



LABORATORI NAZIONALI DI FRASCATI
SIS – Pubblicazioni

LNF-95/010 (P)
8 Marzo 1995

Detection of Gravitational Waves with Spherical Antennas

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Abstract

We report on an investigation of the possibilities offered by solid resonant spheres to detect gravitational radiation. A spherical detector has isotropic sensitivity and can determine the source direction and wave polarisation providing vetoes on General Relativity. A deconvolution procedure for a general gravitational wave is described, along with tests and vetoes on the trace and transversality properties of gravitational waves. We find that the sphere's gravitational wave absorption cross section is large at *two* frequencies and that if the wave direction is known, a spherical detector can determine all the six polarisation states predicted by the most general metric theory. Assuming General Relativity, a two sphere observatory completes the deconvolution process, removing the 180° ambiguity in the source position, enabling the determination of the gravitational wave speed and providing powerful vetoes on local disturbances. We discuss the sensitivity of spheres of different materials and frequencies of operation. A 3 metres diameter Al5056 sphere is shown to be able to reach a 1 ms burst sensitivity of $h \simeq 6 \times 10^{-22}$ if operated at the quantum limit.

PACS.: 04.80.+z

Invited talk

to the "11th Italian Conference on General Relativity and Gravitational Physics"
Trieste, 26–30 September, 1994

1 Introduction

One of the most remarkable properties of almost any metric theory of gravity is their prediction of the existence of propagating solutions of the field equations, or *gravitational waves* (GWs). The distinguishing characteristics of such waves, i.e., *propagation speed, polarisation states and multipolar structure*, vary from one theory to another [1], and so direct measurement of the effect of a GW is potentially a powerful tool to select a candidate theory against its competitors. So far, only weak field, near zone tests have been performed—but this helps little, if at all, to discern between competing alternatives. Direct observation of GWs will provide a new testing ground for gravity theories in the *far zone* regime.

Quite independently of which is the “correct theory” of the gravitational field, the analysis of GWs will also open a new window for the observation of the universe, thereby founding a new Astronomy. GWs will produce unique information about the coherent bulk motion of the matter generating the waves “revealing features of the source which none could ever learn from electromagnetic, cosmic ray or neutrino physics” [2]. The strongest detectable events are thought to be bursts generated by a variety of possible sources: coalescence of binary neutron stars to produce black holes, the collapse of white dwarfs to form neutron stars, or neutron stars to produce black holes, the echo of a supernova event in which a collapse to a dense object happens at the centre and, likely, a wealth of other so far unsuspected sources.

Direct detection of GWs has thus become one of the great challenges of contemporary experimental physics.

The binary pulsar PSR 1913+16 has now been observed for twenty years. Evidence based on the decay rate of the orbital period of the system provides the most compelling argument so far in favour of the GW phenomenon (see [3] and references therein). In spite of the truly remarkable precision with which General Relativity (GR) fits the experimental data —0.4%— the binary pulsar observations so far can only confirm one aspect of GW physics: the *back-action* effect of the emitted GWs on the emitter, as revealed by the pulsar’s pulse arrival time pattern. But there are many other features of GWs which cannot be probed with those data. *So we need an experiment which enables the measurement of other specific properties of GWs.*

On the other hand, after more-than-twenty-years work of development, several GW detectors are now in *continuous observational mode* with the unprecedented burst sensitivity $h = 6 \times 10^{-19}$. Such a sensitivity should allow to detect gravitational collapses in our Galaxy and in the Local Group [4], but still appears to be insufficient for more remote (and therefore more numerous) sources.

All these detectors are of the cryogenic resonant-mass type (cylindrical bars). Since the fundamental noise sources of this type of detector are of thermal origin, cooling to low temperatures was the key action for obtaining high sensitivity. *The underlying non-gravitational physics associated with these detectors is today reasonably understood, and further improvements appear based on solid technological guidelines.*

There is one way to improve sensitivity and performance of resonant mass detectors which is independent of the detector noise: it is to change its shape and symmetry. In particular, we want to reexamine why *spherical detectors* constitute the natural step towards a resonant antenna GW observatory.

Resonant antennas can be characterized by their symmetry properties and classified by the symmetry group consisting of the symmetry operations of their structure [5]. The antenna eigenmodes serve as the basis of the matrix representations of the symmetry group and are categorised by the nature of the representations. A certain antenna mode can be GW-inactive or active depending on its symmetry characteristics. In GR, a particular antenna mode is GW-active only if it has a nonvanishing quadrupole moment. In other theories, modes with different polarities may also be GW-active. Multimode detection appears then a natural way to investigate different gravitation theories. In addition, since every mode has its own radiation pattern, the simultaneous detection of a sufficient number of modes can also be used to determine the source direction and the wave polarisations.

Suppose *e.g.* that GWs are correctly described by GR. Then the wave has two polarisations with the force patterns rotated 45° from each other in the transverse plane. There are five unknown quantities that one would like to determine: the amplitudes of the two polarisations (h_+ , h_\times), the two angles of the source direction (θ , φ), and the polarisation angle in the wavefront (ψ). These parameters could be determined by using multiple antennas of low symmetry with requisite relative orientations, or by simultaneously detecting several modes of a highly symmetric antenna.

The cylindrical bar antenna, introduced by Weber, has the lowest symmetry and measures only one of the above parameters: the amplitude of a combination of the two polarisations. Likewise, square and torsional antennas have been developed and configured to detect one polarisation. It would take at least five of these antennas to determine all the desired parameters, and six of them to give isotropic sensitivity [6].

The antenna geometry which has the highest degree of symmetry is a sphere. It has *five* degenerate quadrupole modes, which fact allows one, in principle, to determine the five unknowns. Each mode can act as a separate antenna oriented towards a different polarisation and direction, giving a perfectly omnidirectional detector. A spherical detector can detect in a very natural way not only the tensorial waves predicted by GR, but also scalar waves predicted by other metric theories of gravitation, like the theory of Brans and Dicke [7]. This type of radiation could be emitted by time variations of the source monopole moment, like in the radial oscillations of vibrating neutron stars, and should be detected by monitoring the excitations of the monopole mode of the sphere.

It was recognized long ago by R. Forward [8] that a spherical detector can be considered a true GW observatory. In 1975, Ashby and Dreitlein [9] studied the response of an elastic sphere to gravitational radiation. In 1976, Wagoner and Paik [10] found a set of equations for determining source direction and wave polarisations

from the amplitudes of the quadrupole modes of the sphere for scalar-tensor theories. Comparing a sphere to a cylindrical bar of the same material and same resonant frequency (this condition is roughly given by the equality between the bar's length and the sphere's diameter), the sphere has a larger cross section because of its larger mass *and* because it is omnidirectional.

These facts have been overlooked for many years. The cylindrical bar geometry was a more practical choice, because of the easier mounting of the transducer and ease of manufacture. Today, new facts give the experimentalist the necessary confidence to start a new project:

- the reliability reached by the cryogenic resonant antennas;
- the feasibility of the cooling to below 0.1 K for the new generation of resonant antennas, demonstrated by NAUTILUS [11];
- the feasibility of a nodal point suspension to support a large resonant mass [12];
- the determination of a clear method for the orientational deconvolution of the signal from a set of transducers coupled to a spherical resonant mass [13], and
- the possibility of making large resonant antennas using new bonding methods [14].

In section 2, we briefly review the sensitivity of cylindrical bars. In section 3, we reexamine the sensitivity of an elastic sphere to GWs, extending and perfecting previous analyses. In that section we also assess the signal deconvolution problem. In section 4, we present the features of an observatory constituted by *two* spheres in terms of sensitivity and of tests of gravitation theory. Also, we discuss the frequency of operation in terms of sources of astrophysical interest. In section 5 the main keypoints of a feasibility study are discussed and conclusions are drawn.

2 Review of cylindrical bar sensitivity

The problem of detection of short bursts of GWs by a resonant antenna has been clarified in its main aspects since many years [15]–[16]. The antenna responds to the oscillating field of GWs and a suitable sensor perceives its vibrations. If the detector is assumed to be a single large mass, the energy absorbed (ΔE_a), due to GWs, can be calculated by means of the detector cross section σ . For a *thin* cylindrical bar (length much longer than radius), the most popular shape, and for its most favourable mode of vibration (the first longitudinal one), the cross section takes the form [17]:

$$\sigma_{abs}(\omega) = \frac{8}{\pi} \frac{GMv_s^2}{c^3} \sin^4 \theta \cos^2 2\varphi \frac{\Gamma}{(\omega - \omega_0)^2 + \Gamma^2/4} \quad (2.1)$$

where $v_s=(Y/\rho)^{1/2}$ is the speed of sound in the bar material, with Y the Young modulus, M the mass of the bar, θ the angle between the direction of propagation of the GW and the cylinder longitudinal axis, φ the azimuthal angle of the wave's polarisation ellipse and Γ the linewidth of the mechanical resonance of the first longitudinal mode at frequency ω_0 .

The energy absorbed by the detector can be written as:

$$\Delta E_a = F(\omega_0) \int \sigma_{abs}(\omega) d\omega \quad (2.2)$$

where $F(\omega_0)$ is the value of the spectrum energy density of the GW burst at the resonance frequency ω_0 . For a burst consisting in a sinusoidal oscillation of amplitude h and duration τ_g , at the frequency of the first longitudinal mode ω_0 , we can write

$$F(\omega_0) = \frac{c^3 \omega_0^2 h^2 \tau_g^2}{16\pi G} \quad , \quad \tau_g \ll \Gamma^{-1} \quad (2.3)$$

The energy absorbed, ΔE_a , must be compared to the variance of the noise energy fluctuations, ΔE_n , of the detector. This energy noise can be written as the sum of two contributions, originating from two uncorrelated Gaussian processes: the thermal noise in the resonant mass and the electronic noise of the readout system [18]. We assume that a fraction β of the antenna energy is converted into electromagnetic energy by a noiseless transducer; the signal at each transducer is then fed to an amplifier, assumed to have an additive noise at its output, whose spectral density is S_q ; the amplifier also exerts a back action noise force onto the antenna mode with spectral density S_f . The energy fluctuations in the readout system can then be expressed by the noise temperature $T_n = k^{-1}(S_f \omega^2 S_q)^{1/2}$. If antenna and transducer are correctly matched, it is found that

$$\Delta E_n = \frac{kT}{\beta Q} + kT_n \quad (2.4)$$

where T is the thermodynamic temperature of the cylindrical bar and Q is the quality factor of the vibration mode. It is customary to express the energy noise as the Boltzmann constant times a *detector noise temperature* T_{eff} , writing $\Delta E_n = kT_{eff}$.

Most groups make use of cylindrical bars made with the high Q aluminium alloy Al 5056 ($v_s = 5.1 \times 10^5 \text{ ms}^{-1}$), whose length, of about 3 metres, is fixed to get the resonant frequency $\nu_0 (\equiv \omega_0/2\pi)$ around 1 kHz, where the signal energy density is expected to be largest. The typical mass is a few tons. The detector's signal-to-noise ratio (SNR) is defined by $SNR = \Delta E_a/kT_{eff}$.

The sensitivity of the detector, i.e., the minimum detectable GW amplitude h_{min} ($SNR=1$), can be written as:

$$h_{min} = \sqrt{\frac{4}{\pi} \frac{GkT_{eff}}{c^3\sigma}} \quad (2.5)$$

The detector bandwidth, defined by the width of the *SNR* resonance curve, is of order $\beta\omega_0$, much larger than the purely mechanical resonance linewidth $\Gamma = \omega_0/Q$ of the vibrational mode.

Since the beginning of the GW research, all the efforts for improving the sensitivity have been focused on the reduction of the effective noise temperature T_{eff} . The use of cryogenic technologies permitted to reduce T_{eff} by a factor of 10^4 , from the tens of Kelvin to a few millikelvin. This result was obtained by cooling the detectors to liquid helium temperatures and adopting electromechanical readout systems making use of superconducting devices.

Further developments are in progress in order to improve the energy resolution up to one vibrational quantum ($\hbar\omega_0$ at 1 kHz corresponds to $T_{eff} \approx 10^{-7}K$). The quantum limited sensitivity reachable by the ultralow temperature ($T < 0.1$ K) cylindrical bar antennas under development is $h_{min} = 3 \times 10^{-21}$.

In order to reach this goal, the experimental parameters which determine the detector noise must be pushed to the extreme limit of the existing conventional and quantum technologies.

3 The elastic sphere as a GW antenna

In this section we reassess the question of how an elastic solid sphere can be made to work as a resonant GW antenna. We shall assume that Einstein's General Relativity (GR) is the correct theory to describe GWs. This—or, indeed, any other assumption as to which is the “correct” theory—is an unavoidable requirement in order to *deconvolve* the signal which will eventually be found to have hit the antenna; one can then find tests and/or vetoes on the assumed theory. We first study the antenna's cross section for absorption of GW energy, then address the problem of signal deconvolution.

3.1 The sphere's spectrum and cross section

Let an incoming flux $F(\omega)$ of GWs of (angular) frequency ω impinge on a given solid body which is intended to be used as a GW antenna. A part of the energy carried by the incoming radiation, say $\Delta E_a(\omega)$, will be absorbed by the antenna, thereby its vibrational modes being excited. It is convenient to quantify the energy that the solid can absorb in terms of an absorption energy per unit flux, or *absorption cross section* (see earlier):

$$\sigma_{abs}(\omega) = \frac{\Delta E_a}{F(\omega)} \quad (3.1)$$

For GWs whose wavelength λ is much larger than the antenna's typical dimension R , and propagating in a space with a radius of curvature much larger than λ , we can follow theory developed by Weinberg (see [19]) to find that, for a *spherical* elastic antenna,

$$\sigma_{abs}(\omega) = \frac{10\pi\eta c^2}{\omega^2} \frac{\Gamma^2/4}{(\omega - \omega_0)^2 + \Gamma^2/4}, \quad (3.2)$$

where ω_0 is one of the sphere's resonant frequencies, assumed close to ω , Γ is the linewidth of the mechanical resonance and $\eta = \Gamma_{grav}/\Gamma$, with

$$\Gamma_{grav} \equiv \frac{\dot{E}_{GW}}{E_{oscill}} \quad (3.3)$$

Γ_{grav}^{-1} is thus the time scale for the damping of the sphere's oscillations due to re-emission of GWs caused by those very oscillations. Formula (3.2) is based on GR, so \dot{E}_{GW} will also be calculated using GR, more specifically (cf. [20]),

$$\dot{E}_{GW} = \frac{G}{5c^5} \ddot{Q}_{ij}^*(t) \ddot{Q}_{ij}(t) \quad (3.4)$$

where

$$Q_{ij}(t) \equiv \int \left(x_i x_j - \frac{1}{3} |x|^2 \delta_{ij} \right) \rho(x, t) d^3x \quad (3.5)$$

is the antenna's quadrupole moment.

To evaluate these quantities, as well as the precise motion induced by the GWs on the detector, the full problem of the elastic vibrations of the sphere has to be solved. We shall assume that the displacements and velocities of the solid are sufficiently small that the equations of motion are those of the non-relativistic linearised theory of elasticity (as described e.g. in [21]).

Let $u(x, t)$ be the field of displacements in the solid. If it is driven by the volume force density $f(x, t)$, its vibrations are given by those solutions to the system of partial differential equations

$$\rho \frac{\partial^2 u}{\partial t^2} - \mu \nabla^2 u - (\lambda + \mu) \nabla(\nabla \cdot u) = f(x, t) \quad (3.6)$$

which satisfy the boundary condition that the solid's surface be free from any tractions or pressures. In (3.6), λ and μ are the usual Lamé coefficients, and

$$f_i(x, t) = -\frac{1}{2} \rho(x, t) \ddot{h}_{ij}(t) x_j \quad (3.7)$$

is the GW-induced *tidal volume force*, expressed in terms of the “electric” components of the GW’s Riemann tensor at the antenna’s location:

$$R_{0i0j} = -\frac{1}{2c^2} \ddot{h}_{ij} \quad (3.8)$$

The absorption cross section, however, only depends on the *homogeneous* solutions to (3.6) —which characterise the solid’s normal modes. From now on we specialise to the case of a *homogeneous* spherical antenna. Its solutions have been known for a long time —cf. e.g. [22]—, and belong to two different families of normal modes. The first family is constituted by the so called *toroidal* modes: these consist of purely *tangential* displacements with no volume and density changes. These modes will never be excited by a metric GW, as any such wave always involves non-zero radial displacement patterns —see Figure 1. The second family is formed by the *spheroidal* modes, and these do contain radial displacements. In vector notation, their analytical expressions are:

$$u_{nlm}(x, t) = u_0 e^{-i\omega_{nl}t} [A_{nl}(r) Y_{lm}(\theta, \varphi) \mathbf{n} - B_{nl}(r) \mathbf{i} \mathbf{n} \times \mathbf{L} Y_{lm}(\theta, \varphi)] \quad (3.9)$$

where

$$A_{nl}(r) \equiv \beta_3(kR) j_l'(qr) - l(l+1) \frac{q}{k} \beta_1(qR) \frac{j_l(kr)}{kr} \quad (3.10.a)$$

$$B_{nl}(r) \equiv \beta_3(kR) \frac{j_l(qr)}{qr} - \frac{q}{k} \beta_1(qR) \frac{1}{kr} \frac{d}{d(kr)} [kr j_l(kr)] \quad (3.10.b)$$

In these expressions \mathbf{n} is the radial outward unit vector, \mathbf{L} is the “angular momentum” operator $\mathbf{L} \equiv -i\mathbf{x} \times \nabla$, $j_l(z)$ is a spherical Bessel function, and $Y_{lm}(\theta, \varphi)$ is a spherical harmonic. u_0 is an arbitrary constant amplitude, R the sphere radius and

$$k \equiv \left(\frac{\rho \omega_{nl}^2}{\mu} \right)^{1/2}, \quad q \equiv \left(\frac{\rho \omega_{nl}^2}{\lambda + 2\mu} \right)^{1/2} \quad (3.11)$$

Finally

$$\beta_1(z) \equiv \frac{d}{dz} \left(\frac{j_l(z)}{z} \right) \quad (3.12.a)$$

$$\beta_3(z) \equiv \frac{1}{2} j_l''(z) + \left(\frac{l(l+1)}{2} - 1 \right) \frac{j_l(z)}{z^2} \quad (3.12.b)$$

The frequencies ω_{nl} are the solutions of a somewhat complicated eigenvalue equation; such equation is *independent* of m —it only depends on l — and we label its solutions with the index n , which runs from 1 to infinite for *each* value of l . In Figure 2 we give a line representation of the first few eigenfrequencies corresponding to the lowest multipole modes $l = 0, 1, 2$. Every one of these frequencies is therefore $(2l + 1)$ – fold degenerate for each n .

There are five quadrupole ($l=2$) modes —all of them with the same frequency of oscillation, as just remarked. A pictorial representation of their radial displacements is given in Figure 3 in terms of real spherical harmonics.

Once the precise form of the resonant modes is known, it is relatively straightforward to calculate Γ_{grav} in (3.3), and thereby obtain the sphere's cross section for each mode. The result can be cast in the form

$$\sigma_{abs}(\omega) = F_{ln} \frac{GMv_s^2}{c^3} \frac{\Gamma}{(\omega - \omega_0)^2 + \Gamma^2/4} \quad (3.13)$$

where F_{ln} is a dimensionless quantity whose calculation involves a number of cumbersome algebraic manipulations. It possesses the remarkable property of being zero for $l \neq 2$, which means that only the quadrupole modes of the sphere can possibly be excited by an incoming GW —an expected result. In (3.13), v_s^2 is the velocity of sound, as in (2.1). By way of example we have, for the first quadrupole mode, $F_{21}=2.99$; this is a factor of 1.17 larger than $8/\pi$, the optimal orientation–optimal polarisation coefficient for a thin cylinder of the same mass as the sphere —cf. (2.1) If we average over directions and polarisations, the sphere's cross section becomes a factor of almost 5 better for equal masses.

In Figure 4 we plot the cross section per unit mass (or *reduced* cross section) for a sphere and an optimally oriented cylinder *versus* mode frequency. We consider the first three modes in each antenna, assuming that the antennas are tuned to have the same fundamental frequency. Remarkably, the *second* sphere's mode still shows a rather high cross section value, only about half the maximum for the fundamental mode. As is well known, only the fundamental cylinder mode shows a significant cross section value.

So a spherical antenna is potentially sensitive to *two* eigenfrequencies within the range of astrophysically interesting events —cf. next section—, this being a *new* advantage of this kind of antenna over cylindrical ones. We underline here that the cross section value at this *second* mode is *higher* by a factor of almost 3 than that of that smaller sphere which has that frequency as its *first* mode. Implications for a network of detectors are obvious.

An also interesting figure is the *integrated* cross section for actual antennas. This is displayed in Table 1 at the indicated frequencies for projected spheres and existing cylinders, always under the assumption of optimal orientation for the latter. Note that the sphere can absorb over 20 times more GW energy than the cylinder in the fundamental mode, and about 8 times in the *second* mode.

Cylinder	Sphere
$\nu_{1c} = 896 \text{ Hz}$	$\nu_{1s} = 896 \text{ Hz}$
$L = 2.9 \text{ metres}$	$\nu_{2s} = 1720 \text{ Hz}$
$D = 0.6 \text{ metres}$	$\phi = 3 \text{ metres}$
$M_c = 2.2 \text{ tons}$	$M_s = 38 \text{ tons}$
$\sigma_{1c} = 4.1 \times 10^{-21} \text{ cm}^2 \text{ Hz}$	$\sigma_{1s} = 8.3 \times 10^{-20} \text{ cm}^2 \text{ Hz}$
	$\sigma_{2s} = 3.2 \times 10^{-20} \text{ cm}^2 \text{ Hz}$
(Optimal orientation)	(Omnidirectional)

Table 1: Integrated cross sections for a standard cylinder in its first longitudinal mode and with *optimal* orientation with respect to the incoming radiation, and for a sphere in its first two quadrupole modes. Antennæ dimensions are also specified.

3.2 The signal deconvolution

Once the vibration modes of the sphere are accurately known, we need a convenient readout system to monitor them. This is accomplished by means of a set of transducers attached to the sphere —*cf.* next subsection. And once we know that and such modes have been excited and how much they have been excited, we want to use this information to trace back the causes of their being excited in that particular way or, as we shall say, to *deconvolve* the signal.

Of course a deconvolution procedure is necessarily bound to depend on a specific GW model. The experimental observations will then produce tests and/or vetoes on *that* model. We shall be assuming GR in this paper, but begin with the description of a somewhat more general framework.

The most general metric theory of gravity contains *six* independent polarisation states for a GW, which correspond to the six “electric” components R_{0i0j} of the Riemann tensor entering equations (3.6) and (3.7). The associated canonical patterns are those of Figure 1. The most general metric GW is thus a linear superposition (in the weak signal regime) of all six such states. A particular theory, however, does not necessarily predict them all: GR for example only predicts the first two, while Brans–Dicke’s also predicts the sixth, etc. Obviously, evidence of non-predicted modes produces a *veto* against all theories predicting their absence; the converse statement, however, is more delicate —*cf.* [1] for a more detailed discussion.

We now describe a deconvolution procedure on the basis of the observation of the vibrations of a spherical antenna.

We first of all recall that any symmetric 3-tensor, like R_{0i0j} , admits the following decomposition:

$$R_{0i0j} = S_{ij} + T \delta_{ij} \quad (3.14)$$

where $T \equiv \frac{1}{3} R_{0i0i}$ is the tensor's trace, and $S_{ij} \equiv R_{0i0j} - T \delta_{ij}$ is its trace-free part. S_{ij} behaves under rotations of the coordinate axes like suitable combinations of the five $l=2$ spherical harmonics, while T is a scalar under the same transformations — thence behaving like the $l=0$ spherical harmonic. Now, since the spherical harmonics happen to be the basis for the sphere's normal modes —see earlier—, we infer that a metric GW can possibly excite the *monopole* ($l=0$) and *quadrupole* ($l=2$) vibration modes only. Analysis of the $l=0$ mode is thus the natural way to probe the *tracelessness* property of GWs.

Let us consider the simpler situation, to begin with, in which we know the direction of incidence of the GW, e.g., because the source is known from astronomical observations or even a network of other GW antennas. Direct measurement of the sphere vibration states will then produce a complete deconvolution of all the coefficients h_{ij} in a purely phenomenological way, i.e., *independently* of any underlying assumption about a particular theory of gravity. The data obtained in this way can then be compared to the predictions of a specific theory in order to either confirm or discard it.

If, more realistically, the direction of incidence is unknown, then knowledge of the sphere's vibrations is insufficient to decide on whether this or that theory is confirmed by the observations made. This is because each theory is characterised by a specific mode pattern, or by a *canonical* form of the matrix h_{ij} . Such canonical form shows in a coordinate frame suitably adapted to the propagation direction of the wave, and this frame will be in general be *rotated* with respect to the laboratory frame. Not knowing the rotation angles is therefore a strong limitation to establish the validity of a given theory.

A possible way out consists in *assuming* a certain theory, for example GR, *then* determine the rotation angles on that hypothesis. This idea was suggested by Wagoner and Paik [10], and is as follows ¹.

GR predicts that, in a wave-adapted set of coordinate axes, the metric perturbation h_{ij} takes on the canonical form

$$h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.15)$$

and so does likewise R_{i0j0} —*cf.* equation (3.7). This can also be phrased in the following way: in that particular coordinate system in which (3.15) holds, only the spherical harmonics with $l=2$ and $m=\pm 2$ are needed to make up R_{0i0j} —or the *helicity* of a GW is 2 in GR.

¹The hypothesis that GR is true can be partly checked by looking at the monopole mode: indication of its being excited would be a fatal veto against GR.

The GW-adapted set of axes will be rotated in general by three Euler angles (θ, φ, ψ) . The first two of these angles define the source direction in the laboratory frame, while the third defines the wave's polarisation ellipse with respect to the line of nodes. In the laboratory frame h_{ij} will thus look like a full 3×3 matrix, whose canonical form will still be (3.15) if GR is true. We can now take advantage of this to deconvolve the three angles (θ, φ, ψ) and the two amplitudes (h_+, h_\times) .

Let $a_m(t)$ ($m = -2, \dots, 2$) be the *measured* amplitudes of the five quadrupole modes of the sphere. If a helicity-2 GW is responsible for their excitation —as would happen should GR be true— then we are guaranteed that a rotation of the coordinate axes exists which reduces the set of measured modes to the canonical set $\hat{a}_m(t) = (\hat{a}_{-2}(t), 0, 0, 0, \hat{a}_2(t))$. If, following standard notation [23], we call $\mathcal{D}_{mm'}^{(2)}(\theta, \varphi, \psi)$ the coefficients of the rotation matrix, we can write

$$\hat{a}_m(t) = \sum_{m'=-2}^2 \mathcal{D}_{mm'}^{(2)}(\theta, \varphi, \psi) a_{m'}(t) \quad , \quad m = -2, \dots, 2 \quad (3.16)$$

and so, by setting $\hat{a}_{-1}(t) = \hat{a}_0(t) = \hat{a}_1(t) = 0$ we have a system of three equations to determine the three angles (θ, φ, ψ) . Once these are known, the other two equations uniquely define $\hat{a}_{-2}(t)$ and $\hat{a}_2(t)$, and these in turn determine $h_+(t)$ and $h_\times(t)$ [10]. The deconvolution procedure is thus completed.

It must however be cautioned that the viability of this procedure *is strongly dependent on GR being true*: we can use e.g. (3.16) to determine (θ, φ, ψ) —but this will yield the *wrong* answer for the actual angles if GR fails to be correct. Should that happen, we are expected to find algebraic incompatibilities as we proceed further to evaluate h_+ and h_\times ; such incompatibilities are to be held as *veto*s on the hypothesis that GR is true or, more specifically, on the *transversality* property of GWs: as can readily be seen in Figure 1, the presence of quadrupole polarisation states other than those with $m = \pm 2$ implies some degree of non-transversality.

In order to unambiguously assign a sudden excitation of the detector to a metric GW, several types of tests can be applied. The most general involves monitoring the detector also at frequencies other than the $l=0, 2$ spheroidal modes. One can for instance look at toroidal modes: since these have no radial displacements they cannot possibly be excited by a metric GW —see Figure 1. The lowest quadrupole toroidal frequency is so near the lowest spheroidal (less than 6%) that a veto based on the excitation of this mode is extremely efficient: any event which is seen at this frequency *cannot* be due to a metric GW. This would require a 6% wideband transducer, or an extra number of transducers to monitor the sphere's vibration at the toroidal frequency.

3.3 Readout system and noise

We consider here the problem of measuring the vibration amplitude of the five quadrupole modes of a spherical detector and the prediction of the detector sensi-

tivity in the presence of noise.

The spherical mass must be instrumented with a set of electromechanical transducers, converting the mechanical vibration of the antenna into electrical signals, and a set of amplifiers. How many transducers and in which positions? There are several possibilities.

One is to use five transducers, each one coupled to *one* of the five quadrupolar modes. Examination of the dependence of the modes on the spherical coordinates shows that this is possible if transducers sensitive to motion in one direction only are used [25]. In fact, it is possible to find five positions each one being a node for all the modes but one in a certain direction. For instance, fixing the laboratory frame, only the $(2, 0)$ mode has radial displacements at the north and south poles; only the $(2, \pm 1)$ modes have non-zero displacements along the local meridian at locations $(\theta=\pi/2, \varphi=0)$ and $(\theta=\pi/2, \varphi=\pi/2)$; and only the $(2, \pm 2)$ modes have non-zero displacements along the equator at locations $(\theta=\pi/2, \varphi=\pi/4)$ and $(\theta=\pi/2, \varphi=\pi)$.

A complete set of transducers can then be constituted by one radial plus four tangential transducers. Each quadrupole mode is equipped with its transducer and its amplifier, forming an independent detection channel. The five independent channels act as five independent detectors with different orientations.

In order to calculate the SNR of a single channel we shall assume that the transducers are identical and noiseless, giving an output proportional to the sphere's displacements. For the sake of simplicity, we also assume the sphere's thermal noise is negligible (as it should if the quantity $T/\beta Q$ is sufficiently small), and consider only the electronic noise due to identical amplifiers having noise temperature T_n . The SNR in a single channel of a spherical detector can be as large as

$$SNR = \frac{\Delta E_a}{kT_n} \quad (3.17)$$

where ΔE_a is the total energy absorbed by the detector, and is proportional to the detector cross section.

Compared to an optimally oriented cylindrical bar of the same resonant frequency and the same noise temperature, the improvement in SNR is equal to the improvement in cross section.

A second possible transducer arrangement consists in a system of more than five identical transducers, all sensitive to the same type of motion. Through a proper linear transformation, the sensors can form five independent channels which are the readout of the five quadrupolar modes.

Forward's initial suggestion [8], for instance, was to put nine electromechanical strain transducers at the intersections of great circle paths. Suitable linear combinations of the nine transducers form five tensors constituting a complete, orthonormal set, giving five physically independent outputs, each one proportional to the amplitude of one mode.

A lower number of identical transducers can be used. A natural solution is to arrange the transducers to match a set of N equivalent axes, like the ones going

from the centre of the sphere to the vertices of a regular polyhedron concentric with the sphere itself. In this way the celestial sphere is divided into identical regular polygons. The minimal choice to get at least five independent outputs is to arrange six transducers along the six axes going from the centre to half the face centres of a regular dodecahedron, or to the vertices of a regular icosahedron (see fig. 5). This is the solution proposed by Johnson and Merkwitz [13].

A transducer arrangement is characterized by a pattern matrix B , whose elements are defined by

$$b_{mj} = Y_{2m}(\theta_j, \varphi_j), \quad j = 1, \dots, 6 \quad (3.18)$$

where j identifies the transducer, and (θ_j, φ_j) is the location of the j th transducer.

The special symmetry of the dodecahedral arrangement is signaled by the special property

$$\sum_j b_{mj} b_{m'j} = \frac{3}{2\pi} \delta_{mm'}, \quad m, m' = -2, \dots, 2 \quad (3.19)$$

Under the same general assumption of amplifier noise limit, with identical amplifiers having uncorrelated noise sources, the six amplifier outputs x_j can be linearly combined to form five independent channels y_m :

$$y_m = \sum_j b_{mj} x_j, \quad m = -2, \dots, 2 \quad (3.20)$$

Each channel is a direct readout of the corresponding quadrupole mode. These channels are uncorrelated, because of the particular symmetry of the transducer arrangement. In fact, the cross correlation of two channels is:

$$\begin{aligned} R_{mm'}(\tau) &= E[y_m(t) y_{m'}(t + \tau)] = \sum_{ij} b_{mj} b_{m'i} E[x_j(t) x_i(t + \tau)] \\ &= \sum_{ij} b_{mj} b_{m'i} R(\tau) \delta_{ij} = \frac{3}{2\pi} R(\tau) \delta_{mm'} \end{aligned} \quad (3.21)$$

We are then back to the case of five independent channels acting as the output of five independent detectors, and thus to the SNR (3.17). Actually, as a result of their noise analysis, Johnson and Merkwitz [13] found that the improvement in SNR with respect to the optimally oriented cylinder is a factor equal to the cross section ratio, this indicating that this six transducer readout system does not suffer SNR penalties due to the complications introduced by the coupling of each transducer to all five quadrupole modes.

The results of this section are displayed in Figures 6 and 7, where Al5056 has been taken as the antenna's material. Figure 6 shows the minimum GW burst

sensitivity detectable by a single channel of quantum limited noise (i.e., $kT_n = \hbar\omega_0$) spherical detectors of different dimensions. The burst is a wave packet of duration τ_g , constituted by one sinusoidal cycle. In abscissæ we represent the sinusoidal frequency $\nu_0 = \tau_g^{-1}$. The smoothness of these sensitivity curves and the little gain (only at high frequency) predicted for the smaller spheres are consequences of the particular shape assumed for the burst.

One can also plot the sensitivity in terms of the strain noise spectrum \tilde{h} , as in [13], using the same matching parameters. This is done in Figure 7, where we report the sensitivities of the fundamental and second quadrupolar modes of the 3 metre diameter sphere and of the fundamental mode of the 1.56 metre sphere. For comparison, the corresponding results for typical (length/radius = 10) cylinders and for the first generation LIGO detector are given as well.

3.4 A two sphere observatory

There are certain details in the signal deconvolution process which cannot be resolved with a *single* sphere. For example, if the direction of incidence is unknown and the procedure described in section 3.2 is applied, there is an unavoidable ambiguity: one cannot possibly tell a given source from a source in its antipode in the sky.

Also, nothing can be said with a single GW antenna about the propagation speed of GWs.

An array of two spheres provides the necessary means to tackle these problems [24]: if the two antennas are placed in strategical places on the earth's surface so that most potential sources are seen under sufficiently different angles, this would remove the direction ambiguity; on the other hand, if the signal arrival time can be determined accurately, then the time delay between detection at the two antennas, together with the information on the source position, enables the direct determination of the GW propagation speed. This measurement may be used as veto on some metric theories. The speed of GW v_g is in fact determined by the detailed structure of the field equations of each metric theory of gravity. In vector-tensor theories, as well as in the Rosen's bimetric theory and in the Rastall theory values of v_g different from c are possible [1]. At present indirect experiments limit v_g/c to be within 0,01 of unity [1]. It is easy to see that if the arrival time of a GW on a detector is determined within few μ s [6], than coincidences between two detectors 1000 Km apart may improve the present limits of about one order of magnitude.

A two sphere array will help solving these problems, but it will also produce redundant information —of course. The latter can be used as a *local disturbance* veto on possible signals, thus improving detection probability. Let us be more specific on this point.

A coincidence experiment requires that all the detectors have signals above a given threshold at the same arrival time within a certain time window. An observatory of two spherical detectors has two advantages over other proposed observatories, such as the one constituted by six cylindrical bars [6] or that consisting of three laser

interferometers. First, in the presence of a signal the two spheres will measure the same energy. Six bars or three interferometers will have different individual orientations, and therefore cannot use this criterion for vetoing possible signals — except at larger SNR 's [25]. And second, the application of the orientational deconvolution procedure to the two individual spheres provides the additional criterium of equal source direction.

These properties of a two spheres observatory provide criteria which can be used to reduce the false event rate. Also, when accidental events from cosmic rays are considered (several thousands are expected per day in a multiton resonant mass detector operating at the quantum limit [26]), it is enough to place just one detector in an underground laboratory to reduce this false event rate to about one in three year. We can briefly summarise the essential additional features of a two-sphere observatory:

- it enables unambiguous determination of the source position in the sky,
- it enables determination of the GW propagation speed, and
- it provides powerful vetoes against local disturbances.

Obviously, an observatory with more than two antennas will further reliability of detection. Also, a network of several antennas can independently determine the direction of the wave from the time delays, which will facilitate the procedure described in the previous section.

4 Design criteria for spherical detectors

A GW detector should be designed looking to the features of potential sources giving a reasonable rate of observable events. The subject of astrophysical sources of GWs is widely discussed in the literature [2]. It is generally accepted that the most intense GWs reaching the Earth must come from dynamic deformed systems near their gravitational radius. Perhaps the most favourable source is a star collapsing across its gravitational radius in a highly non-spherical process.

Such kind of source involves a considerable mass compressed to very high density in a very short timescale. In particular, the usual assumption about *supernovæ* is that they produce a burst of radiation in a timescale characteristic of the bounce, of the order of 1 millisecond. This would result in a burst at about 1 kHz. It is possible, however, that considerable radiation from a collapse event emerges at a frequency below 1 kHz, if rotation is involved. In fact rotational effects slow down the collapse and thereby lower the dominant frequency at which the radiation comes out. The radiation amplitude produced in a galaxy a distance r from the Earth by a collapse in which an energy E is converted into GWs in a time τ , can be estimated as [27]

$$h = 1.4 \times 10^{-21} \left(\frac{E}{10^{-2} M_{\odot} c^2} \right)^{1/2} \left(\frac{\nu}{1 \text{ kHz}} \right)^{-1} \left(\frac{\tau}{1 \text{ ms}} \right)^{-1/2} \left(\frac{r}{15 \text{ Mpc}} \right)^{-1} \quad (4.1)$$

Assuming that the duration of the burst is the timescale of the rebound, i.e., about one millisecond, and that the strongest possible burst would emit the entire binding energy of a neutron star, around $0.1 M_{\odot} c^2$, then this event would produce an amplitude of 3×10^{-18} if it occurred in our Galaxy, and 3×10^{-21} if in the Virgo cluster. A more moderate and plausible event, converting to GWs $0.01 M_{\odot} c^2$, would give an amplitude of about 8×10^{-22} at 20 Mpc. The expected rate of these events is a few per year.

These numbers are relevant to our discussion of the spherical detector design. They support the design of a detector sensitive in the range of, say, 500 Hz to a few kHz. This frequency range requires large size and mass antennas. The resulting cross sections should be large enough to achieve the goal of direct detection and analysis of GWs.

The SNR ratio reported in equation (3.17) depends on the absorbed energy and on the amplifier noise temperature T_n , which is largely independent of the sphere parameters.

The energy absorbed by the detector is proportional to its cross section and to the value of the energy spectral density $F(\omega_0)$ at the sphere's resonant frequency ω_0 . The cross section depends on the detector material through the product Mv_s^2 , which must be maximised for optimum detector design.

The *speed of sound* in a solid can be expressed as $v_s^2 = Y/\rho$ where ρ is the material density and Y is the Young modulus. Then, putting $M = \rho V$, we can write:

$$Mv_s^2 = VY \quad (4.2)$$

In order to maximize the cross section and design an optimal spherical detector, two different philosophies can be followed: i) maximise the cross section at a fixed frequency, and ii) maximise cross section without specific hypotheses on the signal frequency.

We will follow i) in the case of a predicted strongly frequency dependent behaviour for the energy spectral density $F(\omega)$ emitted by particular sources, or if a xylophone of detectors at various frequencies is envisaged. In this case one can write the cross section as a function of frequency:

$$Mv_s^2 \propto \rho R^3 Y \propto \frac{Y^{5/2}}{\omega^3 \rho^{3/2}} \quad (4.3)$$

The material-dependent quantity $\chi = Y^{5/2}/\rho^{3/2}$ is reported in Table 2 for various materials, along with density and Young modulus.

Material	ρ (10^3 Kgm^{-3})	Y (GPa)	χ	$d_{1\text{KHz}}$ (m)	$M_{1\text{KHz}}$ (tons)
Al5056	2.7	70	1	2.64	26
CuBe	8.2	130	0.92	2.07	37
Mo	10.2	325	6.3	2.90	130
Nb	8.6	105	0.49	1.81	26
Ti	4.54	116	6.8	2.61	42
W	19.3	411	4.4	2.38	136

Table 2: Density, Young modulus, and specific cross section (normalised to Al5056) of some selected high Q materials. In the last two columns we report diameter and mass of spheres working at 1 kHz.

Material	d (m)	M (ton)	ν (Hz)
Al5056	3.0	38	880
CuBe	2.4	60	850
Mo	1.8	31	1608
Nb	2.6	80	696
Ti	2.53	38	1033
W	1.7	47	1432

Table 3: Fundamental resonance frequency, mass, and diameter of spherical detectors which have the same cross section

In the case ii), we can maximise the cross section by maximising the product VY . So the simple prescription is to use material having *high Young modulus* and fabricable in large volume. In Table 3 we show the diameter and mass of spheres made with the indicated selected materials, all having the same cross section.

Table 2 shows which are in principle the best GW detector materials. However, in order to use any material listed in Table 2, two practical requirements must be met. First, the material should be commercially accessible in large amounts, and second, a method for fabricating large mass spheres, preserving the high quality factor Q , should be available. We find that **these** requirements practically limit the choice of material to aluminum and **copper alloys**.

5 Conclusions

We have investigated the possibilities offered in principle by the multimode capabilities of massive resonant spheres in the search for GWs. A spherical antenna has large GW absorption cross section at the two lowest quadrupole mode frequencies, isotropic sky coverage, and can be used to determine the source direction and polarisation state of the incoming wave. Also, it adds the ability to provide in a natural way tests and vetoes on gravitation theories. The large cross section values probably make a 1 kHz resonant sphere the most sensitive detector of GW bursts in the foreseeable future. A two sphere observatory also offers the possibility of resolving the antipode ambiguity in the wave direction and of determining the speed of the waves by monitoring the arrival times at the two detectors, as well as providing strong vetoes on local disturbances.

Various questions, however, have been left aside from this investigation. Complications in the readout system may arise due to departures from spherical symmetry caused by the suspensions, transducer arrangement or internal effects; these would slightly modify the sphere's eigenmodes and eigenfrequencies. A related problem is the transfer of energy between degenerate modes (negligible, though, if the time for such transfer is small compared with the mode damping time). All these details can be experimentally studied in a smaller scale prototype.

Other technical aspects are under study, such as the design of a cryogenic apparatus capable of cooling in a reasonable time an up to 100 ton mass [28] or the determination of the sphere's fabrication technique.

Let us finally emphasise the complementarity of a GW observatory based on resonant spheres and one based on large laser interferometers. They are sensitive in different frequency ranges: 10 to 10^3 Hz for the latter [29, 30], and above 1 kHz for resonant spheres. Interferometers should be superior in determining wave forms because of their inherently broad frequency bandwidth, while spheres should perform better in determining wave direction. Both will eventually use very different technologies.

Acknowledgements

We would like to acknowledge E. Iarocci, I. Modena and G. Pizzella for continuous encouragement and support, and M. Cerdonio and S. Vitale for helpful discussions. V. Fafone, G. Mazzitelli and M. Montero have kindly helped in some of the calculations reported in this paper.

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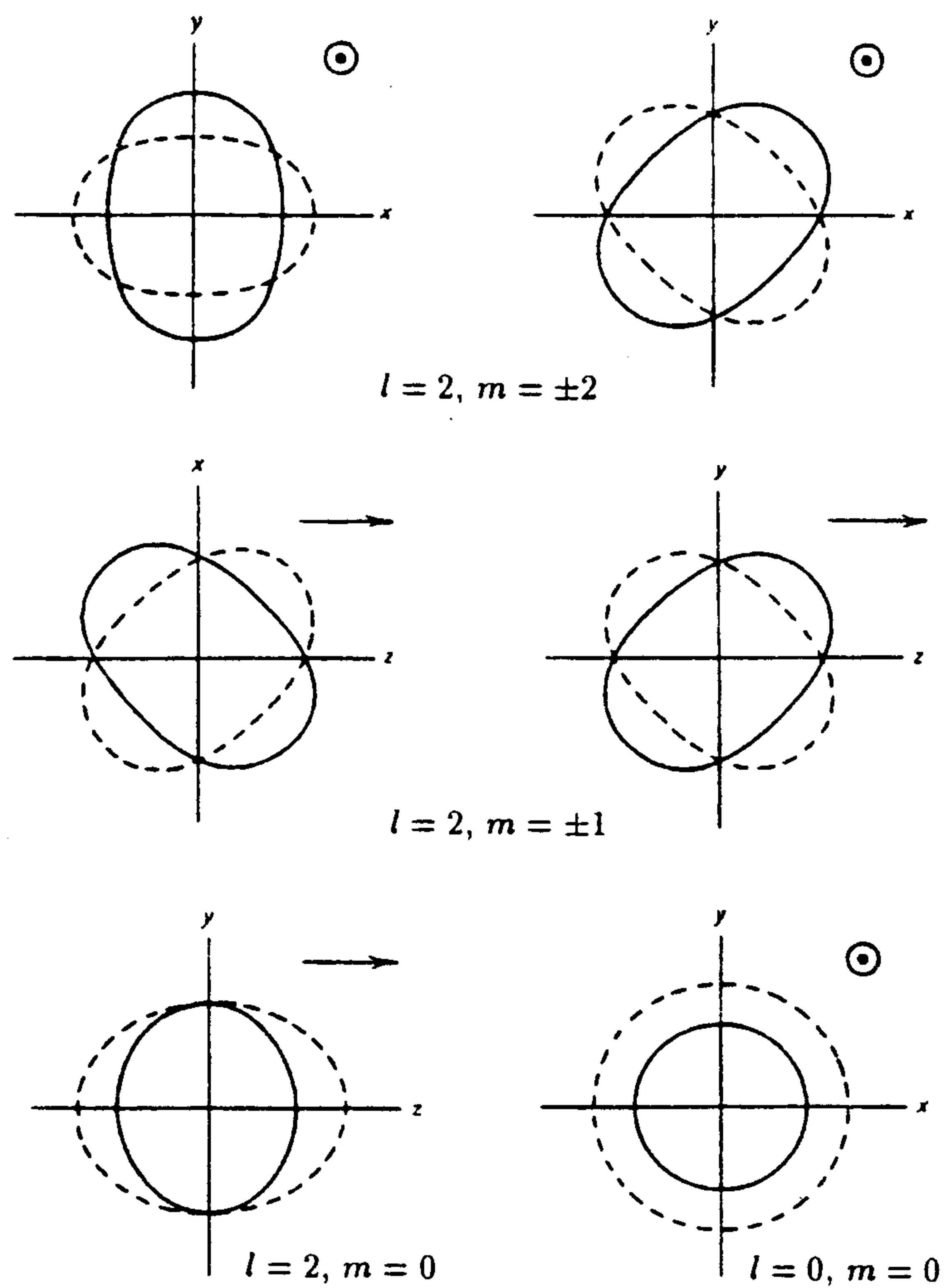


Figure 1 The six polarisation patterns associated to the most general weak metric GW. Five of these are quadrupole modes, while the sixth is a monopole. The values of l and m are displayed, along with the direction of incidence of the GW.

The elastic sphere frequency spectrum

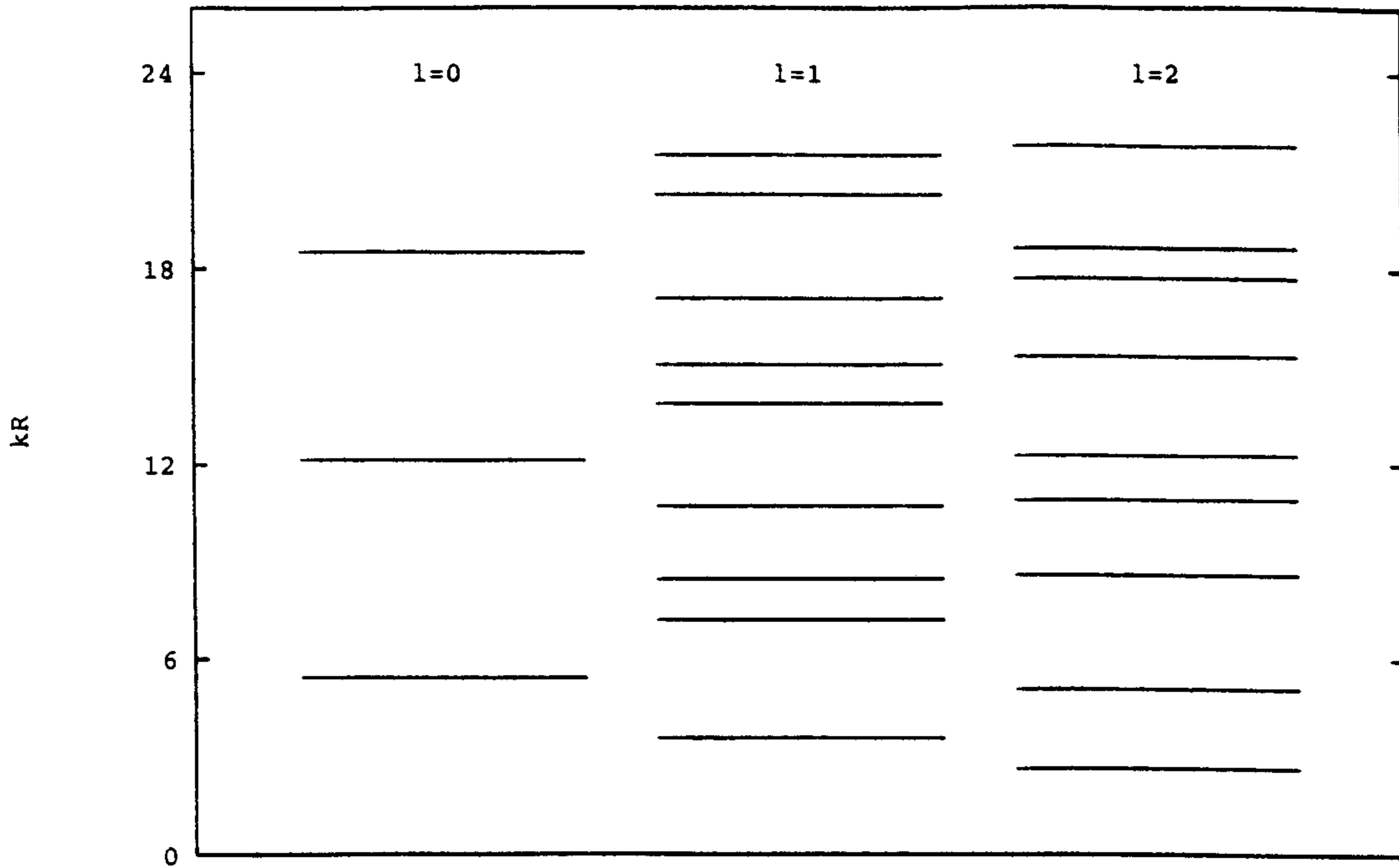


Figure 2 Line spectrum of a homogeneous sphere: the first few monopole, dipole and quadrupole modes are displayed. In ordinates we represent the eigenvalue kR , which is independent of the material of the sphere. In order to get actual frequencies, the plotted value must be multiplied by $[2\pi^2(1 + \sigma)]^{-1/2} (v_s/D)$, where σ is the Poisson ratio and D the sphere's diameter. For Al5056 this is $1060/D$, with D in metres.

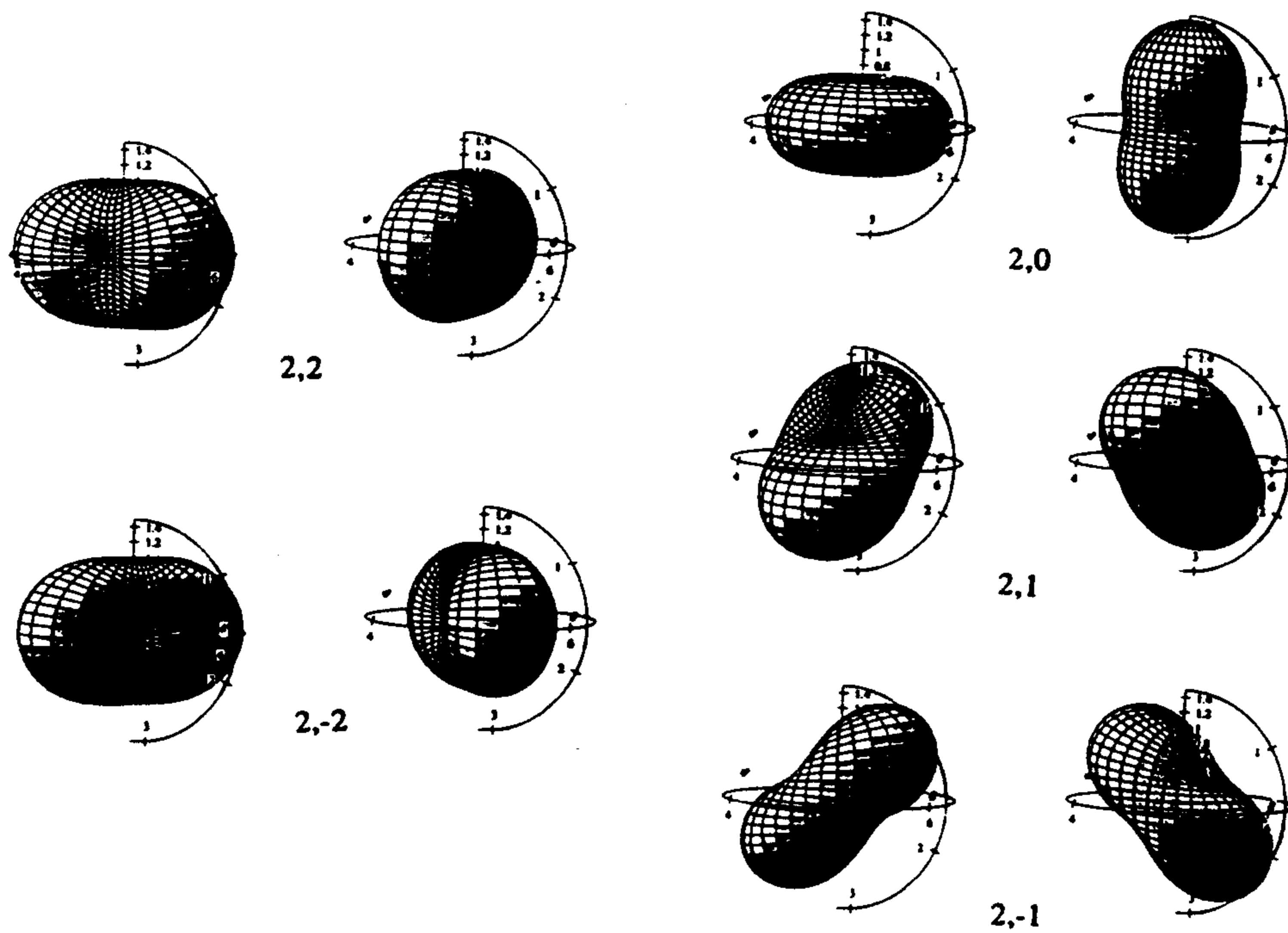


Figure 3 Three dimensional representation of the radial motion in each of the quadrupole modes of the sphere. Real spherical harmonics have been used in order to make the plot possible.

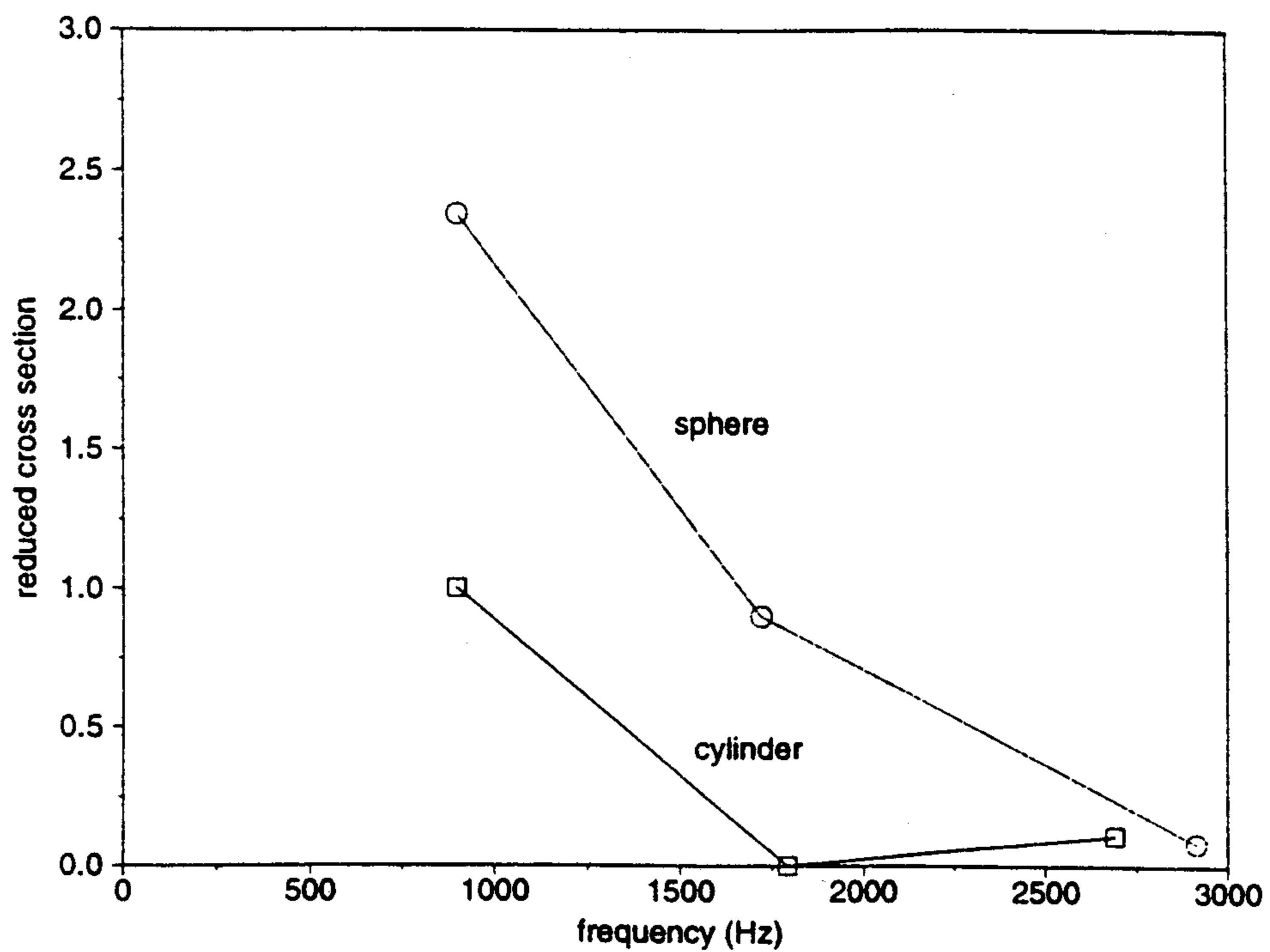


Figure 4 Reduced cross section (i.e., per unit mass) for a sphere and a cylinder having the same fundamental frequency. Note that the second sphere mode still shows a remarkably high cross section, while the third and subsequent (not shown) decay sharply.

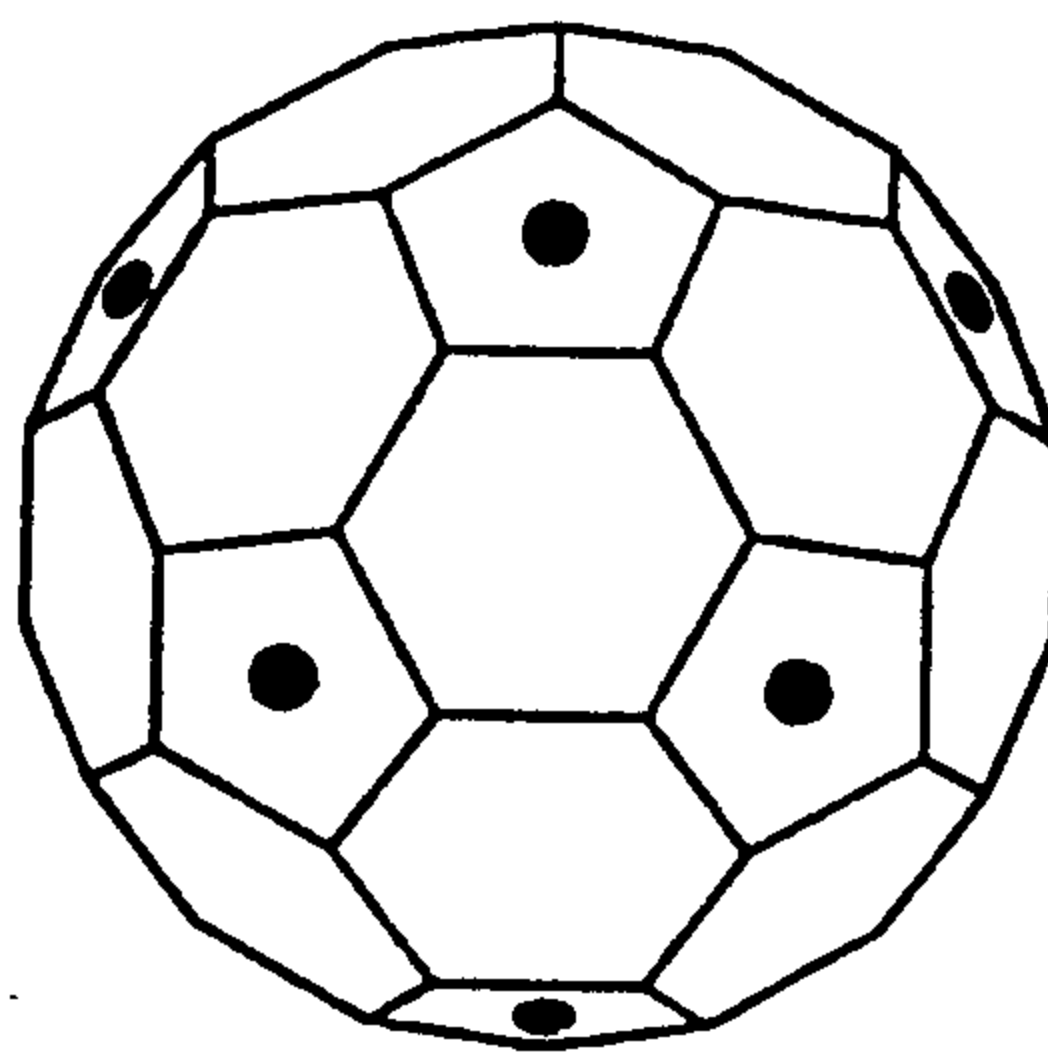


Figure 5 A truncated icosahedron, proposed as shape for a g.w. detector. The transducer locations are indicated. This configuration has the same symmetry of the dodecahedral array simplifying the problem of the orientational deconvolution of the signals.

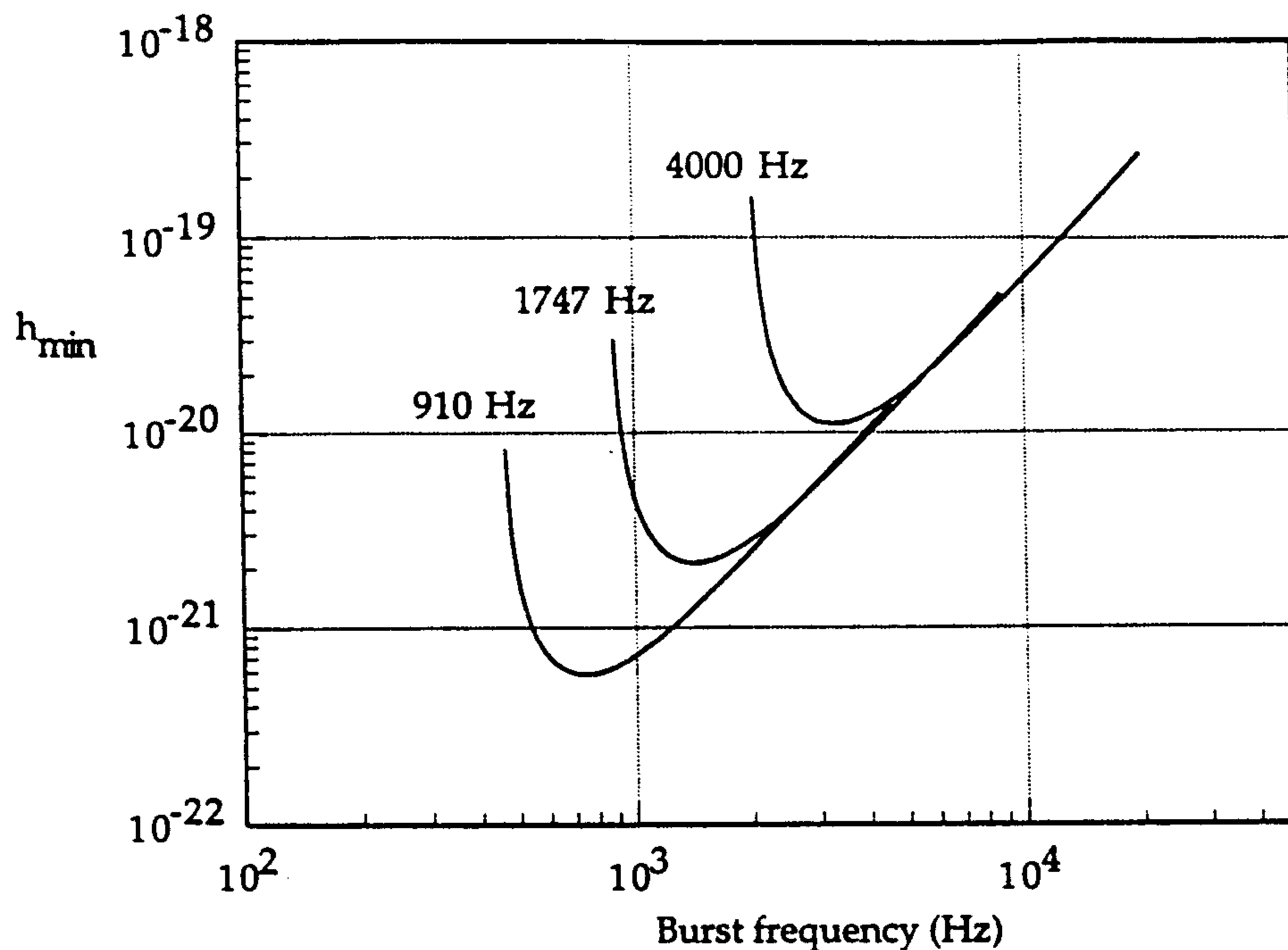


Figure 6 Sensitivity curves for Al5056 spherical antennae of different sizes and quantum limited noise. Note that the smaller spheres have their maximum sensitivity at higher frequencies but, remarkably, such sensitivity is *not* better than that of the larger spheres.

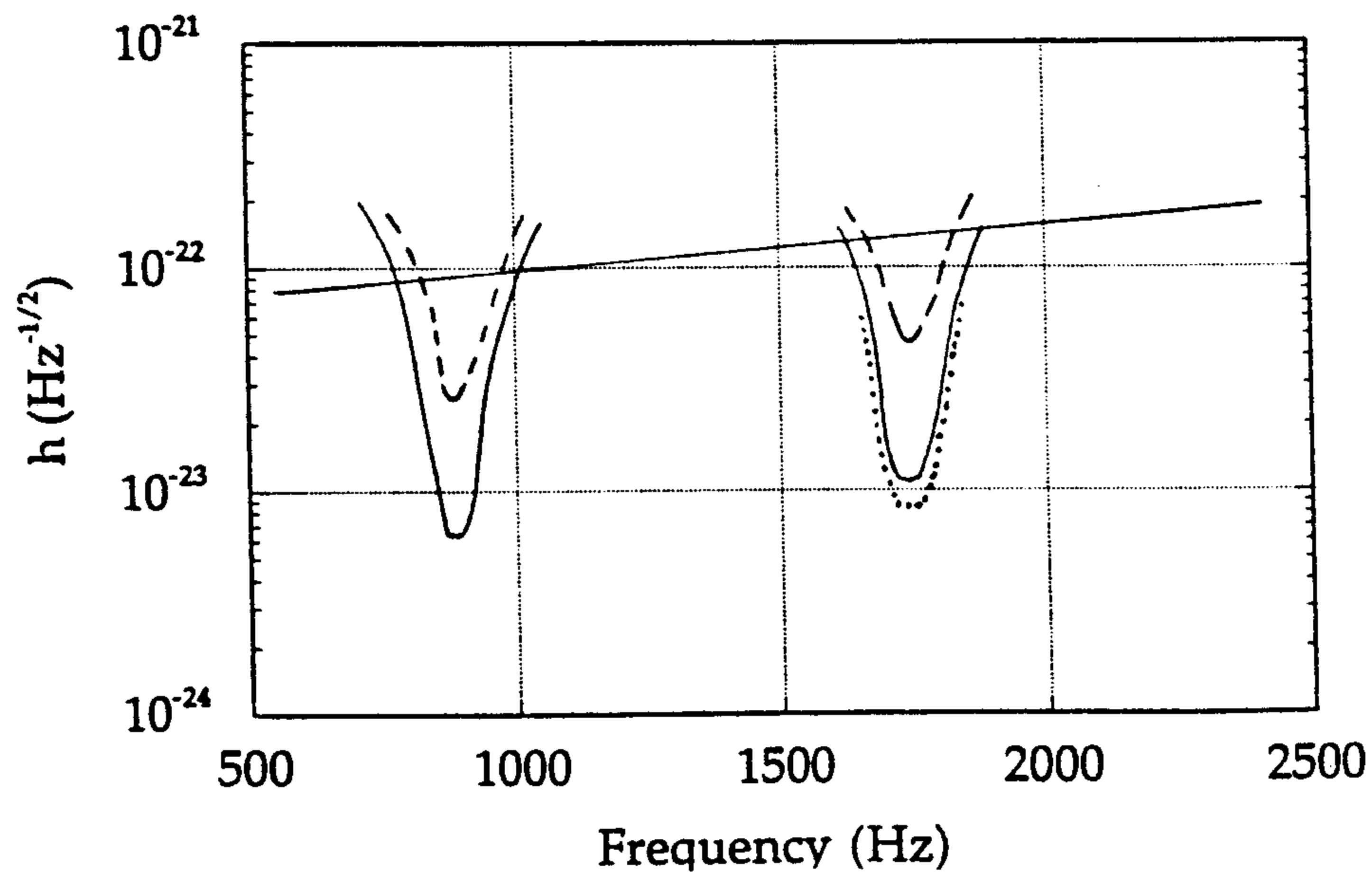


Figure 7 Calculated strain noise spectrum \tilde{h} for various detectors at the quantum limit: *solid* lines for the lowest quadrupole mode of Al5056 spherical detectors 3 and 1.56 metres diameter respectively; *dotted* line for the second quadrupole mode of the 3 metre sphere, and *dashed* line for the equivalent cylindrical bar optimally oriented. Also shown is the thin line for the first generation optimally oriented LIGO detector.