LABORATORI NAZIONALI DI FRASCATI
SIS – Pubblicazioni

LNF–96/001 (IR)
8 Gennaio 1996

Detection of Impulsive, Monochromatic and Stochastic Gravitational Waves with Resonant Antennas

P. Astone
INFN, Sezione di Roma "La Sapienza"

G. V. Pallottino
Dipartimento di Fisica "La Sapienza"
INFN, Sezione di Roma "La Sapienza"

G. Pizzella
Dipartimento di Fisica "Tor Vergata"
Laboratori Nazionali di Frascati dell' INFN

Abstract

The sensitivity of a gravitational wave resonant detector for various types of waves is given in terms of the detector noise spectrum $S_h(\omega)$. It is shown that for g.w. short bursts the sensitivity is determined by $S_h(\omega)/\delta\nu$, where $\delta\nu$ is the detector bandwidth that depends strongly on the electromechanical transducer and associated electronics.

The sensitivity for a g.w. stochastic background depends, at the detector resonance, only on $S_h(\omega)$ that, in turns, for a given material, depends essentially on T/MQ, where T is the bar temperature, M is the bar mass and Q the merit factor. Thus a non sophisticated transducer with a bandwidth of a few Hz appears sufficient for measuring, at a particular frequency, the g.w. background.

It is shown that the very nature of a resonant bar gives a good sensitivity at the resonance.

A bar with M=2300 kg, Q=5 10^6, T=0.1 K and $\delta\nu$=0.9 Hz can detect a g.w. background with amplitude 7 $10^{-23}/\sqrt{\text{Hz}}$. Cross-correlating for one hundred days two identical detectors installed in the same location, the sensitivity improves to 1.3 $10^{-24}/\sqrt{\text{Hz}}$.

PACS.: 0.40.80.+z

1. INTRODUCTION

In this note we express in a simple and unitary form, although sometimemes with approximations only aimed to help clarity, the sensitivity of resonant antennas to various types of gravitational waves. As a matter of fact, in the last years, some of those detectors begun to operate with very satisfactory performance and high duty cycle over relatively long periods of time [1, 2, 3], and more are close to operate [4, 5].

As a model for these detectors, we shall consider the simplest resonant antenna, a cylinder of high Q material, strongly coupled to a non resonant transducer followed by a very low noise electronic amplifier.

In practice, the detectors now operating use resonant transducers (and therefore there are two modes coupled to the gravitational field) to obtain high coupling and high Q, followed by a dc SQUID superconducting amplifier or by a microwave parametric amplifier.

The equation for the end bar displacement ξ is

$$\ddot{\xi} + 2\,\beta_1\dot{\xi} + \omega_o^2 = \frac{f}{m} \qquad (1)$$

where f is the applied force, m the oscillator reduced mass (for a cylinder m = M/2) and $\beta_1=\omega_o/2Q$.

We consider here only the noise which can be easily modeled. The noise of the detector is the sum of two terms : the thermal (Brownian) noise of the basic detector and the electronic noise contributed by the readout system. By referring the overall noise to the displacement of the bar ends, we obtain [6] the power spectrum:

$$S_\xi^n(\omega) = \frac{S_f}{m^2}\;\frac{1 + \Gamma\;[Q^2(1-(\frac{\omega}{\omega_o})^2)^2 + (\frac{\omega}{\omega_o})^2\,]}{(\omega^2 - \omega_o^2)^2 + \frac{\omega^2\,\omega_o^2}{Q^2}} \qquad [\frac{m^2}{Hz}] \qquad (2)$$

$$S_f = \frac{2\,\omega_o}{Q}\,m\,k\,T_e$$

where T_e is the equivalent temperature which includes the effect of the back-action from the electronic amplifier and Γ is the spectral ratio between electronic and brownian noise [7]

$$\Gamma \approx \frac{T_n}{\beta\,Q\,T_e} \qquad (3)$$

T_n is the amplifier noise temperature and β the transducer coupling to the bar ($\beta \approx 10^{-2}$-10^{-3}). The power spectrums are expressed in two-sided form.

When a gravitational wave with amplitude h and optimum polarization impinges perpendicularly to the bar axis, the bar displacement corresponds [8] to the action of a force

$$f = \frac{2}{\pi^2}\,m\,L\,\ddot{h} \qquad (4)$$

The bar end spectral displacement due to a continuous spectrum of g.w. is similar to that due to the action of the Brownian force. Therefore, if only the Brownian noise were present, we would have an infinite bandwidth, in terms of signal to noise ratio (SNR).

2. THE POWER SPECTRUMS

For a g.w. excitation with power spectrum $S_h(\omega)$, the spectrum of the corresponding bar end displacement is

$$S_\xi(\omega) = \frac{4\, L^2\, \omega^4\, S_h}{\pi^4} \frac{1}{(\omega^2 - \omega_o^2)^2 + \frac{\omega^2 \omega_o^2}{Q^2}} \qquad [\frac{m^2}{Hz}] \tag{5}$$

We can then write the SNR

$$SNR = \frac{S_\xi}{S_\xi^n} = \frac{4\, L^2\, \omega^4\, S_h}{\pi^4 \frac{S_f}{m^2}} \frac{1}{1 + \Gamma\, [Q^2(1-(\frac{\omega}{\omega_o})^2)^2 + (\frac{\omega}{\omega_o})^2]} \tag{6}$$

The g.w. spectrum that can be detected with SNR=1 (that is the detector noise spectrum referred to the input) is obtained by introducing this condition in the above:

$$S_h = kT_e \frac{L^2\, \omega}{v^4\, MQ} (\frac{\omega_o}{\omega})^4 \{1 + \Gamma\, [Q^2(1-(\frac{\omega}{\omega_o})^2)^2 + (\frac{\omega}{\omega_o})^2]\} \tag{7}$$

where v is the sound velocity in the bar material (v=5400 m/s in aluminum).

We remark the the best spectral sensitivity, obtained at the resonance frequency of the detector, only depends, according to eq. (7), on the temperature T, on the mass M and on the quality factor Q of the detector, provided $T=T_e$, that is the coupling between bar and read-out system is sufficiently small. Note that those conditions are rather different from that required for optimum pulse sensitivity (see later).

In fig. 1 we plot the above function for the case of the Nautilus antenna, as we plan to have in the near future.

The bandwidth, in this case, estimated from the figure at half-height of the power spectrum, is Δv=0.9 Hz, as can be also calculated [6] with the formula

$$\Delta\nu = \frac{\nu_o}{Q} \frac{1}{\sqrt{\Gamma}} \tag{8}$$

It is expected that the bandwidth would become of the order of 10 Hz by improving the amplifier noise temperature T_n from 20 μK to 0.5 μK

$$\{\ \Gamma \approx \frac{5\ 10^{-7}\ K}{10^{-2}\ \ 5\ 10^{6}\ \ 0.1\ \ K} = 10^{-10}\ \}$$

We come now into applying the Wiener optimum filter for detecting small signals in the noise. It can be shown [9] that the SNR for a gravitational wave signal h(t) whose Fourier transform we indicate with $H(\omega)$ is given by

$$\mathrm{SNR} = \int_{-\infty}^{\infty} \frac{|H(\nu)|^2}{S_h(\nu)} d\nu \tag{9}$$

with $S_h(\omega)$ given by (7).

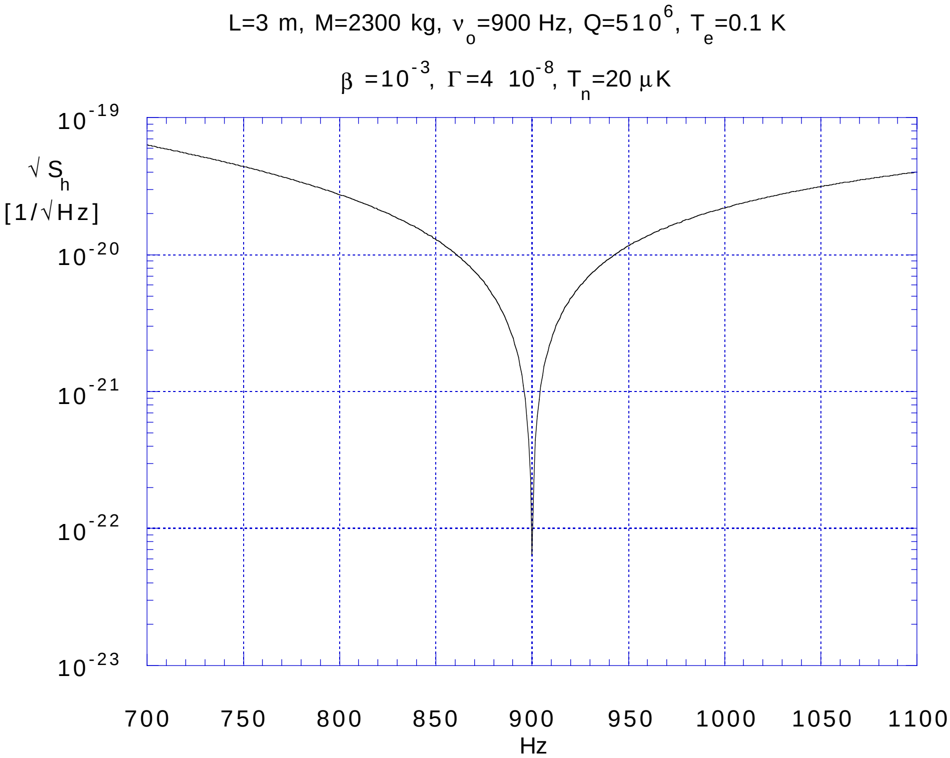

Figura 1 – Sensitivity of Nautilus. The advanced Nautilus sensitivity will have a larger frequency bandwidth.

3. THORNE DEFINITIONS [10]

Thorne defines, for broad band detectors, the following characteristic frequency of the g.w.

$$\nu_c = \frac{\int_0^{\infty} \nu \frac{|H(\nu)|^2}{S_h(\nu)} d\nu}{\int_0^{\infty} \frac{|H(\nu)|^2}{S_h(\nu)} d\nu} \tag{10}$$

and a characteristic strength :

$$h_c = \sqrt{[S_h(\nu_c)\, \nu_c]} \tag{11}$$

Thus

$$h_c^2 = \int_0^\infty \frac{S_h(\nu_c)}{S_h(\nu)} \; |H(\nu)|^2 \; \nu \; d\nu \tag{12}$$

as can be seen by putting in (9) and (10) SNR=1.

4. BURSTS

4.1 Resonant detectors

We solve (9) with SNR=1 by noticing that S_h has a minimum around the resonance (see fig. 1) and that for a short burst $H(\omega)$=constant=H_o [11]. From (9) we obtain

$$H_o^2 = \frac{S_h(\omega_o)}{\pi \, \Delta\nu} \tag{13}$$

with $\Delta\nu$ given by (8). The factor of π has been introduced because we need the equivalent frequency band-width for a bilateral H_o.

Introducing (7) we get

$$H_o = \sqrt{kT_e \frac{L^2 \, \omega}{v^4 \, M \, Q} \frac{Q\sqrt{\Gamma}}{2 \, \pi \, \nu_o}} = \frac{L}{v^2} \sqrt{\frac{k \, T_{eff}}{M}} \tag{14}$$

where

$$T_{eff} = 4 \; T_e \sqrt{\Gamma} \tag{15}$$

if the effective temperature [12]. Formula (14) is just what obtained by the Rome group in previous papers [1,4].

With the values given in fig.1 we get T_{eff}=0.8 mK.

With the antenna Explorer [1] we have already reached values of the order of 5 mK (with T_e = 2.5 K) in close agreement with our expectation.

4.2 Interferometers

Formula (13) is still valid with $2\pi\Delta\nu \approx \nu_o = \nu_c$ as can be seen from (9)

$$SNR = \frac{H_o^2}{S_h} \nu_o$$

We have roughly for SNR=1

$$h_c = H \, \nu_o = \sqrt{(S_h \, \nu_o)} \tag{16}$$

like (11), which is the widely used formula. We have to remark that the definition (11) is not consistent with the case of a resonant detector.

5. MONOCHROMATIC WAVES

5.1 Resonant detectors

For a total measuring time t_m we could detect, with SNR=1, a monochromatic g.w. with strength [6]

$$h = \sqrt{\frac{2\, S_h}{t_m}} = \sqrt{\frac{2\pi^2\, k\, T_e}{M\, Q\, v^2\, \omega_o\, t_m}} \qquad (17)$$

The first equality is valid for all frequencies, the second one only at the resonance. The factor of 2 takes care of the fact that S_h is two-sided.

This formula can be derived also from (9). For a total measuring time t_m the monochromatic wave is just like a wave packet of duration t_m, whose Fourier transform has maximum $h_o t_m/2$. Thus from (9) we get

$$SNR = \frac{h_o^2\, t_m^2/4}{S_h(\omega_o)}\, \frac{2}{t_m} \qquad (18)$$

which gives (17) for SNR=1.

5.2 Interferometers

The first equality of formula (17) holds also for interferometers.

5.3 The case of splitting the total period tm in several sub periods

In the practical case it is often not possible to calculate the Fourier spectrum of the experimental data over the entire period of measurement t_m, either because the number of steps in the spectrum would be to large for a computer or because the physical conditions change like, for instance, the change in frequency due to the Doppler effect for a monochromatic wave. It is then necessary to divide the period t_m in n several sub periods of length $\Delta t = t_m/n$. For the search of a monochromatic wave we have then to consider two cases:

a) The wave frequency is exactly known. In this case we can combine the n Fourier spectra in one unique spectrum taking into consideration also the phase of the signal. The final spectrum has then the same characteristics of the spectrum over the entire period t_m and then formula (17) still applies.

b) The exact frequency is unknown. In this case when we combine the n spectra we lose the information on the phase. The result is that the final combined spectrum over the entire period has a larger variance. In this case the left part of formula (17) has to be changed in

$$h = \sqrt{\frac{2\, S_h}{\sqrt{\Delta t\; t_m}}} = \sqrt{\frac{2\, S_h}{t_m}\sqrt{n}} \qquad (19)$$

as can be understood with the aid of formula (21) in the next section.

6. STOCHASTIC WAVES [13]

Using one detector, the measurement of the noise spectrum corresponding to eq. (7) (see also fig.1) only provides an upper limit for the g.w. stochastic background spectrum. The estimation of this spectrum can be considerably improved by employing two (or more) antennas, whose output signals are crosscorrelated.

Let us consider two antennas, that may in general be different, installed in very close locations with transfer functions T_1 and T_2, and displacements ξ_1 and ξ_2: the displacement crosscorrelation function

$$R_{\xi\xi}(\tau) = \int \xi_1(t)\,\xi_2(t+\tau)\,dt \qquad (20)$$

only depends on the common excitation of the detectors, as due to the g.w. stochastic background spectrum S_{gw} acting on both of them, and is not affected by the noises acting independently on the two detectors. Note that the above result only holds if the crosscorrelation function is evaluated over an infinite time. Otherwise there is a residual statistical error, due to the noise, whose amount decreases with the duration of the observation period.

The Fourier transform of eq. (20) represents the displacement cross spectrum. This spectrum can be expressed as an estimate of the gravitational background S_{gw} by multiplying it by T_1T_2 times $4L^2/\pi^4$, i.e. referring it to the detector input. The estimate, as obtained over a finite observation time t_m, has a statistical error. More precisely, it can be shown [14] that the standard deviation of each sample of the spectrum is:

$$\delta S_{gw}(\omega) = \frac{\sqrt{S_{1h}(\omega)\ S_{2h}(\omega)}}{\sqrt{t_m \delta\nu}} \qquad (21)$$

where t_m is the total measuring time and $\delta\nu$ is the frequency step in the power spectrum. From fig.1 we get the obvious result that, for resonant detectors, the error is smaller at the resonances. If the resonances of the two detectors coincide the error is even smaller. In practice, the best is to have two detectors with the same resonance and bandwidth. If one bandwidth is smaller that the other one than the smallest error occurs in a frequency region overlapping the smallest bandwith.

Note, however, that according to eq. (21) there is no improvement, besides an obvious increase of confidence, by using two detectors instead of one, when the frequency step of the spectrum $\delta\nu$ is chosen equal to $1/t_m$. In this case the statistical improvement factor $\sqrt{(\delta\nu\ t_m)}$ reduces in fact to unity and the sensitivity, for two identical detectors, coincides with that of a single detector, given by (7).

Note, in addition, that one should try to exploit all the a priori information available in order to improve the sensitivity of the experiment. If the background spectrum is expected [13] to be approximately constant over a few hertz or a few tens of hertz near the resonances of the detector, we can shift our attention from a detailed, and statistically expensive, spectral estimation to estimating its intensity over a spectral interval $\Delta\nu$ much larger than the spectral step $\delta\nu$, properly chosen in the region of maximum sensitivity of the detectors, as discussed above. The uncertainty of this estimate is obtained as follows from eq. (21):

$$\delta S_{gw} = \frac{\sqrt{S_{1h}\ S_{2h}}}{\sqrt{t_m\,\Delta\nu}} \qquad (22)$$

where $\Delta\nu$ is the smallest of the two overlapping bandwidths.

For the search of a stochastic background, however, one expects at first just to put upper limits. In this case the estimated spectrum S_{gw} will be zero with a deviation given by (21). And the overall sensitivity of this crosscorrelation experiment, considering an observation bandwidth $\Delta\nu$, will be again given by eq. (22).

We want now to discuss the result obtained in reference [15] when the two detectors have different bandwidths.

In [15] the authors consider a cross-correlation experiment between a bar and an interferometer, thus $\Delta\nu$ in (22) is the bandwidth of the bar. They conclude that the sensitivity of this experiment to stochastic g.w. is independent from the bar bandwidth. The argument is based on the use of the antenna sensitivity to bursts. Let us suppose that a burst has duration τ_g. Therefore, putting $h \approx H_o/\tau_g$, from (13) and (15) we get

$$h_{bar} = \frac{1}{\tau_g}\sqrt{\frac{S_{h,bar}}{2\pi\Delta\nu}}$$

$$h_{inter} = \frac{1}{\tau_g}\sqrt{S_{h,inter}\ \nu_c} \tag{23}$$

Substituting in (22) we obtain

$$\delta S_{gw}(\omega) = \sqrt{\frac{\tau_g^2\ h_{bar}^2\ 2\pi\ \Delta\nu\ \tau_g^2\ h_{inter}^2/\nu_c}{t_m\ \Delta\nu}} \tag{24}$$

independent on $\Delta\nu$ (we might obtain the same result using directly the Fourier transforms $H(\omega)$). In [15] the authors conclude that it is better to correlate an interferometer with a more sensitive bar instead than with another less sensitive interferometer. But it has to be remarked that $\Delta\nu$ is implicit in (24), because a larger value of $\Delta\nu$, obtained with a smaller Γ (that is a smaller elctronic noise) gives a smaller value for h_{bar}.

This result suggests that, for exploring a frequency region around 1 kHz over a band of the order of 10 to 50 Hz, it might be more convenient, in terms of expenses and reliability, to put efforts in improving the sensitivity of a bar, which can be easily installed and oriented at will, instead than constructing a second interferometer.

7. DISCUSSION AND CONCLUSIONS

In the past literature for resonant detectors of gravitational waves the sensitivity has been usually expressed in terms of T_{eff}, which is the minimum energy delivered by a g.w. burst that can be detected by the apparatus.

What really the resonant detectors measure is essentially the Fourier transform (over a certain frequency band) of the g.w. adimensional amplitude, as given in formula (14). This sensitivity depends on both the bar and the transducer with its associated electronics. For monochromatic waves the sensitivity is calculated at the detector resonance and is usually expressed in terms of the adimensional h, the minimum g.w. amplitude that can be detected.

For studying the operation of a resonant antenna as detector for a g.w. stochastic background we had to deal with noise spectrums. This has brought us to reconsider the sensitivity to bursts and other types of g.w. in a somewhat different manner, that improves our understanding of the role played by the electromechanical transducer and its associated electronics.

The noise spectrum of the apparatus is expressed by formula (7), that also gives directly the sensitivity for the g.w. background. We notice that the optimum sensitivity, S_{opt}, is obtained at the resonance. S_{opt} depends essentially on the ratio T_e/MQ, for a given material. The transducer and electronics determine in practice only the bandwidth of the apparatus, expressed by (8). Their effect at resonance is wiped out except for the influence on the quantity T_e, which is the thermodynamic temperature of the antenna plus the backaction from the transducer. It turns out that the backaction effect, when using a dcSQUID amplifier, can be neglected or at most introduces a small correction, as it is of the order of less than 10 mK.

For the measurement of a g.w. background that, reasonably, should not change drastically in a frequency band of a few Hz it might be therefore sufficient to make use of a very simple transducer and electronics with a small bandwidth. The use of a sophisticated transducer can give a larger bandwidth with a better sensitivity to g.w. bursts, as illustrated by formula (13). As far as the search for monochromatic waves a larger bandwidth is better, in the sense that allows a larger frequency region where to search.

Finally we estimate the sensitivity that can be obtained with two identical antennas, each one having the sensitivity shown in fig.1, operating continuously in the same location for one hundred days. We get from (22) at the resonance

$$= = \qquad (25)$$

This sensitivity is attainable with the present resonant detectors. Increasing the bandwidth to 10 Hz would improve the above value only by a factor of 1.8.

We would like to conclude by remarking that the resonant detectors with the smallest possible value for T/MQ seem to be the most suited for measuring the g.w. background, as the bandwidth has, to some extend (power of 1/4 against the power of 1/2 for the bursts), a minor role and S_{opt} turns out to have, by the resonance own nature, very small values without special experimental efforts.

ACKNOWLEDGMENTS

We thank Massimo Cerdonio and Eugenio Coccia for useful and stimulating discussions.

REFERENCES

1. P.Astone, M.Bassan, P.Bonifazi, P.Carelli, M.G.Castellano, G.Cavallari, E.Coccia, C.Cosmelli, V.Fafone, S.Frasca, E.Majorana, I.Modena, G.V.Pallottino, G.Pizzella, P.Rapagnani, F.Ricci, M.Visco, "Long-term operation of the Rome Explorer cryogenic gravitational wave detector". Physical Review D, 47, 2, January (1993).
2. Z.K.Geng, W.O.Hamilton, W.W.Johnson, E.Mauceli, S.Merkowitz, A.Morse, N.Slomonson, "Operation of the Allegro detector at LSU", First Edoardo Amaldi Conference on Gravitational wave Experiments, Frascati, 14-17 June 1994.
3. D.G.Blair, I.S.Heng, E.N.Ivanov, F.van Kann, N.P.Linthorne, M.E.Tobar, P.J.Turner, "Operation of the Perth cryogenic resonant-bar gravitational wave detector", First Edoardo Amaldi Conference on Gravitational wave Experiments, Frascati, 14-17 June 1994.
4. P.Astone, M.Bassan, P.Bonifazi, M.G.Castellano, P.Carelli, E.Coccia, C.Cosmelli, S.Frasca, V.Fafone, E.Majorana, A.Marini, G.Mazzitelli, I.Modena, G.Modestino, G.V.Pallottino, M.A.Papa, G.Pizzella, P.Puppo, P.Rapagnani, F.Ricci, F.Ronga, R.Terenzi, M.Visco, L.Votano, "The Nautilus experiment", First Edoardo Amaldi Conference on Gravitational wave Experiments, Frascati, 14-17 June 1994.
5. M.Cerdonio, L.Franceschini, G.Fontana, R.Mezzena, S.Paoli, G.A.Prodi, S.Vitale, J.P.Zendri, M.Biasotto, M.Lollo, F.Bronzini, R.Macchietto, G.Maron, A.Ortolan, M.Strollo, G.Vedovato, M.Bonaldi, P.Falferi, E.Cavallini, P.L.Fortini, E.Montanari, L.Taffarello, A.Colombo, D.Pascoli, B.Tiveron, "Status of the Auriga gravitational wave antenna", First Edoardo Amaldi Conference on Gravitational wave Experiments, Frascati, 14-17 June 1994.
6. G.V.Pallottino, G.Pizzella, "Sensitivity of a Weber type resonant antenna to monochromatic gravitational waves". Il Nuovo Cimento, 7C, 155 (1984).
7. G.V.Pallottino, G.Pizzella, "Matching of Transducers to Resonant Gravitational-Wave Antennas". Il Nuovo Cimento, may-june 1981, vol.4C, pp. 237-283.
8. G.Pizzella, "Gravitational-Radiation experiments". Il Nuovo Cimento 5, 369 (1975).
9. A.Papoulis, "Probability, Random Variables, and Stochastic Processes", McGraw-Hill, Singapore (1984), pag 298-300.
10. K.S.Thorne, in 300 years of Gravitation, S.W.Hawking and W.Israel, Eds. (Cambridge Univ. Press, Cambridge, 1987).
11. If we want take into account the possibility that the direction and polarization are not optimum, then we should average and take a SNR= 15/4. An additional [12] factor of the order of 20 is needed if we desire to detect, with a sufficient probability not to be an accident, one burst event in one year of antenna operation.
12. G.Pizzella, "Optimum filtering and sensitivity for resonant gravitational wave antennas". Il Nuovo Cimento 2C, 209 (1979).
13. The interest for stochastic g.w. in the kHz region has been recently aroused by: R.Brustein, M.Gasperini, M.Giovannini and G.Veneziano, "Relic Gravitational Waves from String Cosmology", CERN-TH/95-144, June 1995.
14. J.S.Bendat and A.G.Piersol, "Measurement and analysis of random data", John Wiley & Sons, New York, 1966.
15. P.Astone, J.A.Lobo and B.F.Schutz, "Coincidence experiments between interferometric and resonant detectors of gravitational waves", Class.Quantum Grav. 11 (1994), 2093-2112.