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Nilpotent Commuting Fields

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Nilpotent commuting fields

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Bilinear composites of anticommuting constituents are even elements of a Grassmann algebra, which are nilpotent commuting variables (NCV). We define an integral on these variables and start investigating the properties of nilpotent commuting scalar fields.

1. INTRODUCTION

Bilinear composites of fermionic constituents arise in many areas of physics.Well known examples are the Cooper pairs in the theory of superconductivity and nuclear physics, the density fluctuations in the Tomonaga model of the electron gas and the spin waves in ferro-antiferromagnetic metals. In Particle Physics such composites appear for instance in the framework of dynamical symmetry breaking, as well as, of course, in models of composite fields.

In a path integral formulation bilinear composites of anticommuting constituents are even elements of a Grassmann algebra, which are nilpotent commuting variables (NCV). When one is interested in correlation functions which do not involve the constituent fields but only those combinations which define the composites, one would like to be able to treat the composites themselves as independent variables. This would be particularly useful in the presence of confinement, because one could get rid of the constit

We considered a method of treating NCV as independent variables in the framework of a model of composite gauge fields with fermionic constituents [1]. It is based on a definition of integral on even elements of a Grassmann algebra which gives, when even elements are expressed in terms of odd elements, the same results as Berezin integrals on the latters. An investigation of the problem of pairing by this technique is in progress [2]. But before trying to introduce NCV in a Berezin integral defining the partition function of a relativistic theory of fermions to describe composites, we want to see to which extent these variables can be used in actual calculations. For this purpose we investigate some properties of a scalar nilpotent commuting field. This study is per se interesting, if consistent models with such a field can be constructed, because they are expected to have peculiar properties relevant to phenomenology. For instance it is well known that the ϕ^4 theory with negative coupling is perturbatively asymptotically free [3], but it has a euclidean action unbounded from below, so that its partition function is undefined. With NCV the partition function is instead well defined *irrespective* of the sign of the coupling.

Our preliminary results are that connected terms are not more difficult to evaluate than with ordinary variables, but we have *not worked out the non connected ones. Ignoring their contribution*, it turns out that a ϕ^4 theory where the Fourier components of the ϕ-field are NCV of order 1 (see below), is asymptotically free for attractive coupling. For a ϕ-field which is itself a NCV of order 1 we have studied only the propagator, which is equal to that of a selfavoiding random walk. We must emphasize that we have reported the above results only to anticipate which kind of calculations can be done easily, and not because we attach to them any phisical significance at the present stage.

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2. INTEGRAL ON EVEN ELEMENTS OF A GRASSMANN ALGEBRA

Let us start by reporting the definition of integral on NCV. First of all we should mention that these variables are characterized by the order of nilpotency, namely the smallest integer n^* for which $(NCV)^n = 0, for\ n > n^*$. It should be noted that the order of nilpotency is changed by a linear transformation. If a_1, a_2, are nilpotent of order 1, for instance, their linear combination $a_1 + a_2$ is of order 2.

For a complex NCV of order 1

$$a\ :\quad a^2 = 0,\quad aa^* = a^*a \tag{1}$$

the integral is defined according to

$$\int da^* da\, a^* a = 1 \tag{2}$$

all other integrals vanishing. If

$$a = c_1 c_2,\quad a^* = c_2^* c_1^*, \tag{3}$$

the c_i's being odd Grassmann variables, the definition (1) gives the same result as Berezin integral over the c_i's

$$\int dc_1^* dc_1 dc_2^* dc_2 c_2^* c_1^* c_1 c_2 = 1. \tag{4}$$

Notice that according to such a definition

$$\int da^* da e^{(a^* a)} = 1 \tag{5}$$

with a plus sign in the exponent.

The generalization to more degrees of freedom

$$a_h\ :\ a_h^2 = 0,\ a_h a_k = a_k a_h,\ a_h^* a_k = a_k a_h^* \tag{6}$$

is obvious and the integral is defined according to

$$\int \prod_h da_h^* da_h a_h^* a_h = 1 \tag{7}$$

all other integrals vanishing. It is easy to see that

$$\int [da^* da] e^{\sum_{h,k} a_h^* A_{h,k} a_k} = per(A) \tag{8}$$

where

$$[da^* da] = \prod_h da_h^* da_h \tag{9}$$

and per(A) is the permanent of the matrix A.

3. ϕ^4 THEORY ON EVEN ELEMENTS OF A GRASSMANN ALGEBRA

Let us consider a nilpotent commuting scalar field in euclidean space. Since the order of nilpotency is changed by Fourier transformation, we must distinguish whether the Fourier components or the field itself are NCV of given order.

3.1. Nilpotent Fourier components of order 1

The Fourier expansion of a scalar field ϕ in a box of side L is

$$\phi(x) = \frac{1}{L^2} \sum_p \tilde{\phi}(p) e^{ipx} \tag{10}$$

where the sum is over the discrete momenta of the first Brilouin zone and the Fourier transform is

$$\tilde{\phi}(p) = \frac{1}{\omega_\Lambda(p)} [a^*(p) + a(-p)]. \tag{11}$$

In the above equation

$$\frac{1}{\omega_\Lambda(p)} = (p^2 + m^2)^{-\frac{1}{2}} e^{\frac{-p^2}{\Lambda^2}} \tag{12}$$

is the inverse of the regulated wave operator appearing in the free action

$$S_0 = \int d^4x\ \phi(x)[-\Box + m^2]_\Lambda \phi(x) \tag{13}$$

and $a^*(p), a(p)$ are NCV of order 1.

Free propagators are defined by

$$< \tilde{\phi}(p_1)\tilde{\phi}(p_2) >= \frac{1}{Z_0} \int [da^* da] \tilde{\phi}(p_1)\tilde{\phi}(p_2) e^{S_0} \tag{14}$$

where

$$Z_0 = \int [da^* da] e^{S_0}. \tag{15}$$

Note the plus sign in the exponent. It is convenient to introduce the variables

$$\begin{aligned} A(p) &= a^*(p) + a(-p) \\ B(p) &= A(p)A(-p) \end{aligned} \tag{16}$$

which simplify the expression of

$$e^{S_0} = [1 + B(0)] \prod_p^* [1 + 2B(p) + 2B^2(p)]. \quad (17)$$

The star in the above equation means that the product must be taken over all directions and absolute values of $p \neq 0$, but not over its orientations. In order to evaluate correlation functions we must arrange the products of $\tilde{\phi}(p)$'s into products of B(p)'s and use the relations

$$\int [da^* da] B(0) e^{S_0} = Z_0, \quad (18)$$

$$\int [da^* da] B(p) e^{S_0} = Z_0, \;\; p \neq 0, \quad (19)$$

$$\int [da^* da] B^2(p) e^{S_0} = \frac{1}{2} Z_0, \;\; p \neq 0. \quad (20)$$

In such a way we get for the free propagator

$$< \tilde{\phi}(p_1)\tilde{\phi}(p_2) >= \delta_{p_1,-p_2} \frac{1}{\omega^2(p_1)}, \quad (21)$$

namely the expression valid for ordinary scalars. Introducing the ϕ^4 interaction, and evaluating the 4-point correlation function to one loop, we find (neglecting non connected terms) the result of the ordinary theory with negative coupling. This is due to the fact that, because of the plus sign in the exponent in Eq. 5, the counterterm has opposite sign w.r. to that of the ordinary theory with the same sign of the coupling.

3.2. Nilpotent complex scalar field of order 1

The partition function is defined according to

$$Z_0 = \int [d\phi^* d\phi] e^{(S_K + S_M)}. \quad (22)$$

The free action is split into the usual hopping term

$$S_K = a^2 \sum_x \sum_\mu \phi^*(x)[\phi(x+\mu) + \phi(x-\mu)] \quad (23)$$

where $\mu_\nu = \delta_{\mu,\nu}$, and a mass term

$$S_M = a^4 \sum_x M^2 \phi^* \phi. \quad (24)$$

Free propagators have the standard definition. Since there is no systematic way of evaluating the permanent of the wave operator, we make recourse to the hopping expansion. Neglecting non connected terms

$$< \phi^*(x)\phi(y) >= \frac{1}{Z_M} \sum_{\tau=0}^{\infty} \frac{1}{\tau!} \int [d\phi^* d\phi] [\phi^*(x)\phi(y)(S_K)^\tau e^{S_M}]_C \quad (25)$$

where the subscript C means connected and

$$Z_M = \int [d\phi^* d\phi] e^{S_M}. \quad (26)$$

It is easy to see that Eq. 25 can be rewritten

$$< \phi^*(x)\phi(y) >= \frac{1}{a^2} \sum_{\tau=0}^{\infty} (aM)^{-2(\tau+1)} r_\tau(x,y), \quad (27)$$

where $r_\tau(x,y)$ is the number of selfavoiding paths of τ links joining the sites x, y. This result can be extended to n-point correlation functions, which turn out to be equal to those of the self-avoiding random walk [4].

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