On the Extension of the Fermi–Watson Theorem to High Energy Diffraction

A. MAŁECKI*

K E N Pedagogical University**, Cracow, Poland

and Laboratori Nazionali di Frascati dell' INFN, Frascati, Italy

Abstract

The Fermi-Watson theorem, established for low energy reactions and then applied to high energy collisions, is revisited. Its use for the processes of inelastic diffraction is discussed. The theorem turns out to be valid in the case of inclusive cross-section of diffractive transitions.

PACS.: 13.85.–t; 03.80.+r; 11.80.–m

---

* On leave of absence from H. Niewodniczański Institute of Nuclear Physics, Kraków.
** Address: Institute of Physics and Computer Science, ul. Podchorążych 2, 30-084 Kraków, Poland.
E-mail: AMALECKI@VSB01.IFJ.EDU.PL and MALECKI@LNF.INFN.IT

The Fermi-Watson theorem allows to express the inelastic transition amplitude in terms of the amplitudes of elastic scattering in the initial and final states. It was established in the fifties [1],[2], concerning reactions at low energies in which the exit channels, being close to their thresholds, are weakly coupled to the initial state [3]. The theorem had been rediscovered in another form in the seventies [4] and applied to inelastic diffraction of high energy particles [5],[6] and to high energy break-up of light nuclei [7],[8].

The representation of the inelastic diffractive amplitude [4] as a *difference* between elastic amplitudes for the entrance and exit channels became a part of the folklore of high energy physics [7]. It is most easily understood in terms of the 'eigenstates of diffraction' [9] which do not mix with each other, undergoing only the elastic scattering caused by their absorption at the expense of inelastic channels. High energy inelastic diffraction arises thus from the differences in absorption probabilities for various components of the wave function of the incident particle. The passage to the Fermi-Watson theorem requires, however, an additional assumption that elastic scattering amplitudes of diffractive eigenstates coincide to a good approximation with those of real physical states. Despite of the experimental fact that the total cross-sections for inelastic diffraction are much smaller than the elastic ones, the above assumption may be put in doubt by invoking the very sense of elastic diffraction as a unitarity-driven shadow scattering or a feed-back process coupled to the inelastic channels. In fact, it would be curious that at high energies, when all channels are open, the intermediate virtual transitions were hidden .

In this note we discuss how to account for the virtual transitions inside the set of diffractive states. It turns out that this diffractive contribution to the shadow of all inelastic transitions affects strongly the elastic amplitude [10] around and above its dip where the value of the cross-section is many orders of magnitudes lower with respect to the forward peak. Thus the violation of the Fermi-Watson theorem for *exlusive* diffractive amplitudes may generally be expected only at large momentum transfers. On the contrary, in the case of *inclusive* cross-section of inelastic diffraction it is just the Fermi-Watson term which saturates the majority of the inelastic diffractive cross-section.

We derive the Fermi-Watson theorem after an elementary recapitulation of the scattering formalism. The complete knowledge of a collision process, including all possible connections among channels allowed by conservation laws, is contained in the *scattering operator* S which is *unitary*: $SS^\dagger = S^\dagger S = 1$. Alternatively one may use the *transition operator* T, where $S \equiv 1 + iT$, which is *normal* and satisfies the relation: $TT^\dagger = T^\dagger T = i(T^\dagger - T)$. The amplitude of transition from the initial state $|i\rangle$ to the final state $|f\rangle$ is then $T_{fi} \equiv \langle f|T|i\rangle$. Dynamics of scattering will not change under a simultaneous *unitary transformation* of the physical states $|j\rangle \rightarrow |Uj\rangle$ $(UU^\dagger = U^\dagger U = 1)$ and of the transition operator $T \rightarrow T_0 = U^\dagger TU$. Such a transformation may be viewed as the choice of a *convenient basis* in the Hilbert space of states. Any physical state can be expanded in terms of these base states, e.g. the

initial state $| i \rangle = \sum_{|j\rangle} U_{ij}^{\star} | Uj \rangle$ where $U_{ij} \equiv \langle i | U | j \rangle$. The transition amplitude can thus be expressed through the matrix elements of the transformation operator U and of the transition operators T or T_0:

$$T_{fi} \equiv \langle f | T | i \rangle = \sum_{|j\rangle,|k\rangle} U_{fk} U_{ij}^{\star} t_{kj} \tag{1}$$

where $t_{kj} \equiv \langle Uk | T | Uj \rangle = \langle k | T_0 | j \rangle$.

The unitary transformation operator may alternatively be written as:

$$U \equiv e^{iM} \equiv 1 - \Lambda \tag{2}$$

where the operator M *is hermitian* $(M = M^{\dagger})$ while the *normal operator* Λ satisfies the relation:

$$\Lambda\Lambda^{\dagger} = \Lambda^{\dagger}\Lambda = \Lambda + \Lambda^{\dagger}. \tag{3}$$

In frequent cases of physical interest the part of the T-matrix, corresponding to a given entrance channel, is *nearly diagonal*:

$$T = T_0 + i\epsilon \tag{4}$$

where T_0 is *diagonal* and the matrix elements of the operator ϵ are *all small*. Such a situation is typical for reactions at relatively *low energy* as compared to the thresholds for the most important inelastic channels; e.g. the Ansatz (4) was used by Fermi [1] in the case of pion production by nucleons.

The above decomposition of the transition operator in 'large' (or 'hard') and 'small' (or 'soft') parts can also be viewed as the result of a suitable unitary transformation which would *diagonalize* operator T subject to condition the transforming operators M or Λ in (2) were 'soft', i.e. their matrix elements were small. While the diagonalization of the operator T through a unitary transformation is always possible, the satisfaction of *both* unitarity and softness (which means $M^2 = \Lambda^2 = 0$) conditions imposes a severe constraint on the transforming operators: $\Lambda = -iM$ *and* $\Lambda = -\Lambda^{\dagger}$ instead of the less stringent relation (3).

In the case $U = e^{iM} \approx 1 + iM$ we obtain the following 'soft' limit of the Haussdorf expansion in terms of multiple commutators of the operators M and T_0:

$$T = e^{iM} T_0 e^{-iM} = T_0 + \sum_{n=1}^{\infty} [M, \ldots, [M, T_0], \ldots,]\frac{i^n}{n!} \approx T_0 + i[M, T_0]. \tag{5}$$

Alternatively, when $U = 1 - \Lambda$ we have :

$$T = (1 - \Lambda)T_0(1 - \Lambda^{\dagger}) = T_0 - \Lambda T_0 - T_0\Lambda^{\dagger} + \Lambda T_0\Lambda^{\dagger} \approx T_0 - [\Lambda, T_0]. \tag{6}$$

Thus the 'soft' non-diagonal operator in (4) reads: $\epsilon = [M, T_0] = i[\Lambda, T_0]$. The antisymmetry of the commutator together with symmetry under time reversal means

that the matrix elements of both M and Λ are *antisymmetric* which implies $ReM_{jk} = Im\Lambda_{jk} = 0$. From Eqs (5) and (6), on the account of diagonality of T_0, one obtains the amplitude of transition:

$$T_{fi} = t_i\delta_{fi} - iM_{fi}(t_f - t_i) = t_i\delta_{fi} + \Lambda_{fi}(t_f - t_i)$$

$$= T_{ii}\delta_{fi} - iM_{fi}(T_{ff} - T_{ii}) = T_{ii}\delta_{fi} + \Lambda_{fi}(T_{ff} - T_{ii}) \tag{7}$$

where $T_{jj} = t_j \equiv \langle j \mid T_0 \mid j \rangle$ is the amplitude of elastic scattering in the state $|j\rangle$, t_j being the eigenvalue of the diagonal operator T_0. Eq. (7) contains the essence of the *Fermi-Watson theorem* . Denoting $T_{jj} \equiv exp(2i\alpha_j)$, α_j being the conventional scattering phase-shift, Eq. (7) can be transformed to the *'product form'* as in the original paper [1]:

$$T_{f \neq i} = i\rho_{fi}\,exp[i(\alpha_i + \alpha_f)], \tag{8}$$

where $\rho_{jk} = 2sin(\alpha_j - \alpha_k)\Lambda_{jk}$ is a real symmetric matrix. In one of its application Eq. (8) allows to relate the phases of pion photoproduction on nucleons to the phase-shifts of the elastic pion-nucleon scattering [3].

Turning now to diffractive processes let us first relax the assumption of 'softness' and then specify precisely that of diagonalization as used in Ref. [4]. The transformed transition operator T_0 is assumed there to be diagonal in a particular class of physical states $[D]$. The states belonging to [D] are called diffractive states and those from its orthogonal complement $[\sim D]$ are referred to as non-diffractive. Thus for any state $| j \rangle \in [D]$ one has :

$$T_0 \mid j \rangle = t_j \mid j \rangle + \sum_{|k\rangle \in [\sim D]} t_{kj} \mid k \rangle. \tag{9}$$

Eq. (9) can be interpreted as the requirement that the base states of the diffractive sector are subject only to elastic scattering which arises from absorption related to the production of non-diffractive states.

If the transition takes place between two diffractive states, i.e. when $| i \rangle \in [D]$ and $| f \rangle \in [D]$, one has then on account of Eqs (1) and (9):

$$T_{fi} = \sum_{|j\rangle \in [D]} U_{fj}\,U_{ij}^{\star}\,t_j. \tag{10}$$

Using the Ansatz (2) the diffractive transition amplitude (10) is

$$T_{fi} = t_i\delta_{fi} - \Lambda_{fi}\,t_i - \Lambda_{if}^{\star}\,t_f + \sum_{|j\rangle \in [D]} \Lambda_{fj}\,t_j\Lambda_{ij}^{\star}$$

or equivalently

$$T_{fi} = t_i\delta_{fi} + \frac{1}{2}(\Lambda_{fi} - \Lambda_{if}^{\star})\,(t_f - t_i) + \sum_{|j\rangle \in [D]} \Lambda_{fj}\,\Lambda_{ij}^{\star}\,[t_j - \frac{1}{2}(t_f + t_i)]. \tag{11}$$

In particular, the elastic scattering amplitude reads

$$T_{ii} = t_i + \sum_{|j\rangle\in[D]} |\Lambda_{ij}|^2 (t_j - t_i). \tag{12}$$

If all Λ_{kj} were small then retaining only the terms linear in Λ (violating thus the relation (3) resulting from unitarity) one would yield the elastic scattering amplitude equal to the eigenvalue of T_0 in the initial state. Instead, the inelastic diffraction amplitude would be proportional to the difference of these eigenvalues in the initial and final state , which is just the *'difference form'* of the Fermi-Watson theorem (7), as given in Ref.[4].

The last terms of Eqs. (11) and (12) can be considered as the unitarity corrections from the intermediate diffractive states. In fact, the rôle of the virtual transitions inside the set of diffractive states is much greater. In our approach to diffraction [12] we reject the condition (9) regarding the diagonalization of the transition operator. First, this is a redundant assumption since the required division of channels into the two classes may be formulated otherwise. Secondly, considering the most general expression:

$$T_0 \mid j\rangle = \sum_{|k\rangle\in[D]} t_{kj} \mid k\rangle + \sum_{|k\rangle\in[\sim D]} t_{kj} \mid k\rangle \tag{13}$$

one reveals there, besides the absorption of non-diffractive origin, also another source of absorption implied by transitions inside the set of diffractive states.

The fundamental point in the description of diffraction is the presumed existence of two orthogonal subspaces of diffractive and non-diffractive states. This requirement can be rephrased by saying that there exist unitary operators U and $U^\dagger$ which are reducible in the Hilbert space of physical states. This implies the existence of a non-trivial subspace $[D]$ such that for any $|j\rangle \in [D]$ also $|Uj\rangle$ and $|U^\dagger j\rangle$ belong to $[D]$. In consequence, for any state $|k\rangle$ belonging to the orthogonal complement $[\sim D]$ also $|Uk\rangle$ and $|U^\dagger k\rangle$ will belong to $[\sim D]$. In terms of the matrix elements this reads:

$$\langle k|Uj\rangle = \langle k|U^\dagger j\rangle = 0 \tag{14}$$

for any $|j\rangle \in [D]$ and $|k\rangle \in [\sim D]$. A careful inspection of the passage from Eqs. (1) and (9) to Eq. (10) indeed reveals that the above relations of orthogonality were implicitly assumed.

The states $|Uj\rangle$ obtained through the unitary transformation (14) constitute a natural base for the description of diffraction. The amplitude of diffractive transitions follows then directly from Eq. (1) by restricting the summation over states to the class $[D]$ of diffractive states. Using $U = 1 - \Lambda$ one obtains :

$$T_{fi} = t_{fi}\delta_{fi} - \sum_{|k\rangle\in[D]} \Lambda_{fk} t_{ki} - \sum_{|j\rangle\in[D]} t_{fj}\Lambda_{ij}^\star + \sum_{|j\rangle,|k\rangle\in[D]} \Lambda_{fk} t_{kj}\Lambda_{ij}^\star. \tag{15}$$

The three last terms of (15) can be rewritten as follows:

$$\sum_{|k\rangle\in[D]} \Lambda_{fk}\,t_{ki} = N_{fi}(T_0)\Lambda_{fi}t_i$$

$$\sum_{|j\rangle\in[D]} t_{fj}\Lambda_{ij}^\star = \Lambda_{ij}^\star t_f N_{ij}^\star(T_0^\dagger)$$

$$\sum_{|j\rangle,|k\rangle\in[D]} \Lambda_{fk}t_{kj}\Lambda_{ij}^\star = \sum_{|j\rangle\in[D]} N_{fj}(T_0)\Lambda_{fj}t_j\Lambda_{ij}^\star$$

$$= \sum_{|j\rangle\in[D]} \Lambda_{fj}t_j\Lambda_{ij}^\star N_{ij}^\star(T_0^\dagger) \qquad (16)$$

where t_j are the diagonal matrix elements of T_0 :

$$t_j \equiv t_{jj} = \langle j\mid T_0\mid j\rangle \qquad (17)$$

and

$$N_{kj}(T_0) \equiv \frac{1}{\Lambda_{kj}\langle j\mid T_0\mid j\rangle} \sum_{|l\rangle\in[D]} \Lambda_{kl}\langle l\mid T_0\mid j\rangle. \qquad (18)$$

In order to estimate the undimensional quantities N_{kj} we rewrite (18) in the form:

$$\frac{1}{N_{kj}} = 1 - \sum_{|l\rangle\neq|j\rangle\in[D]} \Lambda_{kl}t_{lj}\Big[\sum_{|l\rangle\in[D]} \Lambda_{kl}t_{lj}\Big]^{-1}. \qquad (19)$$

If the class of diffractive states $[D]$ contains a huge number of states, then the second term in (19) will approach unity. This means $N_{kj} \equiv N \to \infty$ for any pair of states $|k\rangle$ and $|j\rangle$.

We have thus discovered the *infinite dimension* of the diffractive subspace which was hidden inside the quantities N_{kj}. This leads to an enormous simplification of Eq.(15) :

$$T_{fi} = t_i\delta_{fi} - N\Big(\Lambda_{fi}t_i + \Lambda_{ij}^\star t_f - \sum_{|j\rangle\in[D]} \Lambda_{fj}t_j\Lambda_{ij}^\star\Big) \qquad (20)$$

which is almost identical, except for the factor N, with the 'diagonal' Eq. (11). But this difference turns out to be essential. In other words, the effect of non-diagonal transitions can be factorized: the diffractive transition amplitude has the form $N\Delta t$ where Δt represents a diversity of diagonal matrix elements of the transition operator over the set of diffractive states. Such expressions are to be considered in the *'diffractive limit'* [11]: $N \to \infty, \Delta t \to 0$ under requirement that $N\Delta t$ is finite. Diffraction thus arises as *infinite sum* of the *infinitesimal* contributions from all intermediate states belonging to the diffractive sector.

In the case of elastic scattering Eq.(20) becomes:

$$T_{ii} = t_i + N\sum_{|j\rangle\in[D]} |\Lambda_{ij}|^2(t_j - t_i) \qquad (21)$$

which, in contrast to Eq. (12), is to be considered in the 'diffractive limit', assuring the second term in (21) to remain finite. We refer to this term as the diffractive contribution to elastic scattering since it originates from the action of the operator Λ which filters as intermediate states only those diffractive, i.e. equivalent to the initial state. The numerical analysis of the elastic scattering of high energy hadrons reveals the importance of the diffractive term around and above the dip of differential cross-section [10, 12]. By contrast, the first term in (21) which is mostly feed by the shadow of non-diffractive transitions, is dominating at small momentum transfers and negligible above the dip.

Making use of completness of diffractive states in their subspace one may obtain from (20) the inclusive cross-section of inelastic diffraction:

$$\sum_{|f\rangle \neq |i\rangle} |T_{fi}|^2 = N^2 [\sum_{|f\rangle} |\Lambda_{if}|^2 |t_f|^2 - 2(\, Re(\Lambda_{ii})^2 - Re(\Lambda_{ii}) + |\Lambda_{ii}|^2)\, |t_i|^2$$
$$- |\sum_{|f\rangle} |\Lambda_{if}|^2 t_f\, |^2 - 2(1 - 2Re(\Lambda_{ii}))\, Re(t_i \sum_{|f\rangle} |\Lambda_{if}|^2 t_f^*)]. \quad (22)$$

Applying now the identity : $|\,\Lambda_{ii}\,|^2 + Re(\Lambda_{ii})^2 = 2[Re(\Lambda_{ii})]^2$, Eq.(22) can be written in the form:

$$\sum_{|f\rangle \neq |i\rangle} |T_{fi}|^2 = N^2 \sum_{|f\rangle \in [D]} |\Lambda_{if}|^2 |t_f - t_i|^2 - |t_i - T_{ii}|^2. \quad (23)$$

Another way of writing the inclusive cross-section (23) is :

$$\sum_{|f\rangle \neq |i\rangle} |T_{fi}|^2 = N^2 \sum_{|f\rangle \in [D]} |\Lambda_{if}|^2 \Big|t_f - \frac{1}{g_i} \sum_{|j\rangle \in [D]} |\Lambda_{ij}|^2 t_j\Big|^2 + \frac{1 - g_i}{g_i}|t_i - T_{ii}|^2 \quad (24)$$

where $g_i \equiv \sum_{|j\rangle \in [D]} |\Lambda_{ij}|^2 = 2Re(\Lambda_{ii})$ is the normalization constant. The first term of Eq. (24) represents a dispersion of the diagonal matrix elements of T_0 while the second term is proportional to the square of diffractive contribution in the elastic scattering amplitude (21).

Rather a small value of g_i would be expected to reflect the experimental fact that the cross-sections for inelastic diffractive processes are about one order of magnitude smaller than elastic ones. Our analysis [12] of the inclusive inelastic cross-sections obtained at the ISR and SPS colliders [13, 14] confirms this expectation : $g_i < 0.07$. Now by comparing Eqs (23) and (24) one may conclude that the majority of the inelastic diffractive cross-section is contained in the first term of Eq. (23) . But it is just this term which, being suitably corrected for the intermediate virtual transitions, corresponds to the Fermi-Watson approximation (7) of the inelastic transition amplitude.

References

[1] E. Fermi, Nuovo Cimento Suppl. **2** (1955) 17.

[2] K. M. Watson, Phys. Rev. **95** (1954) 228.

[3] B. T. Feld in **Models of Elementary Particles** (Blaisdell, Waltham, 1969), p. 73.

[4] A. Białas, W. Czyż and A. Kotański, Ann. of Phys. **73** (1972) 439.

[5] W. Czyż, Phys. Rev. **D 8** (1973) 3219.

[6] H. Miettinen and J. Pumplin, Phys. Rev. **D 18** (1978) 1696.

[7] J. V. Noble, Phys. Rev. **C 8** (1973) 2508.

[8] A. Foursat, E. Lyovshin and K. Sailer, Nucl. Phys. **A 392** (1983) 399.

[9] M. L. Good and W. D. Walker, Phys. Rev.**120** (1960) 1857.

[10] A. Małecki, Phys. Lett. **B 267** (1991) 523.

[11] E. Etim, A. Małecki and L. Satta, Phys. Lett. **B 184** (1987) 99.

[12] A. Małecki , **LNF-94/075 (1994)**- to appear in Phys.Rev. D.

[13] M. G. Albrow et al., Nucl. Phys. **B 108** (1976) 1; J. C. Armitage et al., Nucl. Phys. **B 194** (1982) 365.

[14] D. Bernard et al., Phys. Lett. **B 186** (1987) 227.