LABORATORI NAZIONALI DI FRASCATI
SIS – Pubblicazioni

LNF–95/030 (P))
1 Giugno 1995

On the Connection between CESÀRO and ABEL Summabilities

M. Pallotta
INFN – Laboratori Nazionali, P.O. Box 13, I–00044, Frascati, (Roma) Italy

Abstract

Abel summability is obtained as a special limit from the sequence of Cesàro moments.

1. CESÀRO AND ABEL SUMMABILITIES AND THEIR RELATIONSHIP

In this paper we combine two well known facts about Cesàro summability and obtain an analytic continuation by means of a Laplace transform, whit the help of which one implements Abel summability [1].

Let

$$f(z) := \int_0^\infty dt\ u\ (t,\ z) \tag{1}$$

be an integral defined formally by the symbol on the right hand side of Eq. (1). It is convenient in what follows to introduce the notation

$$u^{(0)}(T, z) \equiv u\ (T, z) \tag{2}$$

PACS.: number(s): 02.30.Lt

(Submitted to Journal of Math. Phys.)

and iterate upon the definition of the partial sum

$$u^{(1)}(T, z) := \int_0^T dt\, u^{(0)}(t, z) \qquad (3)$$

to get

$$u^{(k+1)}(T, z) := \int_0^T dt\, u^{(k)}(t, z) = \frac{T^k}{k!} \int_0^T dt \left(1 - \frac{t}{T}\right)^k u(t, z) \qquad k = 0, 1, 2...... \quad (4)$$

Next, introduce the moments known also as Cesàro means

$$f^{(k)}(T, z) := \frac{k!}{T^k} u^{(k+1)}(T, z) = \int_0^T dt \left(1 - \frac{t}{T}\right)^k u(t, z) \qquad (5)$$

For k = 0, Eq. 5) reduces to the Cauchy partial sum

$$f^{(0)}(T, z) := \int_0^T dt\; u(t, z) \qquad (6)$$

of the integral in Eq. (1). By Cauchy summability one means the application of the limit

$$f^{(0)}(z) := \lim_{T \to \infty} \int_0^T dt\; u(t, z) \qquad (7)$$

If this limit exists one then says that the original integral in Eq. (1) is Cauchy summable and has the value given by the limit

$$f(z) \equiv f^{(0)}(z) = \lim_{T \to \infty} \int_0^T dt\; u(t, z) \qquad (8)$$

It follows from Eq. (5) that when this happens the limits

$$f^{(k)}(z) := \lim_{T \to \infty} \int_0^T dt \left(1 - \frac{t}{T}\right)^k u(t, z) \qquad (9)$$

of all the higher moments $f^{(k)}(T, z)$ ($k = 1, 2, ...$) exist and have a common value $f^{(0)}(z)$, i.e.

$$f(z) = f^{(k)}(z) = f^{(k+1)}(z)\,;\; k = 0, 1, 2, \qquad (10)$$

If, on the other hand, the limit in Eq. (7) does not exist, that is, it is ambiguous or outright divergent, the higher moments $f^{(k)}(T, z)$ $(k \geq 1)$ still may tend to a definite limit. The convergence of the moments for $T \to \infty$ has therefore to be investigated. Let the first convergent moment in this case be the n–th. Again based on Eq. (5) one finds that the limits of all the moments $f^{(n+k)}(T, z)$ $(k \geq 0)$ exist and tend to a common value while the lower moments $f^{(n-k)}(T, z)$ $(k = 1, 2, \ldots, n)$ do not converge for $T \to \infty$. In this case one says that the integral in Eq. (1) is Cesàro summable at order n. The value of the integral is given by the common value of the moments $f^{(n+k)}(T, z)$ for $T \to \infty$, i.e.

$$f(z) = f^{(n+k)}(z) = \lim_{T \to \infty} \int_0^T dt \; \left(1 - \frac{t}{T} \right)^{n+k} u(t, z); \tag{11}$$

Now consider the limit of $k \to \infty$ in Eq. (11) and the feasibility of interchanging the order of the limits $T \to \infty$ followed by $k \to \infty$ on the right hand side. The interchange yields a non–zero result only if k and T tend to infinity at fixed ratio

$$\lambda = \frac{k}{T} \tag{12}$$

Effecting these limits on the integral on the right hand side of Eq. (11) yields the function

$$f(\lambda, z) := \lim_{\substack{k\,;\,T \to \infty \\ \lambda = \frac{k}{T}\ \text{fixed}}} \int_0^T dt \; \left(1 - \frac{t}{T} \right)^{n+k} u(t, z) \quad k \geq 0. \tag{13}$$

This becomes the Laplace transform

$$f(\lambda, z) = \int_0^\infty dt \;\; e^{-\lambda t} \, u(t, z) \tag{14}$$

upon making use of the formula

$$\lim_{k \to \infty} \left(1 - \frac{\lambda t}{k} \right)^k = e^{-\lambda t} \tag{15}$$

Making use of (13) and (14) in Eq. (11) one obtains agreement with the left hand side if the parameter λ subsequently tends to zero. i. e.

$$f(z) = \lim_{\lambda \to 0} \int_0^\infty dt \;\; e^{-\lambda t} \, u(t, z). \tag{16}$$

Thus if the integral f(z) in Eq. 1) is Cesàro summable at order n it follows that is also summable per continuity of the Laplace transform $f(\lambda, \dot{z})$ for $\lambda \rightarrow 0$.

Note that $f(\lambda = 0, z)$ coincides with the originally divergent integral f(z). The Laplace transform $f(\lambda, z)$ is thus a particular continuation of the function f(z) into a larger domain, i.e. the half line $\lambda \geq 0$. The continuation is completely determined by the Cesàro moments $f^{(k)}(T, z)$ $(k \geq 0)$. A divergent integral f(z) admitting such a parametric continuation $f(\lambda, z)$ not necessary through Cesàro moments, and such that the limit $f(\lambda \rightarrow 0, z)$ exist is said to be Abel summable. Thus Eqs. (11), (15) and (16) state that an integral that is Cesàro summable is also Abel summable. The converse of this statement is false. Abel summability emerges here then as a special limit performed on the sequence of Cesàro moments. The sequence of moments $f^{(k)}(T, z)$ forms the hierarchy of convergence tests, referred to earlier, for $T \rightarrow \infty$.

ACKNOWLEDGMENTS

The author would like to thank Etim Etim for many stimulating and helpful discussions.

REFERENCES

[1] G. H. Hardy, Divergent Series (Oxford University Press), London, 1956.