity correction to $\pi - \pi$ re–scattering are already included in the constant G_8.

If CP is conserved then $K_L \to \gamma\gamma$ does not receive any contribution at $O(p^4)$: at this order the pole diagrams with π^0 and η exchange (§ fig. 15) cancel each other. Due to the large branching ratio of the process, this cancellation implies that, contrary to the $K_S \to \gamma\gamma$ case, $O(p^6)$ operators have to generate large effects. Since the CP–violating phase of these operators contribute to $\eta_\perp$, it is reasonable assume

$$|\epsilon'_{\|}| < |\epsilon'_\perp|. \tag{7.5}$$

However, since[164] $|\epsilon'_\perp|^{(4)} \sim |\epsilon'|$, we expect that also $|\epsilon'_\perp|$ is dominated by local $O(p^6)$ contributions.

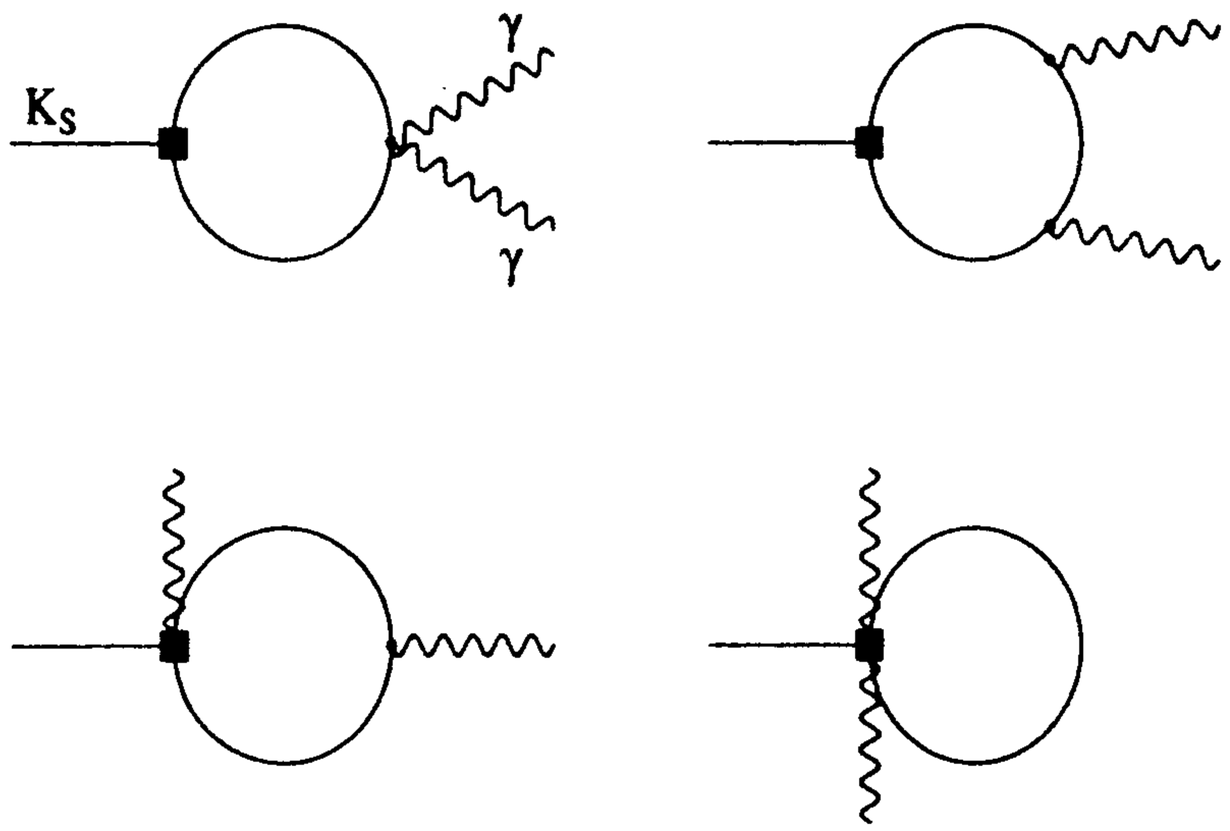

Figure 14: One–loop diagrams for the transition $K_S \to \gamma\gamma$.

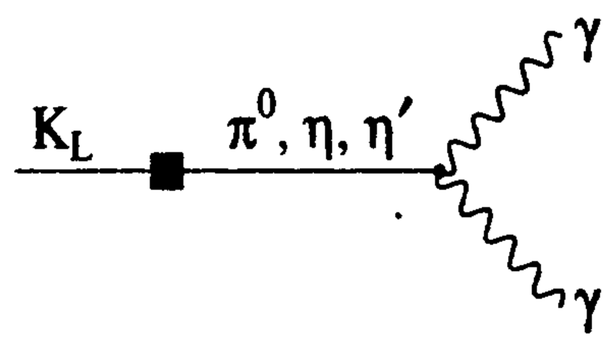

Figure 15: Polar diagrams for the transition $K_L \to \gamma\gamma$. The $P \to \gamma\gamma$ vertices of order p^4 are generated by the anomalous–functional Z_{WZW}.

Neglecting for the moment short distance effects, analogously to $K \to 3\pi$ and $K \to \pi\pi\gamma$ cases, we find

$$|\epsilon'_\perp| \lesssim \left|\frac{\Im m A_0}{\Re e A_0}\right|. \tag{7.6}$$

For what concerns short distance contributions, due to the suppression (7.3), even if the new operators had a CP–violating phase of order one, their effect on $\epsilon'_\perp$ and $\epsilon'_\parallel$ could not overcome the limit (7.6). Results near to this limit have been obtained for instance in Refs.[164,165].

7.2 $K_L \to \pi^0\gamma\gamma$.

$K^0(\bar{K}^0) \to \pi^0\gamma\gamma$ transitions are not very interesting by themselves for the study of CP violation. However, the process $K_L \to \pi^0\gamma\gamma$ has an important role as intermediate state in

the decay $K_L \to \pi^0 e^+ e^-$, that is very interesting for the study CP violation (§ sect. 8.1).

The CP-invariant decay amplitude of $K_L \to \pi^0\gamma\gamma$ can be decomposed in the following way:

$$M(K_L(p) \to \pi^0(p')\gamma(q_1,\varepsilon_1)\gamma(q_2,\varepsilon_2)) = \varepsilon_{1\mu}\varepsilon_{2\nu}M^{\mu\nu}(p,q_1,q_2)\ , \tag{7.7}$$

where

$$\begin{aligned} M^{\mu\nu} &= \frac{A(y,z)}{M_K^2}(q_2^\mu q_1^\nu - q_1q_2g^{\mu\nu}) \\ &+\frac{2B(y,z)}{M_K^4}(-pq_1pq_2g^{\mu\nu} - q_1q_2p^\mu p^\nu + pq_1q_2^\mu p^\nu + pq_2p^\mu q_1^\nu) \end{aligned} \tag{7.8}$$

and the variables y and z are defined by

$$y = p(q_1 - q_2)/M_K^2 \qquad \text{and} \qquad z = (q_1+q_2)^2/M_K^2\ . \tag{7.9}$$

Due to Bose symmetry $A(y,z)$ and $B(y,z)$ must be symmetric for $q_1 \leftrightarrow q_2$ and consequently depend only on y^2.

The physical region in the dimensionless variables y and z is given by the inequalities

$$|y| \le \frac{1}{2}\lambda^{1/2}(1,z,r_\pi^2)\ , \qquad 0 \le z \le (1-r_\pi)^2\ , \tag{7.10}$$

where $r_\pi = M_\pi/M_K$ and $\lambda(a,b,c)$ is a kinematical function defined by

$$\lambda(x,y,z) = x^2+y^2+z^2-2(xy+yz+zx)\ . \tag{7.11}$$

From (7.7) and (7.8) we obtain the double differential decay rate for unpolarized photons:

$$\frac{d^2\Gamma}{dy\,dz} = \frac{M_K}{2^9\pi^3}\left\{z^2|A+B|^2 + \left[y^2 - \frac{1}{4}\lambda(1,z,r_\pi^2)\right]^2|B|^2\right\}. \tag{7.12}$$

We remark that, due to the different tensor structure in (7.8), the A and B parts of the amplitude give rise to contributions to the differential decay rate which have different dependence on the two–photon invariant mass z. In particular, the second term in (7.8) gives a non–vanishing contribution to $\dfrac{d\Gamma(K_L \to \pi^0\gamma\gamma)}{dz}$ in the limit $z \to 0$. Thus the kinematical region with collinear photons is important to extract the B amplitude, that plays a crucial role in $K_L \to \pi^0 e^+ e^-$ (§ sect. 8.1).

Analogously to $K_S \to \gamma\gamma$, also $K_L \to \pi^0\gamma\gamma$ receive $O(p^4)$ contributions only by loops, which thus are finite[166,167] and generate only an A-type amplitude. The diagrams are very similar to the ones of $K_S \to \gamma\gamma$ (§ fig. 14) in the diagonal basis of Ref.[168]. The shape of the of the photon spectrum at $O(p^4)$ (§ fig. 16), determined by the cut $K_L \to 3\pi \to \pi\gamma\gamma$, is in perfect agreement with the data (§ fig. 17), however the branching ratio

$$BR(K_L \to \pi^0\gamma\gamma)^{O(p^4)} = 0.61 \times 10^{-6}, \tag{7.13}$$

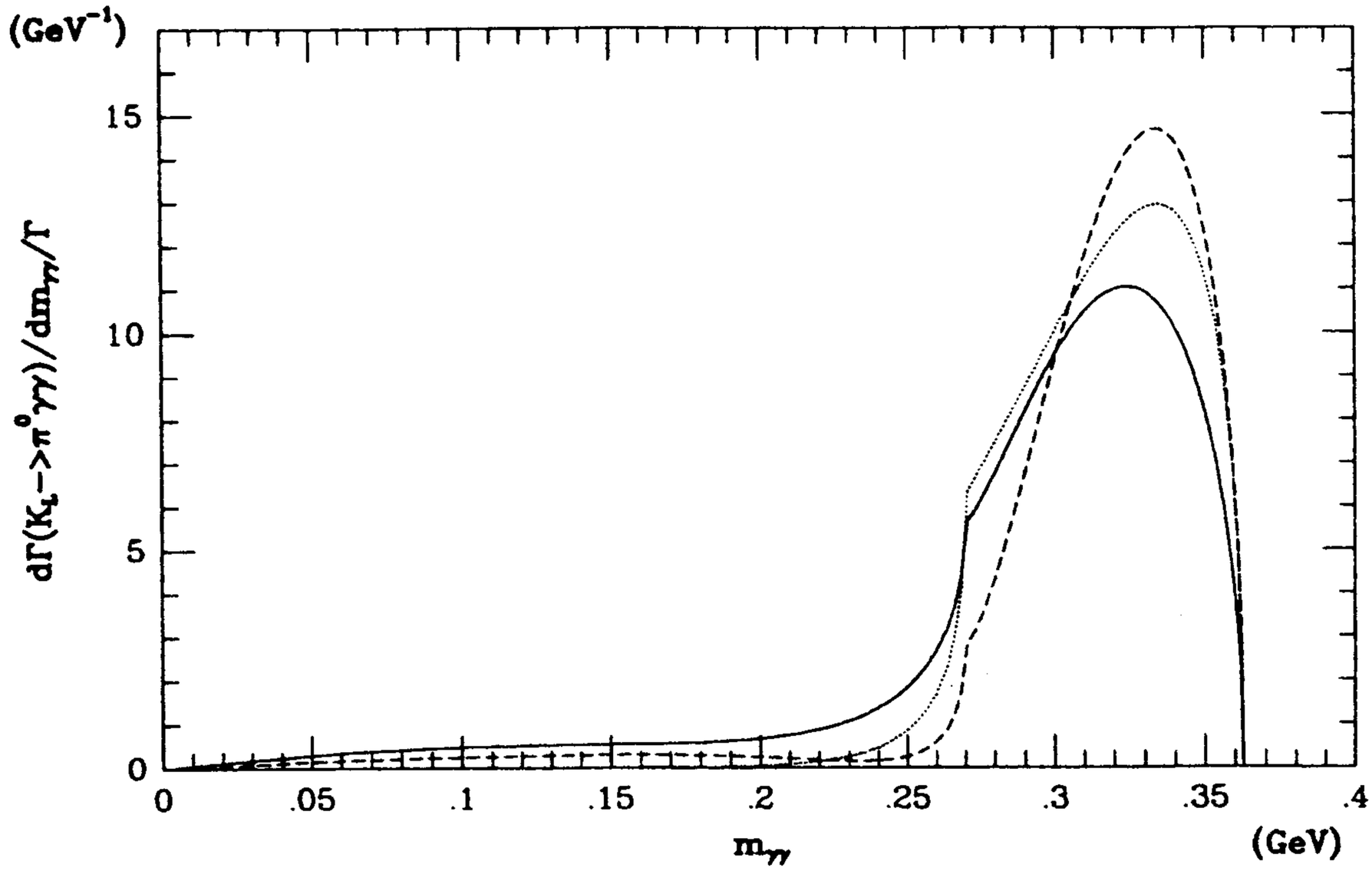

Figure 16: Theoretical predictions for the width of $K_L \to \pi^0\gamma\gamma$ as a function of the two-photon invariant mass. The dotted curve is the $O(p^4)$ contribution, dashed and full lines correspond to the $O(p^6)$ estimates[171,173] for $a_V = 0$ and $a_V = -0.8$, respectively. The three distributions are normalized to the 50 unambiguous events of NA31 (§ fig. 17).

is definitely underestimated (§ tab. 15). This implies that $O(p^6)$ effects are not negligible, nevertheless the B-type contribution should be small. Though the full $\mathcal{O}(p^6)$ calculation is still missing, several authors have considered some $\mathcal{O}(p^6)$ contributions (see, e.g. Ref.[119] and references cited therein). At this order there are counterterms and loops.

Similarly to the strong sector one can assume that nearby resonances generate the bulk of the local contributions, however we do not know the weak coupling of resonances and we have to rely on models. A useful parametrization of the local $O(p^6)$ contributions generated by vector resonances was introduced in Ref.[169], by means of an effective coupling a_V (of order one):

$$A = \frac{G_8 M_K^2 \alpha}{\pi} a_V (3 - z + r_\pi^2) \,, \qquad B = -\frac{2 G_8 M_K^2 \alpha}{\pi} a_V \,. \tag{7.14}$$

Thus, in general vector exchange can generate a B amplitude changing the $O(p^4)$ spectrum, particularly in the region of small z, and contributing to the CP conserving part of $K_L \to \pi^0 e^+ e^-$.

Also non-local contributions play a crucial role. Indeed, the $\mathcal{O}(p^2)$ $K \to 3\pi$ vertex from (4.32), used in the $K_L \to \pi^0\gamma\gamma$ loop amplitude does not take into account the

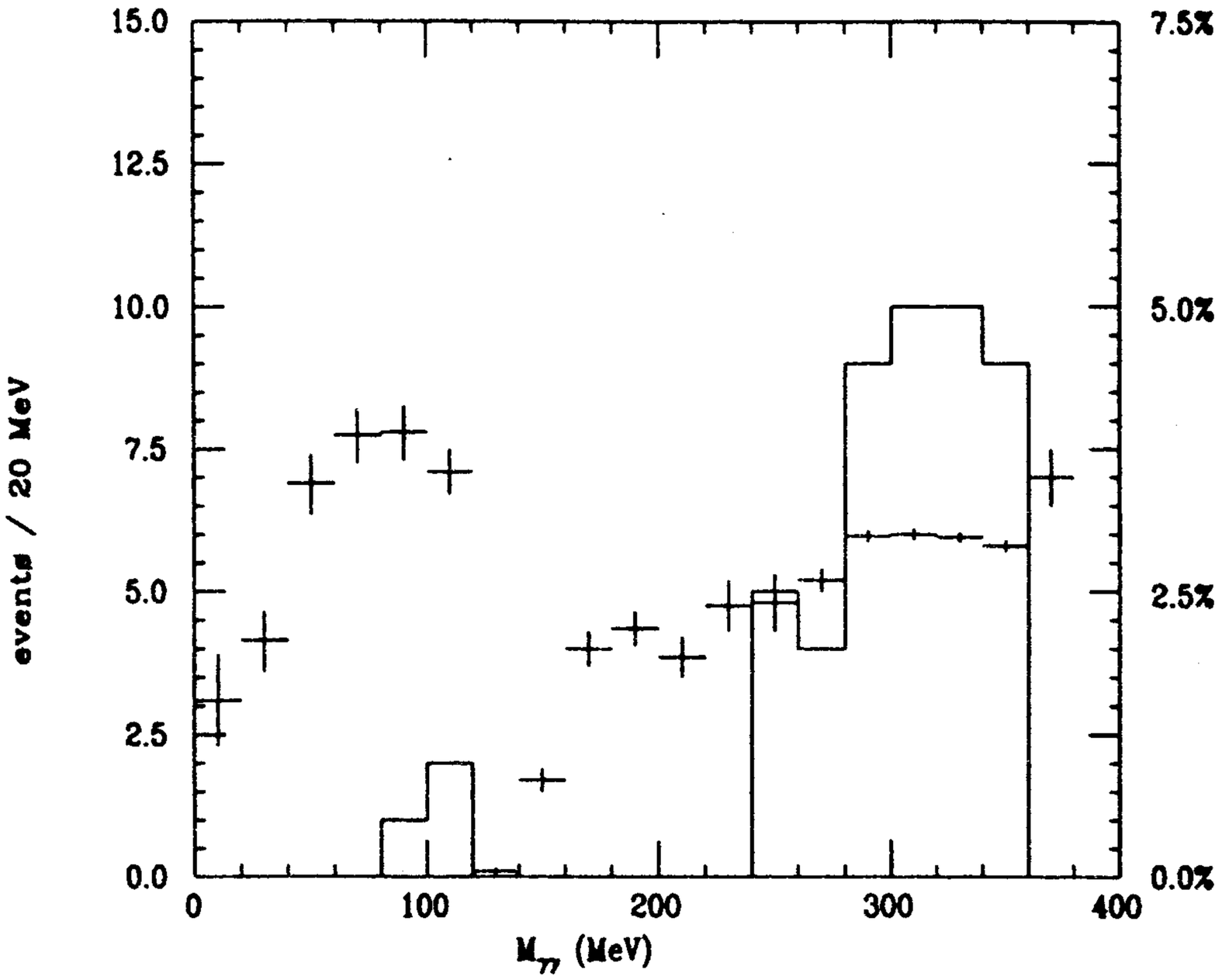

Figure 17: Distributions of the 50 unambiguously $K_L \to \pi^0\gamma\gamma$ events reconstructed by NA31[175] (histograms). Crosses indicate the experimental acceptance (scale on the right).

quadratic slopes of $K \to 3\pi$ (§ Eq. (5.8)) and describes the linear ones with 20%–30% errors (§ tab. 7). Only at $\mathcal{O}(p^4)$ the full physical $K \to 3\pi$ amplitudes are recovered[106]. Using the latter as an effective $K \to 3\pi$ vertex for $K_L \to \pi^0\gamma\gamma$ leads to a 40% increase in the width and a change in the spectrum[170,171], due to the quadratic slopes which generate a B amplitude (§ fig. 16).

Including both local and non–local effects, one can choose appropriately a_V ($a_V \sim -0.9$) to reproduce the experimental spectrum and the experimental branching ratio[171]. Finally, a more complete unitarization of $\pi - \pi$ intermediate states (Khuri–Treiman treatment) and the inclusion of the experimental $\gamma\gamma \to \pi^0\pi^0$ amplitude[172] increases the $K_L \to \pi^0\gamma\gamma$ width by another 10% and the resulting spectrum (§ fig. 16) requires a smaller a_V ($a_V \sim -0.8$)[173]. This value of a_V is consistent with results obtained in the factorization model[173,174].

Experiments test the presence of a B amplitude by studying the spectrum of $K_L \to \pi^0\gamma\gamma$ at low z. Since NA31[175] (§ fig. 17) reports no evidence of a B amplitude, this implies, as we shall see in sect. 8.1, very interesting consequences for $K_L \to \pi^0 e^+ e^-$. In the next section we shall see how the the relative role of unitarity corrections and vector meson contributions can be tested[176] also in $K^\pm \to \pi^\pm\gamma\gamma$.

7.3 Charge asymmetry in $K^{\pm} \to \pi^{\pm}\gamma\gamma$.

Analogously to $K \to \gamma\gamma$ transitions, also $K^{\pm} \to \pi^{\pm}\gamma\gamma$ is dominated by long–distance effects and receive the first non–vanishing contribution at $O(p^4)$. However, since in this case the final state is not a CP eigenstate and contains a charged pion, $K^{\pm} \to \pi^{\pm}\gamma\gamma$ receive contributions not only from loops but also from Z_{WZW} and non–anomalous counterterms.

The $O(p^4)$ decay amplitude can be decomposed in the following way:

$$M(K^+(p) \to \pi^+(p')\gamma(q_1,\epsilon_1)\gamma(q_2,\epsilon_2)) =$$
$$= \epsilon_\mu(q_1)\epsilon_\nu(q_2)\left[A(y,z)\frac{(q_2^\mu q_1^\nu - q_1 q_2 g^{\mu\nu})}{M_K^2} + C(y,z)\epsilon^{\mu\nu\alpha\beta}\frac{q_{1\alpha}q_{2\beta}}{M_K^2}\right], \qquad (7.15)$$

where $C(y,z)$ is the anomalous term. The variables y and z and their relative phase space are defined in (7.9) and (7.10). The $O(p^4)$ result for $A(y,z)$ and $C(y,z)$ is[168]:

$$A(y,z) = \frac{G_8 M_K^2 \alpha}{2\pi z}\left[(r_\pi^2 - 1 - z)F\left(\frac{z}{r_\pi^2}\right) + (1 - r_\pi^2 - z)F(z) + \hat{c}z\right], \qquad (7.16)$$

$$C(y,z) = \frac{G_8 M_K^2 \alpha}{\pi}\left[\frac{z - r_\pi^2}{z - r_\pi^2 + i r_\pi \frac{\Gamma_{\pi^0}}{M_K}} - \frac{z - \frac{2+r_\pi^2}{3}}{z - r_\eta^2}\right], \qquad (7.17)$$

where $r_i = M_i/M_K$ $(i = \pi, \eta)$, $F(z)$ is defined in the appendix and $\hat{c}$ is a finite combination of counterterms:

$$\hat{c} = \frac{128\pi^2}{3}[3(L_9 + L_{10}) + N_{14} - N_{15} - 2N_{18}]\,. \qquad (7.18)$$

Very similarly to $K_L \to \pi^0\gamma\gamma$ (§ sect. 7.2), at $O(p^6)$ there are i) unitarity corrections from the inclusion of the physical $K \to 3\pi$ vertex in the loops, and ii) corrections generated by vector meson exchange[176,173]. Differently from $K_L \to \pi^0\gamma\gamma$ one expects[169,176,173] that the $O(p^6)$ vector meson exchange is negligible. However, unitarity corrections are large here too and generate a B-amplitude (see Eq. (7.8)) as in $K_L \to \pi^0\gamma\gamma$. The resulting diphoton spectrum is shown in fig. 18 for two values of $\hat{c}$: 0.0 and -2.3, corresponding to the theoretical predictions of the weak deformation model (WDM) and of the naive factorization (NF), respectively. In fig. 19 is shown the $BR(K^+ \to \pi^+\gamma\gamma)$ as function of $\hat{c}$. Brookhaven[158] has actually now 30 candidates for this channel with a tendency to $\hat{c} = 0$.

Since loops generate an absorptive contribution, if $\hat{c}$ has a non–vanishing phase, the condition [2] of sect. 2 is satisfied and is possible to observe direct CP violation. Indeed, from Eqs. (7.15-7.16) it follows[168]:

$$\Gamma(K^+ \to \pi^+\gamma\gamma) - \Gamma(K^- \to \pi^-\gamma\gamma) = \frac{\Im m\hat{c}|G_8\alpha|^2 M_{K^+}^5}{2^{10}\pi^5} \times$$
$$\times \int_{4r_\pi^2}^{(1-r_\pi)^2} dz\lambda^{\frac{1}{2}}(1, z, r_\pi^2)(r_\pi^2 - 1 - z)z\Im m F(z/r_\pi^2), \qquad (7.19)$$

where $\lambda(a,b,c)$ is the kinematical function defined in (7.11). Unitarity corrections to this formula have been taken into account in Ref.[170] and lead to

$$|\delta\Gamma| = \frac{|\Gamma(K^+ \to \pi^+\gamma\gamma) - \Gamma(K^- \to \pi^-\gamma\gamma)|}{\Gamma(K^+ \to \pi^+\gamma\gamma) + \Gamma(K^- \to \pi^-\gamma\gamma)} \lesssim |\Im m\hat{c}|. \qquad (7.20)$$

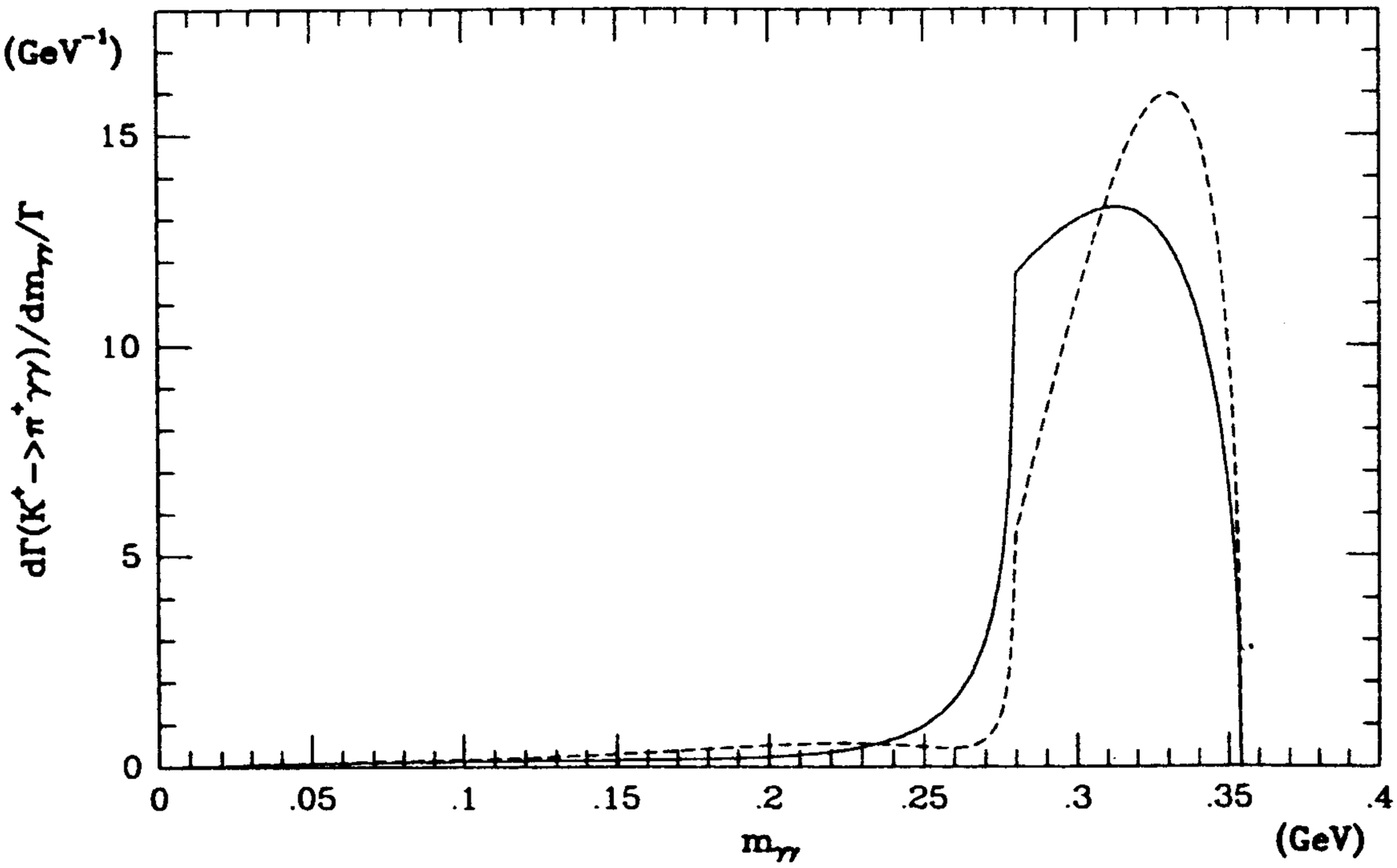

Figure 18: Theoretical predictions for the normalized width of $K^+ \to \pi^+\gamma\gamma$ as a function of the two–photon invariant mass for $\hat{c} = -2.3$ (NF, dashed line) and $\hat{c} = 0$ (WDM, full line) [176].

For what concerns the estimate of $\Im m\hat{c}$, the situation is completely similar to the $K \to \gamma\gamma$ case. Since short–distance contributions[127,161] are suppressed at least by a factor 10^{-4}, we argue

$$\frac{|\Gamma(K^+ \to \pi^+\gamma\gamma) - \Gamma(K^- \to \pi^-\gamma\gamma)|}{\Gamma(K^+ \to \pi^+\gamma\gamma) + \Gamma(K^- \to \pi^-\gamma\gamma)} < 10^{-4}. \tag{7.21}$$

Since $BR(K^+ \to \pi^+\gamma\gamma) \lesssim 10^{-6}$, the above result implies that also this asymmetry is far from the near–future experimental sensitivities.

8 Decays with two leptons in the final state.

8.1 $K \to \pi f \bar{f}$.

$K \to \pi f \bar{f}$ decays can be divided in two categories:

$$\text{(a) } K \to \pi l^+ l^- \qquad \text{and} \qquad \text{(b) } K \to \pi\nu\bar{\nu}, \tag{8.1}$$

where $l = e, \mu$. Even if the branching ratios of these processes (§ tab. 16) are very small compared to those considered before, the different role between short– and long–distance contributions make them very interesting for the study of CP violation.

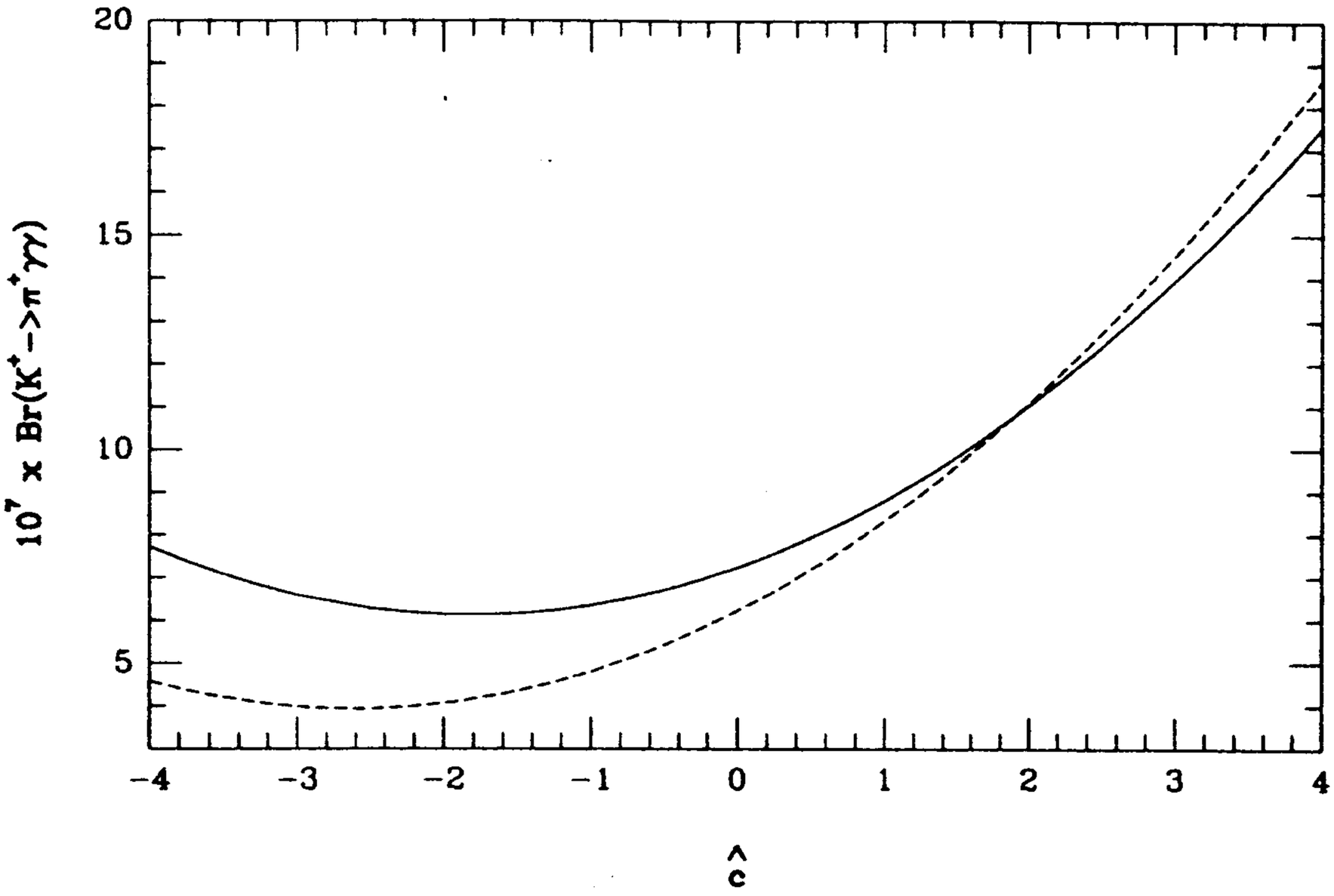

Figure 19: $BR(K^+ \to \pi^+\gamma\gamma)$ as a function of $\hat{c}$. The dashed line corresponds to the $\mathcal{O}(p^4)$ CHPT amplitude. The full line corresponds to the amplitude including the evaluated $\mathcal{O}(p^6)$ corrections[176].

Short–distance contributions are generated by the loop diagrams in fig. 20, which give rise to the following local operators:

$$O^V_{f\bar{f}} = \bar{s}_L\gamma_\mu d_L \bar{f}\gamma^\mu f, \tag{8.2}$$

$$O^A_{f\bar{f}} = \bar{s}_L\gamma_\mu d_L \bar{f}\gamma^\mu\gamma_5 f. \tag{8.3}$$

Due to the GIM suppression, the dominant contribution to the Wilson coefficients of these operators is generated by the quark top and is proportional to λ_t. Thus short–distance contributions carry a large CP–violating phase.

There are two kinds of long–distance contributions. First of all $K \to \pi\gamma^*(Z^*)$ transitions, ruled by the non–leptonic weak hamiltonian (3.30). Secondly, but only for case (a), $K \to \pi\gamma\gamma$ transitions followed by $\gamma\gamma \to l^+l^-$ re–scattering.

Both sort–distance and $K \to \pi\gamma^*(Z^*)$ contributions produce the lepton pair in a $J^{CP} = 1^{--}$ or 1^{++} state, so that $CP|\pi^0 f\bar{f}) = +|\pi^0 f\bar{f})$. As a consequence, in $K_L \to \pi^0 l^+l^-(\nu\bar{\nu})$ these two contributions violate CP. Since the phase of λ_t is of order one and the phase of the weak hamiltonian (3.30) is very small ($\sim \Im m A_0/\Re e A_0$), the short–distance contribution is essentially a direct CP violation whereas the long–distance contribution is dominated by indirect CP violation. Only the re–scattering $\gamma\gamma \to l^+l^-$, that is however very suppressed, generates a CP–invariant contribution.

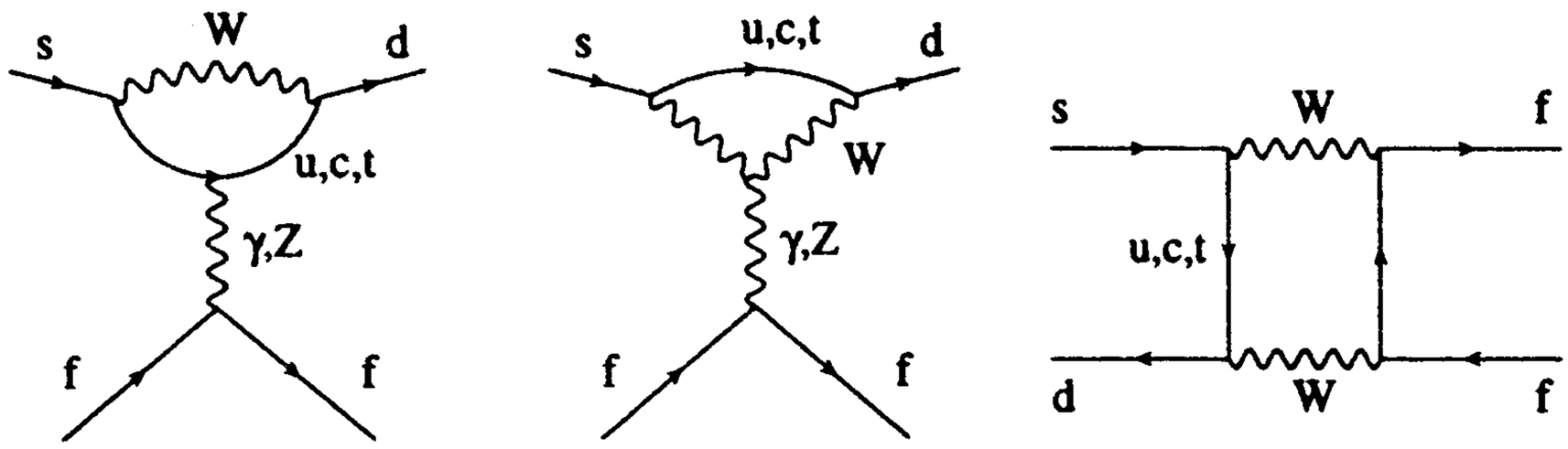

Figure 20: Short–distance contributions to $K \to \pi f \bar{f}$ decays.

decay	branching ratio
$K^{\pm} \to \pi^{\pm} e^+ e^-$	$(2.74 \pm 0.23) \times 10^{-7}$
$K^{\pm} \to \pi^{\pm} \mu^+ \mu^-$	$< 2.3 \times 10^{-7}$
$K^{\pm} \to \pi^{\pm} \nu\bar{\nu}$	$< 5.2 \times 10^{-9}$
$K_L \to \pi^0 e^+ e^-$	$< 4.3 \times 10^{-9}$
$K_L \to \pi^0 \mu^+ \mu^-$	$< 5.1 \times 10^{-9}$
$K_L \to \pi^0 \nu\bar{\nu}$	$< 2.2 \times 10^{-4}$
$K_S \to \pi^0 e^+ e^-$	$< 1.1 \times 10^{-6}$

Table 16: Experimental data on $K \to \pi f \bar{f}$ decays[18].

$K_L \to \pi^0 l^+ l^- (\nu\bar{\nu})$ decays have not been observed yet and certainly have very small branching ratios (§ tab. 16). However, if the short–distance contribution was dominant then an observation of these decays would imply the evidence of direct CP violation[177]. In the following we will try to analyze under which conditions this is true.

8.1.1 Direct CP violation in $K_L \to \pi^0 f \bar{f}$.

The effective hamiltonian describing short distance effects in these decays is[73]:

$$\mathcal{H}_{eff}^{|\Delta S|=1;f\bar{f}} = \frac{2G_F \alpha_{em}}{\sqrt{2}} \sum_{q=u,c,t} \lambda_q (\bar{s}_L \gamma_\mu d_L) \bar{f} \gamma^\mu \left(V_{f\bar{f}}^q + A_{f\bar{f}}^q \gamma_5 \right) f + \text{h.c.} \qquad (8.4)$$

As anticipated, in spite of the λ_t suppression, the dominant contribution in (8.4) is obtained for $q = t$. The coefficients $V_{f\bar{f}}^t$ and $A_{f\bar{f}}^t$ have been calculated including next–to–leading–order QCD corrections[178], for $\mu \sim 1$ GeV the result is

$$V_{l\bar{l}}^t = 3.4 \pm 0.1 \quad V_{\nu\bar{\nu}}^t = -A_{\nu\bar{\nu}}^t = \frac{1}{2} A_{l\bar{l}}^t = 1.6 \pm 0.2. \qquad (8.5)$$

Differently that in (3.30), in this case the μ–dependence and the uncertainties related to α_s are quite small: the error is dominated by the uncertainty on m_t.

The hadronic part of O_{ll}^V and O_{ll}^A matrix elements is well known because is related, by isospin symmetry, to the matrix element of $K^+ \to \pi^0 e^+ \nu_e$:[a]

$$\langle \pi^0(p_\pi)|\bar{d}\gamma_\mu s|K^0(p_K)\rangle = \frac{f_+(q^2)}{\sqrt{2}}(p_K + p_\pi)_\mu + \frac{f_-(q^2)}{\sqrt{2}}(p_K - p_\pi)_\mu; \tag{8.6}$$

where

$$f_+(q^2) = 1 + \lambda\frac{q^2}{M_{\pi^+}^2} \qquad \text{and} \qquad \lambda = (0.030 \pm 0.002). \tag{8.7}$$

Using the previous equations, in the limit $m_f = 0$, we find:[b]

$$\begin{aligned} A(K_2 \to \pi^0 f\bar{f}) &= iG_F\alpha_{em}\Im m\lambda_t f_+(q^2) \times \\ &\quad \times (p_K + p_\pi)_\mu \bar{u}(k)\gamma^\mu \left[V_{ff}^t + A_{ff}^t\gamma_5\right] v(k'), \end{aligned} \tag{8.8}$$

which implies

$$BR_{CP-dir}(K_L \to \pi^0 f\bar{f}) = (1.16 \times 10^{-5})\left[(V_{ff}^t)^2 + (A_{ff}^t)^2\right](A^2\lambda^5\eta)^2. \tag{8.9}$$

Using for A, λ and η the values of tab. 2 and summing over the three neutrino families, we finally obtain

$$\begin{aligned} 1.3 &< 10^{12} \times BR_{CP-dir}(K_L \to \pi^0 e^+ e^-) < 5.0, \\ 0.8 &< 10^{12} \times BR_{CP-dir}(K_L \to \pi^0 \nu\bar{\nu}) < 3.3, \\ 0.3 &< 10^{12} \times BR_{CP-dir}(K_L \to \pi^0 \mu^+ \mu^-) < 1.0. \end{aligned} \tag{8.10}$$

8.1.2 Indirect CP violation in $K_L \to \pi^0 f\bar{f}$.

Neglecting the interference pieces among the different terms, the indirect–CP–violating contribution to the branching ratio is given by

$$\begin{aligned} BR_{CP-ind}(K_L \to \pi^0 f\bar{f}) &= |\epsilon|^2\frac{\Gamma_S}{\Gamma_L}BR(K_S \to \pi^0 f\bar{f}) \\ &= 3 \times 10^{-3} \times BR(K_S \to \pi^0 f\bar{f}). \end{aligned} \tag{8.11}$$

Unfortunately, present limits on $K_S \to \pi^0 f\bar{f}$ branching ratios (§ tab. 16) are not sufficient to establish the relative weight between Eq. (8.11) and Eq. (8.10). Thus we need to theoretically estimate $\Gamma(K_S \to \pi^0 f\bar{f})$.

This process receives contributions both from the effective hamiltonian of Eq. (8.4) and from the one of Eq. (3.30). The former is negligible since generates a width of the same order of $\Gamma_{CP-dir}(K_L \to \pi^0 f\bar{f})$ estimated in the previous subsection. The long–distance contribution generated by $\mathcal{H}_{eff}^{|\Delta S|=1}$ must be evaluated in the CHPT framework. The lowest

---

[a] It is possible to derive the same result also by means of CHPT (§ sect. 4.2.1), but is necessary to keep also $O(p^4)$ contributions to obtain the correct form–factor behaviour at $q^2 \neq 0$.

[b] The contribution of $f_-(q^2)$ is proportional to m_f and thus negligible in the case of e^+e^- and $\nu\bar{\nu}$ pairs.

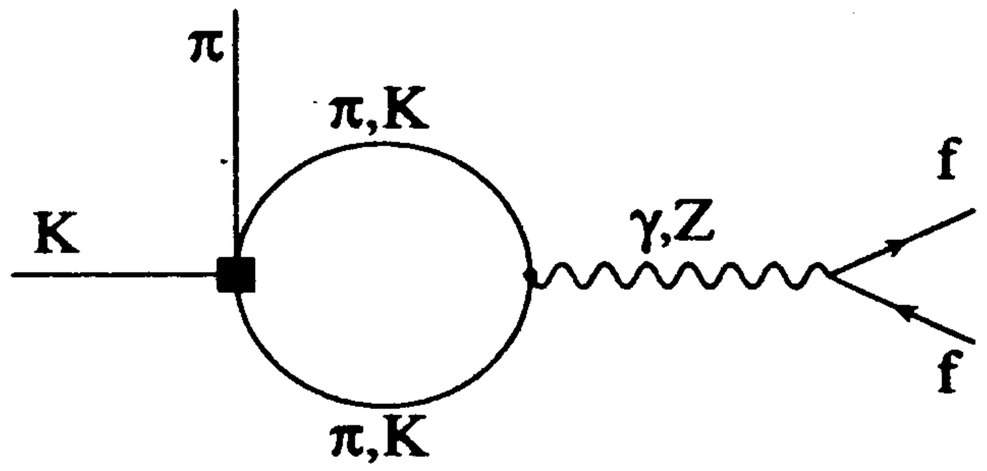

Figure 21: One–loop diagram (in the diagonal basis of Ecker, Pich and de Rafael[168]) relevant to $K \to \pi f \bar{f}$ transitions.

order result vanishes both in the $K_S \to \pi^0\gamma^*$ case[179] and in the $K_S \to \pi^0 Z^*$ one[180]. The first non–vanishing contribution arises at $O(p^4)$.

Before going on with the calculation, we note that the one–loop diagram of fig. 21 with a Z^* is heavily suppressed ($\sim (M_K/M_Z)^2$) respect to the corresponding one with γ^*, thus

$$\frac{BR_{long-d}(K_S \to \pi^0\nu\bar{\nu})}{BR_{long-d}(K_S \to \pi^0 e^+e^-)} \lesssim \left(\frac{M_K^2\alpha_{em}}{M_Z^2\alpha_W}\right) < 10^{-7}. \tag{8.12}$$

This result, together with the experimental limit on $K_S \to \pi^0 e^+e^-$ (§ tab. 16), let us state that *the dominant contribution to $K_L \to \pi^0\nu\bar{\nu}$ is generated by direct CP violation.* Unfortunately the observation of this process is very difficult: requires an extremely good π^0–tagging and an hermetic detector able to eliminate the background coming from $K_L \to \pi^0\pi^0$ (with two missing photons), that has a branching ratio eight orders of magnitude larger. The KTeV[181] expected sensitivity for $K_L \to \pi^0\nu\bar{\nu}$ is $\sim 10^{-8}$.

Coming back to the $O(p^4)$ calculation, the result for $K_S \to \pi^0 e^+e^-$ is[179]:

$$A^{(4)}(K_S \to \pi^0 e^+e^-) = \frac{G_8\alpha_{em}}{4\pi}\left[2F\left(\frac{q^2}{M_K^2}\right) + \omega_S\right] \times$$
$$\times \left\{(p_K + p_\pi)_\mu - (M_K^2 - M_\pi^2)\frac{q^\mu}{q^2}\right\}\bar{u}(K)\gamma^\mu v(k'), \tag{8.13}$$

where $F(z)$ is defined in the appendix and

$$\omega_S = \frac{2}{3}(4\pi)^2\left[2N_{14}^r(\mu) + N_{15}^r(\mu)\right] - \frac{1}{3}\log\left(\frac{M_K^2}{\mu^2}\right). \tag{8.14}$$

Also in this case the counterterm combination is not experimentally known.[c]

[c] Contrastingly to the cases analyzed before, the counterterm combination which appears in $K_S \to \pi^0 e^+e^-$ is not finite. However, the constant ω_S of Eq. (8.14) is by construction μ–independent.

Expressing $K_S \to \pi^0 e^+ e^-$ branching ratio as a function of ω_S, leads to[119]

$$BR(K_S \to \pi^0 e^+ e^-) = \left[3.07 - 18.7\omega_S + 28.4\omega_S^2\right] \times 10^{-10}. \tag{8.15}$$

Within the factorization model (§ sect. 4.4) is possible to relate ω_S to the counterterm combination that appears in $K^+ \to \pi^+ e^+ e^-$ (§ sect. 8.1.4), the only $K \to \pi f \bar{f}$ channel till now observed[116,119]:

$$\omega_S = (0.5 \pm 0.2) + 4.6(2k_f - 1). \tag{8.16}$$

In the factorization model earlier proposed in Ref.[179], the choice $k_f = 1/2$ was adopted. This value of k_f essentially minimize Eq. (8.15) and for this reason in the literature has been sometimes claimed that indirect CP violation is negligible in $K_L \to \pi^0 e^+ e^-$ (see e.g. Ref.[37]). Though supported by a calculation done in a different framework[182], this statement is very model dependent (as can be easily deduced from Eq. (8.16)). Indeed the choice $k_f > 0.7$, perfectly consistent from the theoretical point of view, implies $BR(K_S \to \pi^0 e^+ e^-) > 10^{-8}$, i.e. $BR_{CP-ind}(K_L \to \pi^0 e^+ e^-) > 3 \times 10^{-11}$. In our opinion, the only model independent statement that can be done now is:

$$\begin{array}{rcccl} 5 \times 10^{-10} & < & BR(K_S \to \pi^0 e^+ e^-) & < & 5 \times 10^{-8}, \\ 1.5 \times 10^{-12} & < & BR_{CP-ind}(K_L \to \pi^0 e^+ e^-) & < & 1.5 \times 10^{-10}, \end{array} \tag{8.17}$$

thus *today is not possible to establish if* $K_L \to \pi^0 e^+ e^-$ *is dominated by direct or indirect* CP *violation*. The only possibility to solve the question is a direct measurement of $\Gamma(K_S \to \pi^0 e^+ e^-)$, or an upper limit on it at the level of 10^{-9}, within the reach of KLOE[135,39]. We stress that this question is of great relevance since the sensitivity on $K_L \to \pi^0 e^+ e^-$ which should be reached at KTeV is $\sim 10^{-11}$ (for a discussion about backgrounds and related cuts in $K_L \to \pi^0 e^+ e^-$ see Ref.[183]).

8.1.3 CP–invariant contribution of $K_L \to \pi^0 \gamma\gamma$ to $K_L \to \pi^0 e^+ e^-$.

As anticipated, the decay $K_L \to \pi^0 e^+ e^-$ receives also a CP–invariant contribution form the two–photon re–scattering in $K_L \to \pi^0 \gamma\gamma$ (contribution that has been widely discussed in the literature, see e.g. Refs.[184–187,167,171]).

As we have seen in sect. 7.2, the $K_L \to \pi^0 \gamma\gamma$ amplitude can be decomposed in to two parts: A and B (§ Eq. (7.8)). When the two photons interact creating an $e^+ e^-$ pair the contribution of the A amplitude is negligible being proportional to m_e.

In the parametrization of Ecker, Pich and de Rafael[169] (§ Eq. (7.14)) the absorptive contribution generated by on–shell photons coming from the B term is given by[171,187,173]:

$$BR_{CP-cons}(K_L \to \pi^0 e^+ e^-)|_{abs} = 0.3 \times 10^{-12}, \qquad a_V = 0, \tag{8.18}$$

$$BR_{CP-cons}(K_L \to \pi^0 e^+ e^-)|_{abs} = 1.8 \times 10^{-12}, \qquad a_V = -0.9. \tag{8.19}$$

Using these results we can say that the CP–invariant contribution should be smaller than the direct–CP–violating one. At any rate, even in this case a more precise determination of $\Gamma(K_L \to \pi^0 \gamma\gamma)$ at small z (definitely within the reach of KLOE) could help to evaluate better the situation.

Another important question is the dispersive contribution generated by off–shell photons, which is more complicated since the dispersive integral is in general not convergent. A first estimate of this integral was done in Ref.[187], introducing opportune form factors which suppress the virtual–photon couplings at high q^2. The dispersive contribution estimated in this way is of the same order of the dispersive one, but is quite model dependent. A more refined analysis is in progress[188].

Finally, we remark that the different CP-conserving and CP-violating contributions to $K_L \to \pi^0 e^+ e^-$ could be partially disentangled if the asymmetry in the electron–positron energy distribution[187] and the time–dependent interference[189,190] of $K_{L,S} \to \pi^0 e^+ e^-$ would be measured in addition to the total width.

8.1.4 $K^\pm \to \pi^\pm l^+ l^-$.

Another CP–violating observable that can de studied in $K \to \pi f \bar{f}$ decays is the charge asymmetry of $K^\pm \to \pi^\pm e^+ e^-$ widths. As we have seen in sect. 2, in order to have a non–vanishing charge asymmetry is necessary to consider processes with non–vanishing re–scattering phases. This happens in $K^\pm \to \pi^\pm e^+ e^-$ decays due to the absorptive contribution of the loop diagram in fig. 21.

Analogously to the $K^0 \to \pi^0 e^+ e^-$ case, the $O(p^4)$ amplitude of $K^+ \to \pi^+ e^+ e^-$ is given by[179]:

$$A^{(4)}(K^+ \to \pi^+ e^+ e^-) = \frac{G_8 \alpha_{em}}{4\pi} \left[-F\left(\frac{q^2}{M_K^2}\right) - F\left(\frac{q^2}{M_\pi^2}\right) - \omega_+ \right] \times$$
$$\times \left\{ (p_K + p_\pi)_\mu - (M_K^2 - M_\pi^2)\frac{q^\mu}{q^2} \right\} \bar{u}(K)\gamma^\mu v(k'), \tag{8.20}$$

where

$$\omega_+ = \frac{4}{3}(4\pi)^2 \left[N_{14}^r(\mu) - N_{15}^r(\mu) + 4L_9^r(\mu) \right] - \frac{1}{3}\log\left(\frac{M_K^2}{\mu^2}\right). \tag{8.21}$$

Thus the charge asymmetry is given by:

$$\Gamma(K^+ \to \pi^+ e^+ e^-) - \Gamma(K^- \to \pi^- e^+ e^-) = \Im m w_+ \frac{|G_8|^2 \alpha_{em}^2 M_K^5}{192\pi^5} \times$$
$$\times \int_{4r_e^2}^{(1-r_\pi^2} \mathrm{d}z \lambda^{3/2}(1, z, r_\pi^2) \sqrt{1 - 4\frac{r_e^2}{z^2}} \left(1 + 2\frac{r_e^2}{z^2}\right) \Im m F\left(\frac{z}{r_\pi^2}\right), \tag{8.22}$$

where $z = q^2/M_K^2$, $r_i = m_i/M_K$ and $\lambda(x, y, z)$ is defined in Eq. (7.11). Integrating Eq. (8.22) and using the experimental value of $\Gamma(K^+ \to \pi^+ e^+ e^-)$ we finally get

$$\frac{\Gamma(K^+ \to \pi^+ e^+ e^-) - \Gamma(K^- \to \pi^- e^+ e^-)}{\Gamma(K^+ \to \pi^+ e^+ e^-) + \Gamma(K^- \to \pi^- e^+ e^-)} \simeq 10^{-2} \Im m w_+. \tag{8.23}$$

The real part of w_+ is fixed by the experimental information on the width and the spectrum of the decay[194]:

$$\Re e w_+ = 0.89^{+0.24}_{-0.14}. \tag{8.24}$$

On the other hand, we expect that $\Im m w_+$ is theoretically determined by short distance contributions. Comparing Eq. (8.20) with the amplitude obtained by $\mathcal{H}_{eff}^{|\Delta S|=1;f\bar{f}}$, we find:

$$|\Im m w_+| \simeq 4\pi A^2 \lambda^5 \eta V_{f\bar{f}}^t, \tag{8.25}$$

which implies

$$|\delta\Gamma| = \frac{|\Gamma(K^+ \to \pi^+ e^+ e^-) - \Gamma(K^- \to \pi^- e^+ e^-)|}{\Gamma(K^+ \to \pi^+ e^+ e^-) + \Gamma(K^- \to \pi^- e^+ e^-)} \simeq 2 \times 10^{-4} \times A^2 \eta. \tag{8.26}$$

Unfortunately, due to the small branching ratio of the process, the statistics necessary to test this interesting prediction is beyond near–future experimental programs.

Beside the asymmetry of $K^\pm$ widths, in $K^+ \to \pi^+ \mu^+ \mu^-$ (and in $K^- \to \pi^- \mu^+ \mu^-$) is possible to measure also asymmetries which involve muon polarizations. These can be useful both to study T violation[191] and provide valuable information about CKM matrix elements[192]. However, these measurements are not easy from the experimental point and thus we will not discuss them in detail (for accurate analyses see Refs.[191–193]).

8.1.5 $K^+ \to \pi^+ \nu\bar{\nu}$.

As in the $K_L \to \pi^0 \nu\bar{\nu}$ case, also this decay is by far dominated by short distance. This process is not directly interesting for CP violation but is one of the best channels to put constraints on CKM parameters ρ and η.

Using the short distance hamiltonian (8.4) we find, analogously to Eq. (8.8),

$$A(K^+ \to \pi^+ \nu\bar{\nu}) = G_F \alpha_{em} f_+(q^2) \sum_q \lambda_q V_{\nu\bar{\nu}}^q (p_K + p_\pi)_\mu \bar{u}(k) \gamma^\mu [1 - \gamma_5] v(k'). \tag{8.27}$$

As in the previous cases the top contribution is dominant, however, since $K^+ \to \pi^+ \nu\bar{\nu}$ is a CP conserving amplitude, in this case the charm effect is not completely negligible. Following Buras et al.[195] we can parametrize the branching ratio in the following way:

$$BR(K^+ \to \pi^+ \nu\bar{\nu}) = 2 \times 10^{-11} A^4 \left[\eta^2 + \frac{2}{3}(\rho - \rho^e)^2 + \frac{1}{3}(\rho - \rho^\tau)^2\right] \left(\frac{m_t}{M_W}\right)^{2.3}, \tag{8.28}$$

where ρ^e and ρ^τ differ from unity because of the presence of the charm contribution; using $\overline{m}_c(m_c) = 1.30 \pm 0.05$ GeV from Buras et al.[195] we find

$$1.42 \leq \rho^e \leq 1.55 \qquad 1.27 \leq \rho^\tau \leq 1.38. \tag{8.29}$$

Present limits on this decay (§ tab. 16) are still more than one order of magnitude far from the Standard Model value, which however should be reached in the near future.[158]

As shown in fig. 22, $K \to \pi f \bar{f}$ measurements would allow in principle a complete determination of sizes and angles of the unitarity triangle introduced in sect. 3.4. Even if these measurements are not easy to perform, it is important to stress that could compete for completeness and cleanliness with those in the B sector.

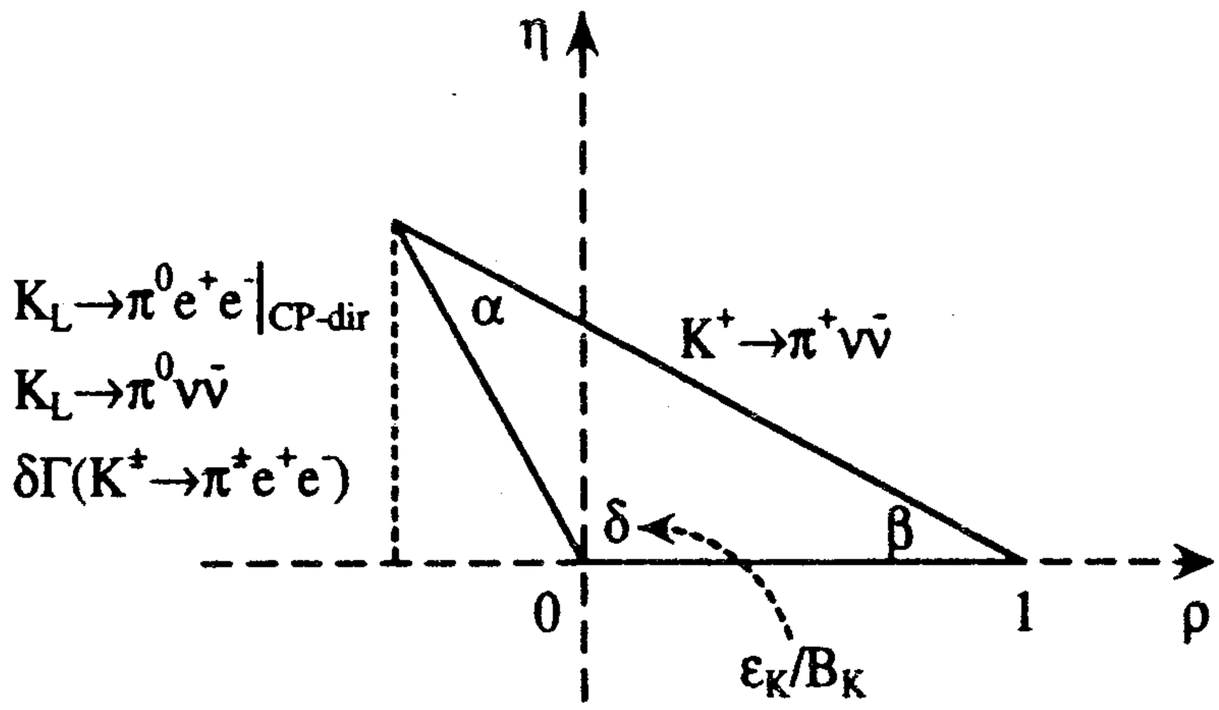

Figure 22: Kaon decays and the unitarity triangle.

decay	branching ratio
$K_L \to \mu^+\mu^-$	$(7.4 \pm 0.4) \times 10^{-9}$
$K_L \to e^+e^-$	$< 4.1 \times 10^{-11}$
$K_S \to \mu^+\mu^-$	$< 3.2 \times 10^{-7}$
$K_S \to e^+e^-$	$< 1.0 \times 10^{-5}$

Table 17: Experimental data[18] on $K_{L,S} \to l^+l^-$.

8.2 $K \to l^+l^-$

The decay amplitude $A(K^0 \to l^+l^-)$ can be generally written as

$$A(K^0 \to l^+l^-) = \bar{u}(k)(iB + A\gamma_5)v(k'), \tag{8.30}$$

then

$$\Gamma(K^0 \to l^+l^-) = \frac{M_K \beta_{(l)}}{8\pi}\left(|A|^2 + \beta_{(l)}^2|B|^2\right), \qquad \beta_{(l)} = \left(1 - \frac{4m_l^2}{M_K^2}\right)^{1/2}. \tag{8.31}$$

Up to now, only the $K_L \to \mu^+\mu^-$ decay has been observed (§ tab. 17).

Analogously to previous decays, also in in this case is convenient to decompose amplitude contributions in short– and long–distance ones. In both channels (K_L and K_S) the dominant contribution is the long–distance one, generated by the two–photon re–scattering in $K_L(K_S) \to \gamma\gamma$ transitions. Since $A(\gamma\gamma \to l^+l^-)$ is proportional to m_l, this explains why only the $K_L \to \mu^+\mu^-$ has been observed till now.

A and B amplitudes have different transformation properties under CP: if CP is conserved $K_2 \to l^+l^-$ receives a contribution only from A whereas $K_1 \to l^+l^-$ only form

B. Thus CP violation in $K \to l^+l^-$ decays can be observed trough the asymmetry

$$\langle P^{(l)} \rangle = \frac{N_+(l^-) - N_-(l^-)}{N_+(l^-) + N_-(l^-)} \propto \Im m(A^* B), \tag{8.32}$$

where $N_\pm(l^-)$ indicates the number of l^- emitted with positive or negative helicity. In the K_L case we obtain:

$$\left|\langle P_L^{(\mu)} \rangle\right| \simeq \beta_{(\mu)} \left|\Im m \left(\frac{B_2 + \tilde{\epsilon} B_1}{A_2}\right)\right|, \tag{8.33}$$

where the subscripts of A and B indicate if the amplitudes belong to K_1 or K_2.

The amplitude B_2, responsible of direct CP violation, is generated in the Standard Model by the short–distance contribution of the effective operator[196] $\bar{s}dH$, where H is the physical Higgs field. Its effect is completely negligible with respect to the indirect–CP–violating one.

The CP–invariant amplitude A_2 has a large imaginary part ($|\Re e A_2| \ll |\Im m A_2|$), because the absorptive contribution to $K_L \to \gamma\gamma \to \mu^+\mu^-$ essentially saturates the experimental value of $\Gamma(K_L \to \mu^+\mu^-)$:

$$BR(K_L \to \mu^+\mu^-)|_{abs} = (6.85 \pm 0.32) \times 10^{-9}. \tag{8.34}$$

As a consequence, if we neglect both direct CP violation and $\Re e A_2/\Im m A_2$, and we assume $\epsilon = |\epsilon| e^{i\pi/4}$, then Eq. (8.33) becomes:

$$\left|\langle P_L^{(\mu)} \rangle\right| \simeq \frac{\beta_{(\mu)} |\epsilon| \sqrt{2}}{|\Im m A_2|} \left|\Im m B_1 - \Re e B_1\right|. \tag{8.35}$$

The amplitude B_1 can be calculated unambiguously in CHPT at the lowest non–vanishing order ($K_S \to \gamma\gamma\ O(p^4) + \gamma\gamma \to l^+l^-\ O(e^2)$) since, as in $K_S \to \gamma\gamma$, the loop calculation is finite[197]. The results thus obtained are

$$\Im m B_1 = +0.54 \times 10^{-12}, \tag{8.36}$$

$$\Re e B_1 = -1.25 \times 10^{-12}, \tag{8.37}$$

and imply[197]

$$\left|\langle P_L^{(\mu)} \rangle\right| \simeq 2 \times 10^{-3}. \tag{8.38}$$

The measurement of $\langle P_L^{(\mu)} \rangle$ is not useful for the study of direct CP violation within the Standard Model. However, the above result tell us that a measurement of $\langle P_L^{(\mu)} \rangle$ at the level of 10^{-3} could be very useful to exclude new CP–violating mechanisms with additional scalar fields[37].

Another interesting question in this channel is the short distance[198] contribution to $\Re e A_2$, which depends on the CKM matrix element V_{td}. Thus $K_L \to \mu^+\mu^-$ could in principle add new information about the unitarity triangle of fig. 22. However, $\Re e A_2$ receives also a long–distance (model dependent) contribution from the dispersive integral $K_L \to \gamma^*\gamma^* \to \mu^+\mu^-$. The smallness of $\Re e A_2$ implies a cancellation among these two

terms. The extent of this cancellation and the accuracy with which one can evaluate the dispersive integral, determine the sensitivity to V_{td}. $K_L \to \gamma l^+ l^-$ and $K_L \to \gamma e^+ e^- \mu^+ \mu^-$ decays (see Refs.[199,200,173] and references therein) could bring some information on the relevant form factor, however the result is still model dependent.

8.3 $K \to \pi l \nu$.

Analogously to the previous case, also the transverse muon polarization in $K \to \pi \mu \nu$ decays ($\langle P_\perp^{(\mu)} \rangle$) is very sensitive to new CP–violating mechanisms with additional scalar fields (for an update discussion see Ref.[86]). $\langle P_\perp^{(\mu)} \rangle$ is a measurement of the muon polarization perpendicular to the decay plane and, by construction, is related to the correlation

$$\langle \vec{s}_{(\mu)} \cdot (\vec{p}_{(\mu)} \times \vec{p}_\pi) \rangle \tag{8.39}$$

which violates T in absence of final–state interactions. In the $K_L \to \pi^- \mu^+ \nu$ case, with two charged particles in the final state, electromagnetic interactions can generate[201] $\langle P_\perp^{(\mu)} \rangle_{FSI} \sim \alpha/\pi \sim 10^{-3}$. However, in $K^+ \to \pi^0 \mu^+ \nu$ this effect is much smaller[202] ($\langle P_\perp^{(\mu)} \rangle_{FSI} \sim 10^{-6}$) and T–violation could be dominant.

In the framework of the Standard Model and in any model where the $K^+ \to \pi^0 \mu^+ \nu$ decay is mediated by vector meson exchanges, $\langle P_\perp^{(\mu)} \rangle$ is zero at tree-level and is expected to be very small[203]. On the other hand, interference between W bosons and CP–violating scalars can produce a large effect. Writing the effective amplitude as[18]

$$A(K^+ \to \pi^0 \mu^+ \nu) \propto f_+(q^2) \left[(p_K + p_\pi)_\mu \bar{\mu} \gamma^\mu (1 - \gamma_5) \nu + \xi(q^2) m_{(\mu)} \bar{\mu} (1 - \gamma_5) \nu \right], \tag{8.40}$$

neglecting the q^2 dependence of $\xi(q^2)$ and averaging over the phase space, leads to[204]:

$$\langle P_\perp^{(\mu)} \rangle \simeq 0.2 (\Im m \xi). \tag{8.41}$$

The present experimental determination of $\Im m \xi$ is[18]:

$$\Im m \xi = -0.0017 \pm 0.025, \tag{8.42}$$

but an on–going experiment at KEK (E246) should improve soon this limit by an order of magnitude[205]. From the theoretical point of view, it is interesting to remark that present limits on multi–Higgs models, coming from neutron electric dipole moment and $B \to X \tau \nu_\tau$, do not exclude a value of $\Im m \xi$ larger than the sensitivity achievable at KEK[86]. Futhermore, an eventual evidence of a non–vanishing $\Im m \xi$ at the level of 10^{-3}, would imply interesting consequence for the next–generation of experiments in the B sector.

8.4 $K \to \pi \pi l^+ l^-$.

The last channels we are going to discuss are $K \to \pi \pi l^+ l^-$ transitions. The dynamic of these processes is essentially the same of $K \to \pi \pi \gamma$ transitions (§ sect. 6), with the

difference that the photon is virtual. For this reason we shall discuss these decays only briefly.

With respect to $K \to \pi\pi\gamma$ transitions, $K \to \pi\pi l^+ l^-$ decays have the disadvantage that the branching ratio is sensibly smaller (obviously the e^+e^- pair is favoured with respect to the $\mu^+\mu^-$ one). Nevertheless, there is also an advantage: the lepton plane furnishes a measurement of the photon polarization vector. This is particularly useful in the case of $K_L \to \pi^+\pi^- e^+e^-$,[d] because let us to measure the CP–violating interference between electric and magnetic amplitudes (§ sect. 6.1.1) also in experimental apparata where photon polarizations are not directly accessible. The observable proportional to this interference is the $\phi_{\pi/e}$–distribution, where $\phi_{\pi/e}$ is the angle between e^+e^- and $\pi^+\pi^-$ planes.

The asymmetry in the $\phi_{\pi/e}$–distribution has been recently estimated in Refs.[149,206] and turns out to be quite large ($\sim 10\%$), within the reach of KLOE. However, since the electric amplitude of $K_L \to \pi^+\pi^-\gamma$ is dominated by the bremsstrahlung of $K_L \to \pi^+\pi^-$, this effect is essentially an indirect CP violation. Elwood et al.[206] have shown how to construct an asymmetry which is essentially an index of direct CP violation. In this case, however, the prediction is of the order of 10^{-4}, far from experimental sensitivities.

9 Conclusions

In table 18 we report the Standard Model predictions discussed in this review for the direct–CP–violating observables of $K \to 3\pi$, $K \to 2\pi\gamma$, $K \to \pi f\bar{f}$, $K \to \gamma\gamma$ and $K \to \pi\gamma\gamma$ decays.

In many cases, due to the uncertainties of next–to–leading order CHPT corrections, it has not been possible to make definite predictions but only to put some upper limits. However, this analysis is still very useful since an experimental evidence beyond these limits would imply the existence of new CP violating mechanisms.

The essential points of our analysis can be summarized as follows:

- In $K \to 3\pi$, $K \to 2\pi\gamma$, $K \to \gamma\gamma$ and $K \to \pi\gamma\gamma$, the presence of several $\Delta I = 1/2$ amplitudes generally let to overcome the $\omega = \Re e A_2/\Re e A_0$ suppression which depresses direct CP violation in $K \to 2\pi$. Thus in the above decays direct–CP–violating observables are usually larger than ϵ' of about one order of magnitude. Nevertheless, due to the small branching ratios, the experimental sensitivities achievable in these decays are well below those of $K \to 2\pi$.

  With respect to some controversial questions in the literature, we stress that charge asymmetries in $K^\pm \to (3\pi)^\pm$, as well as those in $K^\pm \to \pi^\pm\pi^0\gamma$, cannot exceed 10^{-5}.

- In $K_L \to \pi^0 f\bar{f}$ decays, the absence or the large suppression of CP–invariant contributions implies that CP violation plays a fundamental role: the question is whether the direct CP violation dominates over the other contributions.

[d] This decay has not been observed yet, the theoretical branching ratio[119] is $BR(K_L \to \pi^+\pi^- e^+e^-) = 2.8 \times 10^{-7}$.

channel	observable − prediction			σ
$(2\pi)^0$	$\left\lvert\frac{\epsilon'}{\epsilon}\right\rvert$	$<$	10^{-3}	$\gtrsim 10^{-4}$
$(3\pi)^\pm$	$\lvert\delta g\rvert$	$<$	10^{-5}	$\gtrsim 10^{-4}$
$(3\pi)^0$	$\lvert\epsilon^X_{+-0}\rvert, \lvert\epsilon'_{+-0}\rvert$	$<$	5×10^{-5}	$\gtrsim 10^{-4}$
$\pi^+\pi^-\gamma$	$\lvert\eta_{+-\gamma} - \eta_{+-}\rvert$	$<$	5×10^{-6}	$\gtrsim 10^{-5}$
$\pi^\pm\pi^0\gamma$	$\lvert\delta\Gamma_{DE}\rvert$	$<$	10^{-4}	$\gtrsim 10^{-3}$
$\gamma\gamma$	$\lvert\epsilon'_\perp\rvert, \lvert\epsilon'_\parallel\rvert$	$<$	10^{-4}	$\gtrsim 10^{-3}$
$\pi^\pm\gamma\gamma$	$\lvert\delta\Gamma\rvert$	$<$	10^{-4}	$\gtrsim 10^{-2}$
$\pi^0\nu\bar{\nu}$	$BR(K_L\to\pi^0\nu\bar{\nu})$	$=$	$(0.8\div 3.3)\times 10^{-12}$	$\gtrsim 10^{-8}$
$\pi^0e^+e^-$	$BR(K_L\to\pi^0e^+e^-)$	$=$	$(1.3\div 5.0)\times 10^{-12}$ (*)	10^{-11}
$\pi^\pm e^+e^-$	$\lvert\delta\Gamma\rvert$	$=$	$(0.4\pm 0.2)\times 10^{-4}$	$\gtrsim 10^{-2}$

Table 18: Standard Model predictions for direct–CP–violating observables of K decays. In the third column we report a rough estimate of the expected sensitivities, achievable by combining KTeV[181], NA48[207] and KLOE[135] future results. The '$*$' in $K_L\to\pi^0e^+e^-$ concerns the question of the indirect–CP–violating contribution (§ sect. 8.1.2).

In $K_L \to \pi^0 \nu\bar{\nu}$ the direct CP violation is certainly dominant, but the observation of this decay is extremely difficult.

In $K_L \to \pi^0 e^+ e^-$, due to the uncertainties of both $O(p^4)$ effects in $K_S \to \pi^0 e^+ e^-$ and the dispersive contribution from $K_L \to \pi^0 \gamma\gamma$, is impossible to establish without model dependent assumptions which is the dominant amplitude. Only a measurement of $BR(K_S \to \pi^0 e^+ e^-)$ (or an upper limit on it at the level of 10^{-9}) together with a more precise determination of the dispersive contribution form $K_L \to \pi^0 \gamma\gamma$ could solve the question.

To conclude this analysis, we can say that in the near future there is a realistic hope to observe direct CP violation only in $K \to 2\pi$ and $K_L \to \pi^0 e^+ e^-$, but even in these decays a positive result is not guaranteed. Nevertheless, a new significant insight in the study of this interesting phenomenon will certainly start in few years with the next–generation of experiments on B decays[208] and, possibly, with new rare kaon decay experiments[209].

Acknowledgments

We are grateful to G. Ecker and L. Maiani for enlightening discussions and valuable comments on the manuscript. We acknowledge also M. Ciuchini, E. Franco, G. Martinelli, H. Neufeld, N. Paver, J. Portoles and A. Pugliese for interesting discussions and/or collaborations on some of the subjects presented here. G.D. would like to thank the hospitality of the Particle Theory Group at MIT where part of this work was done.

A Loop functions.

The function $\widetilde{C_{20}}(x,y)$, which appear at one loop in $K \to \pi\pi\gamma$ direct–emission amplitudes, is defined as[118]

$$\widetilde{C_{20}}(x,y) = \frac{C_{20}(x,y) - C_{20}(x,0)}{y}, \tag{1.1}$$

in terms of the three–propagator one–loop function $C_{20}(p^2, kp)$ for $k^2 = 0$:

$$\begin{aligned} \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{[l^2 - M_\pi^2][(l+k)^2 - M_\pi^2][(l+p)^2 - M_\pi^2]} \\ = i g^{\mu\nu} C_{20}(p^2, kp) \; + O(p^\mu, k^\mu). \end{aligned} \tag{1.2}$$

The explicit expression for $x,\ x - 2y > 4M_\pi^2$ is

$$\begin{aligned} (4\pi)^2 \Re e \widetilde{C_{20}}(x,y) \;=\; & \frac{x}{8y^2} \left\{ \left(1 - 2\frac{y}{x}\right) \left[\beta \log\left(\frac{1+\beta}{1-\beta}\right) - \beta_0 \log\left(\frac{1+\beta_0}{1-\beta_0}\right) \right] \right. \\ & \left. + \frac{M_\pi^2}{x} \left[\log^2\left(\frac{1+\beta_0}{1-\beta_0}\right) - \log^2\left(\frac{1+\beta}{1-\beta}\right) \right] + 2\frac{y}{x} \right\}, \end{aligned} \tag{1.3}$$

$$(16\pi)\Im m\widetilde{C_{20}}(x,y) = -\frac{x}{8y^2}\left\{\left(1-2\frac{y}{x}\right)[\beta-\beta_0] + \frac{2M_\pi^2}{x}\left[\log\left(\frac{1+\beta_0}{1-\beta_0}\right)-\log\left(\frac{1+\beta}{1-\beta}\right)\right]+2\frac{y}{x}\right\}, \quad (1.4)$$

where

$$\beta_0=\sqrt{1-\frac{4M_\pi^2}{x}} \qquad \text{and} \qquad \beta=\sqrt{1-\frac{4M_\pi^2}{(x-2y)}}. \quad (1.5)$$

The other one–loop function introduced in the text is[168]:

$$F(z)=\frac{5}{18}-\frac{4}{3z}-\frac{1}{3}\left(1-\frac{4}{z}\right)G(z), \quad (1.6)$$

where

$$G(z)=\begin{cases}\sqrt{4/z-1}\arcsin\left(\sqrt{z}/2\right) & z\le 4\\ -\frac{1}{2}\sqrt{1-4/z}\left(\log\dfrac{1-\sqrt{1-4/z}}{1+\sqrt{1-4/z}}+i\pi\right) & z\ge 4\end{cases} \quad (1.7)$$

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