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On the Analytical Solution of the Inverse Diffraction Transform Kernel

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Abstract

In diffraction tomography, optical information processing and, more generally, Fourier optics, the diffraction transform solves both the direct and the inverse boundary value propagation problem for the Helmholtz equation. Its kernel is itself an integral. It is presented an analytical solution.

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I. INTRODUCTION

In classical physics (Helmholtz equation), it is usual to approximate diffraction scattering as a boundary value problem associated with a linear wave equation[1]. According to this picture, a diffractive object (or target) is regarded as a passive geometrical obstacle in the path of the incident wave (or projectile). The incident wave field is specified as a boundary condition over the target profile. Because of the linearity of the wave equation, the scattered field is obtained as an integral transform of the incident field. This is the so - called direct scattering problem. The inverse problem consists in the recovery of the incident field, given the scattered field configuration. This solution provides information about the target profile geometry [2–5]. There are important applications (e.g., diffraction tomography[6], optical information processing and, more generally, Fourier optics[7]) which make extensive use of classical diffraction theory. This theory is, however, saddled with a long - standing, apparently unyielding, divergence problem [8,9]. In a previous paper[10] we solved the divergence with Cesàro summability. But it is possible to give an analytical solution of eq.(13)[10].

II. THE DIFFRACTION TRANSFORM KERNEL

Consider the Helmholtz equation for a scalar field $\varphi(\vec{x})$

$$(\nabla^2 + k^2)\varphi(\vec{x}) = 0 \tag{1}$$

where

$$\nabla^2 \equiv \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} \tag{2}$$

is the Laplace operator in 3 - space with coordinate vector $\vec{x}$ and $k^2 \geq 0$ is a positive parameter. In what follows it will be convenient to view the space spanned by the vector $\vec{x}$ in terms of two - dimensional plane slices orthogonal to the third or z - axis. A point on each such slice is described by the position z of the plane along the z - axis and a 2 - vector coordinate $\vec{b}$ (the impact parameter) on the plane . Thus, the 3 - vector $\vec{x}$ is decomposed into $\vec{x} \equiv (\vec{b}, z)$ and will be so understood throughout the rest of this paper. eq. (1) is to be solved with the boundary condition

$$\varphi(b, z = z_0) = \varphi_0(b). \tag{3}$$

It is well - known that one arrives at the homogeneous integral equation

$$\varphi_\pm(b, z) = \int d^2b'\; G_\pm(b, z|b', z')\; \varphi_\pm(b', z') \tag{4}$$

where

$$G_\pm(b, z|b', z') = \frac{1}{(2\pi)^2} \int d^2q\; e^{\pm i\vec{q}(\vec{b}-\vec{b}')} \cdot \{\theta(k^2 - q^2)\; e^{\pm ip_1(z-z')} + \theta(q^2 - p^2)\; e^{\mp p_2(z-z')}\} \tag{5}$$

and

$$p_1(q) := \sqrt{k^2 - q^2} \quad p^2(q) \geq 0 \tag{6}$$

$$p_2(q) := \sqrt{q^2 - k^2} \quad p^2(q) < 0 \tag{7}$$

To arrive at eq. (4) we have interchanged the order of the $\vec{b}$ - integration. This interchange is usually (Sherman[8] ; Shewell and Wolf[9]) held responsible for the divergence in $G_\pm(b, z|b', z')$. If the integral representation in eq. (5) is regarded as a Cauchy integral, then the forward (backward) propagator $G_+(b, z|b', z')$ ($G_-(b, z|b', z')$) is convergent (divergent) for $z < z'$. If, on the other hand, this integral representation is interpreted as a Cesàro summable integral, then $G_\pm(b, z|b', z')$ exists and is finite for all finite values of $z - z'$. Now we make use of translation invariance and rewrite eq. (5) as

$$G_\pm(b, z|b', z') \equiv G_\pm(r, \zeta) = \int_0^\infty dq\; q\; J_0(qr)\; u_\pm(q, \zeta) \tag{8}$$

where $\vec{r} := \vec{b} - \vec{b}'$, $\zeta := z - z'$ and

$$J_0(qr) := \frac{1}{2\pi} \int_0^{2\pi} d\varphi\; e^{iqr\cos(\varphi)} \tag{9}$$

of order zero. The functions $u_\pm(q,\zeta)$ are defined by

$$u_\pm(q,\zeta) = u_\pm^{(1)}(q,\zeta) + u_\pm^{(2)}(q,\zeta) \tag{10}$$

where

$$u_\pm^{(1)}(q,\zeta) := \frac{1}{2\pi}\theta(k^2 - q^2)e^{\pm ip_1\zeta} \tag{11}$$

$$u_\pm^{(2)}(q,\zeta) := \frac{1}{2\pi}\theta(q^2 - k^2)e^{\mp p_2\zeta} \tag{12}$$

The integrals in eq. (10) involving $u_\pm^{(1)}(q,\zeta)$ are convergent and will, therefore, not be discussed further. We concentrate on the contributions of $u_\pm^{(2)}(q,\zeta)$ to these integrals and restrict attention only to $G_+(r,\zeta)$ since $G_-(r,\zeta) = G_+(r,-\zeta)$. To lighten the notation, we represent this contribution as

$$G_2(r,\zeta) := \frac{1}{2\pi}\int_k^\infty dq\; q\; J_0(qr)\; e^{-p_2\zeta} \;=\; \frac{k^2}{2\pi}\int_0^\infty dt\; t\; J_0(R\sqrt{1+t^2})\; e^{-tY} \tag{13}$$

where $R := kr$ and $Y := k\zeta$.

The integral in eq.(13) is convergent for $\zeta > 0$.For $\zeta = 0$

$$G_2(r,\zeta) \;:=\; \frac{1}{2\pi}\;\delta(r^2) \;-\; \frac{1}{2\pi}\int_0^k dq\; q\; J_0(qr) \tag{14}$$

where

$$\int_0^\infty dq\; q\; J_0(qr) \tag{15}$$

is the Dirac delta function. For $\zeta < 0$, the integral in eq. (13) is Cauchy divergent.

III. AN ANALYTICAL SOLUTION OF THE INVERSE DIFFRACTION TRANSFORM KERNEL

First we find an analytical solution for $\zeta > 0$ for every ν and μ. We have[11]

$$J_\nu(R\sqrt{1+t^2}) \;=\; \sum_{m=0}^{\infty} \frac{(-)^m\; (\frac{1}{2}R\sqrt{1+t^2})^{\nu+2m}}{m!\;\Gamma(\nu+m+1)} \tag{16}$$

and[12]

$$\int_0^\infty t^{s-1}\, e^{-Yt} \;= \frac{\Gamma(s)}{Y^s}. \tag{17}$$

For $\nu = 0$

$$\begin{aligned}
\int_0^\infty t^{\mu-1}\, J_0(R\sqrt{1+t^2})\, e^{-Yt}\, dt \;&=\; \sum_{m=0}^{\infty} \frac{(-)^m(\frac{1}{2}R)^{2m}}{m!\;\Gamma(m+1)} \int_0^\infty t^{\mu-1}\, (\sqrt{1+t^2})^{2m}\, e^{-Yt}\, dt \;= \\
&= \sum_{m=0}^{\infty} \frac{(-)^m(\frac{1}{2}R)^{2m}}{m!\;\Gamma(m+1)} \int_0^\infty t^{\mu-1}\, [1+t^2]^m\, e^{-Yt}\, dt \;= \\
&= \sum_{m=0}^{\infty} \frac{(-)^m(\frac{1}{2}R)^{2m}}{m!\;\Gamma(m+1)} \int_0^\infty t^{\mu-1}\, [\sum_{j=0}^{m} \binom{m}{j}\, t^{2j}]\, e^{-Yt}\, dt \;= \\
&= \sum_{m=0}^{\infty} \frac{(-)^m(\frac{1}{2}R)^{2m}}{m!\;\Gamma(m+1)} \sum_{j=0}^{m} \binom{m}{j} \int_0^\infty t^{\mu+2j-1}\, e^{-Yt}\, dt \;= \\
&= \sum_{m=0}^{\infty} \frac{(-)^m(\frac{1}{2}R)^{2m}}{m!\;\Gamma(m+1)} \sum_{j=0}^{m} \binom{m}{j} \frac{\Gamma(\mu+2j)}{Y^{\mu+2j}}.
\end{aligned} \tag{18}$$

For $\mu = 2$ at last,

$$\begin{aligned}
f(Y) \;&=\; \frac{k^2}{2\pi} \int_0^\infty dt\; t\; J_0(R\sqrt{1+t^2})\, e^{-tY} \;= \\
&= \frac{k^2}{2\pi} \sum_{m=0}^{\infty} \frac{(-)^m(\frac{1}{2}R)^{2m}}{m!\;\Gamma(m+1)} \sum_{j=0}^{m} \binom{m}{j} \frac{\Gamma[2(j+1)]}{Y^{2(j+1)}}.
\end{aligned} \tag{19}$$

The final series converges absolutely, since is justified.[13] The results[13] has been proved only when $R(Y) > 0$ and $|R| < |Y|$; but, so long as merely, $\Re(Y + iR) > 0$ and $\Re(Y - iR)$, then both sides of eq. (19) are analytic functions of R; and so, by the principle of analytic continuation, eq. (19) is true for this more extensive range of value of R. Moreover eq. (19) is even function of Y, therefore is true for $Y < 0$. We illustrate these results graphically in Fig. 1 where $RegG_2(r, \zeta)$ is plotted against ζ for various values of r. We notice that Fig. 1 confirms the Fig. 1 of paper " On the Convergence of the Inverse Diffraction Transform Kernel Using Cesàro Summability"[10].

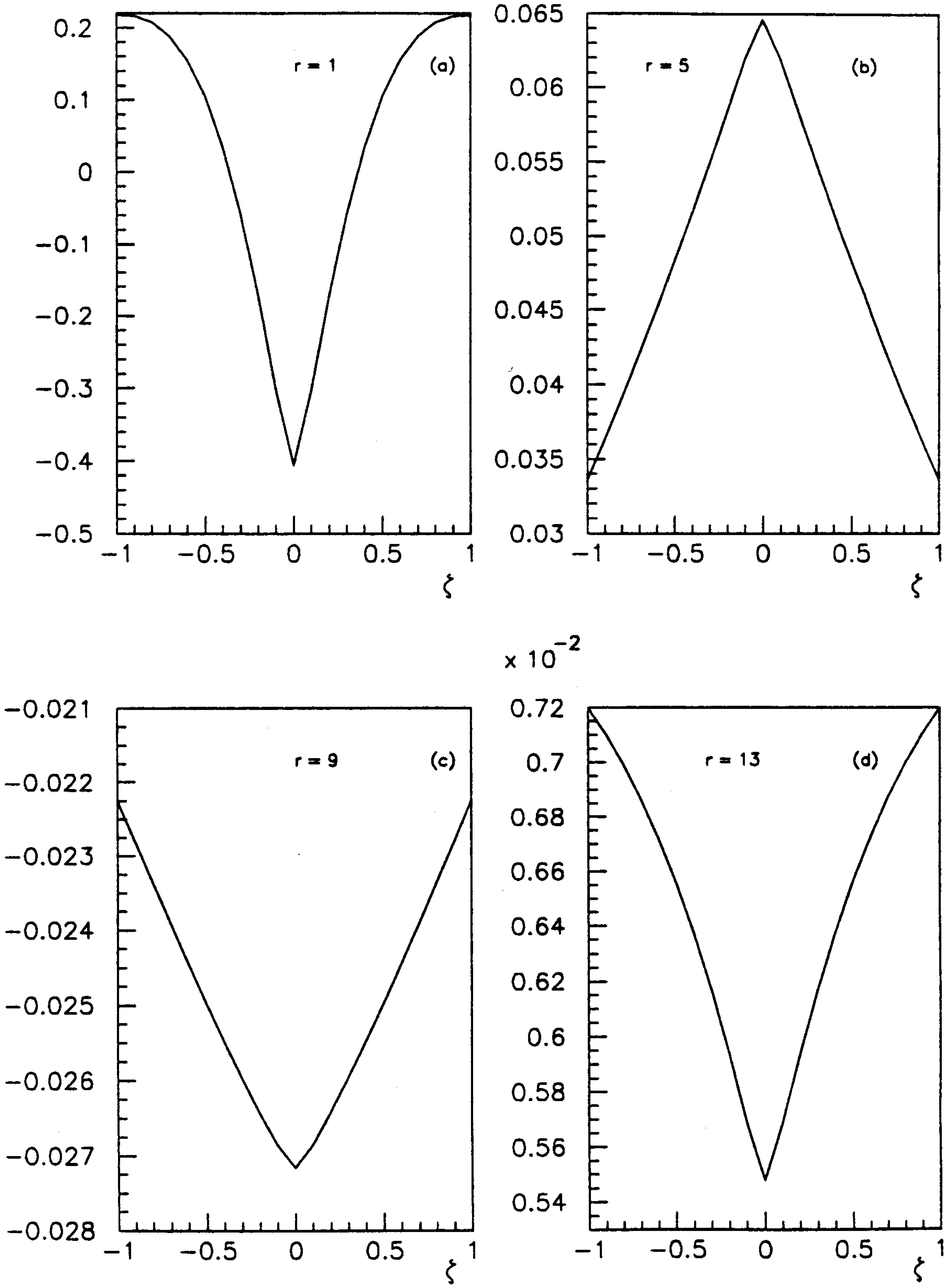

FIG. 1. Plots of the Cesàro integral *Reg* $G_2(r,\zeta)$ as a function of $\zeta(-\zeta \geq \zeta \geq +1)$ for various values of $r(r = 1, 5, 9, 13)$ and the parameter k $(k = 1)$.

IV. CONCLUSIONS

The problem solved in this paper was first clearly formulated by Sherman[8] and by Shewell and Wolf[9] in optical physics. The divergence, according to these authors, arises as a result of the interchange of the order of the $\vec{q} - and - \vec{b}$ integrations. Shewell and Wolf propose to regularize the singularity most simply by means of a cut - off, a so called band - width limitation. The cut - off eliminates the higher frequencies $(q^2 > k^2)$. The arguments in support of this cut - off procedure are not theoretical but rather they make recourse to the behaviour of frequency detectors. The arguments claim that no detector can resolve frequencies that are arbitrarily high. Sherman is much more sophisticated in his regularization programme. He suggests to interpret the divergent integral as a distribution. In any case, the suggestion implies no more than using test functions to operate cut - offs. The work of Sherman and of Shewell and Wolf is widely used in applications. We quote, in this regard, the otherwise interesting paper of Devaney on diffraction tomography[6]. One encounters here too the divergent kernel $G_+(b, z|b', z')(z < z')$. To invert the diffraction transform and recover the object field from a given scattered field configuration, Devaney resorts to the construction of a set of filters upon which are imposed various band - width limitations. The operators corresponding to these filters are then expected to combine and yield a "good approximation" to the unit operator in function space. With this approximation one inverts the diffraction transform and recovers the required object field. The latter is then compared with the experimentally deduced target geometrical profile. The analytical solution presented in the paper eliminates the divergence in the theory. The Helmholtz equation should be viewed as a particular case of more general situation. Starting from the Klein - Gordon equation the latter situation may arise. The solution of the scattering problem would then consist of only evanescent waves. Band - width limitations cannot be invoked to eliminate these waves. One may apply the same procedure to the solution of the classical scattering problem for the Klein - Gordon equation with $m^2 < 0$. The scattering problem for the Helmholtz equation involves essentially the solution of the one dimensional Schrödinger equation with a potential barrier at the boundary plane $z = z_0$. The text book solution of this quantum mechanical problem is well known. One may approach the problem differently by taking issue with the non - Hermiticity of the translation operator conjugate to the constrained variable z. We propose to re - consider the problem from this point of view elsewhere.

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