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Radiative Corrections to $\pi_{\ell 2}$ and $K_{\ell 2}$ Decays

Markus Finkemeier

INFN – Laboratori Nazionali di Frascati, P.O. Box 13, I–Frascati (Roma) Italy

Abstract

We reexamine the radiative corrections to π_{l2} and K_{l2} decays. We perform a matching calculation, using a specific model with vector meson dominance in the long distance part and the parton model in the short distance part. By considering the dependence on the matching scale and on the hadronic parameters, and by comparing with model independent estimates, we scrutinize the model dependence of the results. For the pseudoscalar meson decay constants, we extract the values $f_\pi = (92.1 \pm 0.1)$ MeV and $f_K = (112.4 \pm 0.9)$ MeV. For the ratios R_π and R_K of the electronic and muonic decay modes, we predict $R_\pi = (1.2354 \pm 0.0002) \cdot 10^{-4}$ and $R_K = (2.472 \pm 0.001) \cdot 10^{-5}$.

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1 Introduction

The study of π_{l2} decays has long history, but it is still a current subject both from the theoretical and from the experimental point of view. Historically, these decays played an important role in understanding the structure of the weak interaction. The strong dynamical suppression of the electronic decay mode with respect to the muonic one [1],

$$R_\pi^{(exp)} := \frac{\Gamma(\pi \to e\nu_e(\gamma))}{\Gamma(\pi \to \mu\nu_\mu(\gamma))} = (1.230 \pm 0.004) \cdot 10^{-4} \tag{1}$$

proves that the interaction is (at least very predominantly) of the (V, A) type.

Radiative corrections to R_π are rather large, $O(-4\%)$, and so their understanding is important. They have first been considered by Berman and by Kinoshita [2, 3], using a model with an effective point pion. Later, there have been numerous attempts to improve the simple point meson calculation using various models and leading to a range of predictions, see for example [4, 5].

From a modern point of view, it is important to understand which of these calculations give reasonable approximations to the standard model. In a recent analysis, Marciano and Sirlin [6] have pointed out that some of the models used can not be considered as realistic and that the ratio R_π can in fact be predicted with a precision of the order of $O(10^{-4})$. Therefore measurements of π_{l2} decays offer an important possibility of a low energy precision test of the standard model.

The aim of the present paper is twofold: On the one hand, we scrutinize the issue of model dependent corrections, and on the other hand, we extend the analysis to K_{l2} decays.

In [6], the authors separate the loop integration into a long and a short distance part, which they crudely match at a scale $\mu_{cut} = m_\rho$. In the long distance part, there are the effective point pion contribution and hadronic structure dependent corrections. For the hadronic structure dependent corrections, the authors use the calculation by Terent'ev [4], who determined the the leading logarithms in m_ρ. These leading terms are model independent. For the remaining model dependent terms they only give rough order of magnitude estimates. In the short distance correction, they consider the leading logarithm $2\alpha/\pi \ln(m_Z/\mu_{cut})$, sum up leading terms of $O(\alpha^n)$ by using the renormalization group and add a leading QCD correction. These leading (model independent) short distance corrections are independent of the lepton mass and so cancel in the ratio R_π.

In this paper we will explicitely calculate the model dependent terms using a realistic phenomenological model. Firstly, this allows us to predict an improved central value, which includes terms which have been missing before. Secondly, we estimate the uncertainty of the model dependent terms by considering the dependence on the matching scale and on the hadronic parameters. Thirdly, we find that the relative size of the model dependent terms is very small. Therefore the uncertainty of the theorectical prediction due to model dependence is in fact extremely small.

Essentially we use the approach developed in [7] to calculate radiative corrections to tau decays. For the long distance part, we use a phenomenological model which has pseudoscalar mesons (π, K), vector (ρ, ρ', K^*, ...) and axial vector (a_1, K_1) resonances as explicit degrees of freedom. This allows us to push the matching scale up to $\mu_{cut} = (1 \ldots 2)$ GeV, rendering the calculation of the short distance part more reliable. In the short distance corrections, we include lepton mass dependent corrections, which do not cancel in R_π. Its is obvious that these terms are very small, as they are suppressed by m_μ/μ_{cut}. However, in view of the high precision of the theorectical prediction one would like to know exactely how small they are. We show that there is a small leading model independent contribution. The remaining contribution, which depends on the pion wave function and therefore is model dependent, is neglibigly small for any choice of the pion wave function.

Regarding K_{l2} decays, the experimental precision tends to be less good than in pion decays. However, because of the larger mass of the kaon, effects of non standard model physics might be enhanced by a factor m_K/m_π. Therefore the consideration of K_{l2} decays is also very interesting.

Although we use essentially the model of [7], there are two differences in the analysis. Firstly, in this paper we consider $SU(3)$ flavour symmetry breaking effects which have been neglected in [7]. In the calculation of the tau decay $\tau \to K\nu_\tau(\gamma)$, these could savely be neglected in view of the large overall hadronic uncertainties, but this is not obvious in the case of K_{l2} decays. Secondly, the analysis of the uncertainty of the theoretical predictions differs. In π_{l2} and K_{l2} decays, hadronic and matching uncertainties cancel in the ratios of electronic and muonic decay modes to a very large extent. The resulting precison of the order $O(\alpha)$ prediction is such that leading and next-to-leading $O(\alpha^2)$ corrections have to be considered.

The decay rates $\pi \to \mu\nu_\mu$ and $K \to \mu\nu_\mu$ are also being used to extract the pion and kaon decay constants f_π and f_K, which play an important rôle in chiral perturbation theory [8]. Therefore we will also reconsider these parameters within our framework.

This paper is organized as follows: In the next section we will describe the structure of the radiative corrections. In Sec. 3 we will describe the general approach and the specific parametriztions we use. In Sec. 4, we will then give the numerical results and our predictions. A short summary is given in Sec. 5.

2 Structure of the Radiative Corrections

The decay rate for the semileptonic decay of a pseudoscalar meson M ($M = \pi, K$) is given by

$$\Gamma\Big(M \to (l\nu_l + l\nu_l\gamma + l\nu_l\gamma\gamma + \ldots)\Big) = \frac{G_F^2 V_M^2}{4\pi} f_M^2 m_M m_l^2 \left(1 - \frac{m_l^2}{m_M^2}\right)^2 \left[1 + \frac{\delta\Gamma}{\Gamma_0}\right] . \qquad (2)$$

where G_F is the Fermi coupling constant extracted from muon decay, V_M denotes the respective CKM matrix element,

$$\begin{aligned} V_\pi &= V_{ud} \\ V_K &= V_{us} \end{aligned} \qquad (3)$$

and f_M is the meson decay constant (our convention is such that at the lowest order, $f_\pi \approx$ 93 MeV). The radiative correction $\frac{\delta\Gamma}{\Gamma_0}$ starts at the order $O(\alpha)$ (Γ_0 denotes the Born amplitude).

Because of the infrared divergences, in a calculation of the order α^n, inclusive decay rates into final states with up to n additional photons have to be considered. As we will perform the calculation to the order $O(\alpha)$, we will have to consider one loop corrections to $M \to l\nu_l$ and the radiative decays $M \to l\nu_l\gamma$ at tree level. At this point one can either chose to include only soft photons, using some upper photon energy cut, or to include all photons, soft and hard ones.

The amplitude of the radiative decays can be separated into internal bremsstrahlung (IB) and hadronic structure dependent radiation (SD) [9, 10]

$$\mathcal{M}(M \to l\nu_l\gamma) = \mathcal{M}_{IB} + \mathcal{M}_{SD} \tag{4}$$

The IB contribution, which consists of photon radiation off the lepton, the pion pole contribution and a seagull required by gauge invariance, corresponds to the lowest order $O(P^2)$ in chiral perturbation theory. It is identical to what is obtained assuming a pointlike structureless meson. Thus is completely determined by QED and contains no free parameters beyond f_M. The SD amplitude, on the other hand, involves two form factors F_V and F_A, which parametrize the effects of non perturbative strong interactions and which start at the order $O(P^4)$ in chiral perturbation theory.

As regards the question of hard photons in the inclusive decays rates, we adopt the following convention, which is essentially in accord with the one used by experiments. In the case of pion decays, we include *all* photons in the corrections. However, we will quote separately the contribution arising from hadronic structure dependent radiation Γ_{SD+INT} (where SD is the structure dependent amplitude squared and INT denotes the interference of internal bremsstrahlung and structure dependent radiation). This SD + INT part is mainly responsible for the radiation of hard photons. In the case of kaon decays, we include the full IB contribution, which is strongly dominated by very soft photons, but we exclude completely the structure dependent Γ_{SD+INT} contribution, which is strongly dominated by hard photons.

Note that for $K \to \mu\nu_\mu\gamma$ this structure dependent contribution is completely negligible, but for the electronic mode it is extremely large, $\Gamma_{SD+INT}(K \to e\nu_e\gamma) \approx \Gamma_0(K \to e\nu_e)$. This is due to the fact that the IB radiation amplitude is strongly helicity suppressed, being proportional to the Born amplitude. The SD radiation, however, does not vanish for $m_e \to 0$. Therefore it is usefull to consider the SD radiation as a separate decay mode and not to included it in the radiative corrections to $K \to e\nu_e$.

Thus we will calculate

$$\begin{aligned} \delta\Gamma(\pi \to l\nu_l(\gamma)) &= \delta\Gamma_{virtual} + \Gamma_{IB} + \Gamma_{INT} + \Gamma_{SD} \\ \delta\Gamma(K \to l\nu_l(\gamma)) &= \delta\Gamma_{virtual} + \Gamma_{IB} \end{aligned} \tag{5}$$

where $\delta\Gamma_{virtual}$ denotes the virtual corrections, and the rates Γ_{IB}, Γ_{INT} and Γ_{SD} for the radiative decay are integrated over the full phase space.

Of course it is not possible to tell definitely whether a radiated photon is due to internal bremsstrahlung or to structure dependent radiation. However, if suitable experimental cuts are used, which put a small upper limit onto the photon energy, the measured rate of $K \to e\nu_e(\gamma)$ will include only a very small SD + INT background, and only very little of the IB part will have been discarded. Using the predicted differential distributions [9, 10], the SD +INT background can be subtracted and the missing IB part added. Because of the smallness of this correction, it does not give rise to any important uncertainties.

While for the pion decay rates, the particle data book [1] states very clearly that all photons are included, its statements are not so clear for the kaon decays. Still, from reading the original papers such as [11], we believe that our convention for kaon decays comes close to the one used in the extraction of the experimental data. With increasing experimental precision, it will become important that experimentalists state very clearly how cuts and corrections have been performed.

Note that beyond the order $O(\alpha^0)$, the definition of the pseudoscalar decay constants f_M is no longer unambigous. One could decide to include certain parts of the radiative correction $\delta\Gamma/\Gamma_0$ into f_M by definition.

However, we decide to factor out *all* $O(\alpha)$ effects from f_M, no matter whether they are process dependent or process independent, or whether they are long or short distance corrections. Note that this definition differs from the one used by Holstein [12], who includes process dependent terms proportional to $\ln(m_\pi/m_\mu)$ in the defintion of f_π (cf. the discussion in [6]). However, our definition is identical to the one used in [6].

All theses ambiguities are of course due to the fact that f_π and f_K are not observables. Theses ambiguities cancel in the ratios R_M

$$R_M := \frac{\Gamma(M \to e\nu_e(\gamma))}{\Gamma(M \to \mu\nu_\mu(\gamma))} = \frac{m_e^2}{m_\mu^2}\left(\frac{m_M^2 - m_e^2}{m_M^2 - m_\mu^2}\right)^2 \left(1 + \delta R_M\right) \tag{6}$$

of electronic and muonic decay modes, which can be predicted very precisely. These radiative corrections δR_π and δR_K will be the main subject of interest in this paper.

3 Parametrization of the Amplitudes

We will be brief here and refer to [7] for details.

In the case of the radiative decays $\pi \to l\nu_l$, we use the following parameterizations of the hadronic structure dependent form factors $F_V^{(\pi)}$ and $F_A^{(K)}$:

$$\begin{aligned} F_V^{(\pi)}(t) &= \frac{F_V^{(\pi)}(0)}{1+\lambda+\mu}\left[\mathrm{BW}_\rho(t) + \lambda \mathrm{BW}_{\rho'}(t) + \mu \mathrm{BW}_{\rho''}(t)\right] \\ F_A^{(\pi)}(t) &= F_A^{(\pi)}(0)\mathrm{BW}_{a_1}(t) \end{aligned} \tag{7}$$

where

$$\begin{aligned} F_V^{(\pi)}(0) &= \frac{m_\pi}{4\sqrt{2}\pi^2 f_\pi} = 0.0270 \\ F_A^{(\pi)}(0) &= 0.0116 \pm 0.0016 \end{aligned} \tag{8}$$

$F_V^{(\pi)}(0)$ has been obtained from the ABJ anomaly [13] and $F_A^{(\pi)}(0)$ from the measurement of $\pi \to e\nu_e\gamma$ decays..

$\mathrm{BW}_X(t)$ denotes a Breit-Wigner propagator amplitude, normalized such that $\mathrm{BW}_X(t = 0) = 1$, either with energy dependent widths calculated from the relevant phase space

$$\mathrm{BW}_X(t) = \frac{m_X^2}{m_X^2 - t - i m_X \Gamma_X(t)} \tag{9}$$

or alternatively using a form with fixed width (for details see [7]).

The relative contributions of the higher radial exitations in the vector form factor have been determined in [14] by using four experimental and theoretical constraints:

$$\lambda = 0.136; \qquad \mu = -0.051 \tag{10}$$

In order to estimate the model dependence we vary λ and μ around these central values.

In the case of the radiative kaon decays $K \to l\nu_l\gamma$, we use the following form factor parametrizations:

$$\begin{aligned} F_V^{(K)}(t) &= F_V^{(K)}(0)\mathrm{BW}_{K^*}(t) \\ F_A^{(K)}(t) &= F_A^{(K)}(0)\mathrm{BW}_{K_1}(t) \end{aligned} \tag{11}$$

where

$$\begin{aligned} F_V^{(K)}(0) &= 0.0955 \\ F_A^{(K)}(0) &= 0.0525 \pm 0.010 \end{aligned} \tag{12}$$

Here $F_V^{(K)}(0)$ has been obtained from flavour symmetry and the anomaly and $F_A^{(K)}(0)$ from this value for $F_V^{(K)}(0)$ and the measurement of the sum [1].

In order to estimate the model dependence we also compare with a parametrization of $F_V^{(K)}$ which includes small admixtures of $K^*(1410)$ and $K^*(1680)$.

In order to calculate the virtual corrections we separate the loop integration into a long and a short distance part by splitting the photon propagator

$$\frac{1}{k^2-\lambda^2} = \underbrace{\frac{1}{k^2-\lambda^2}\frac{\mu_{cut}^2}{\mu_{cut}^2-k^2}}_{\text{"long distance"}} + \underbrace{\frac{1}{k^2-\mu_{cut}^2}}_{\text{"short distance"}} \tag{13}$$

using a matching scale $\mu_{cut} = (1\ldots2)$ GeV.

The long distance part, involving a regulated photon propagator, is calculated using a phenomenological model where mesons are the relevant degrees of freedom. The short distance part, involving a massive photon propagator, is calculated using the parton model.

To calculate the long distance corrections, we start from the amplitudes obtained with an effective pointlike meson. These amplitudes are good approximations for very small momentum transfers only. Consider for example the amplitude V^μ for the coupling of a photon to two pions. In the point meson (P.M.) approximation it is given by

$$V^\mu(\pi^+(p)\pi^-(p') \to \gamma)^{(P.M.)} = ie(p-p')^\mu \tag{14}$$

However, this coupling *defines* the electromagnetic form factor F_π of the pion via

$$V^\mu(\pi^+(p)\pi^-(p') \to \gamma) =: ieF_\pi[(p+p')^2](p-p')^\mu \tag{15}$$

Therefore we modify the effective point pion diagrams by multiplying this coupling by F_π. This modification in turn determines by gauge invariance the appropriate modification of the weak-electromagnetic seagull coupling $\pi\gamma W$.

The form factor $F_\pi(Q^2)$ is quite well known experimentally in the relevant $\sqrt{Q^2}$ region below $(1\ldots2)$ GeV. We use a parametrization obtained in [15]

$$F_\pi(t) = \frac{1}{1+\sigma+\rho}[\mathrm{BW}_\rho(t) + \sigma\mathrm{BW}_{\rho'}(t) + \rho\mathrm{BW}_{\rho''}(t)] \tag{16}$$

with

$$\sigma = -0.1; \qquad \rho = -0.04 \tag{17}$$

We also vary these parameters σ and ρ around these central values.

Analogously in the kaonic case, we modify the point kaon coupling by multiplying by it F_K:

$$V^{\mu}(K^+(p)K^-(p') \to \gamma) \longrightarrow ieF_K[(p+p')^2](p-p')^{\mu} \tag{18}$$

In [7] a parametrization of F_K with a simple ρ dominance

$$F_K(t) = \mathrm{BW}_{\rho}(t) \tag{19}$$

was used. However, this assumes exact $SU(3)$ flavour symmetry, $m_\rho = m_\omega = m_\Phi$. We will now drop this assumption. Thus we have to consider the relative contributions of the ρ, the ω and the Φ to the form factor F_K. Assuming ideal mixing, $\Phi = (s\bar{s})$, we obtain

$$F_K(t) = \frac{1}{2}\mathrm{BW}_{\rho}(t) + \frac{1}{6}\mathrm{BW}_{\omega}(t) + \frac{1}{3}\mathrm{BW}_{\Phi}(t) \tag{20}$$

which we will use in the present paper.

In additon to the modified effective point meson diagrams, there are loop diagrams which are obtained from the hadronic structure dependent radiation (SD) by contracting the emitted photon with the lepton. If k^2 is small, where k is the momentum of the virtual photon, the relevant form factors H_V and H_A in these hadronic structure dependent loops will obviously be identical to F_V and F_A determining the radiative decay. However, they can additionally depend on k^2. In the case of the pion decay, we adopt the following ansatz with double vector meson dominance:

$$\begin{aligned} H_V^{(\pi)}(k,p) &= \mathrm{BW}_{\omega}(k^2)F_V^{(\pi)}[(k-p)^2] \\ H_A^{(\pi)}(k,p) &= \mathrm{BW}_{\rho}(k^2)F_A^{(\pi)}[(k-p)^2] \end{aligned} \tag{21}$$

where p is the momentum of the decaying pion. There is no experimental information on these form factors for $k^2 \neq 0$, and certainly not on the contribution of higher radial exitations in the vector meson dominance of the k^2 dependence. Therefore our standard choice here is to use only use the lowest resonances ρ and ω. However, we also compare with the results obtained using small admixtures of the next two higher radials. Furthermore, we compare with the results obtained using a single vector meson dominance version:

$$\begin{aligned} H_V^{(\pi)}(k,p) &= F_V^{(\pi)}[(k-p)^2] \\ H_A^{(\pi)}(k,p) &= F_A^{(\pi)}[(k-p)^2] \end{aligned} \tag{22}$$

However, we find that this leads to an unacceptably large dependence of the radiative correction on the matching scale μ_{cut} and therefore can be excluded.

We will use the corresponding ansatz for the kaonic case:

$$\begin{aligned} H_V^{(K)}(k,p) &= \mathrm{BW}_V(k^2)F_V^{(K)}[(k-p)^2] \\ H_A^{(K)}(k,p) &= \mathrm{BW}_V(k^2)F_A^{(K)}[(k-p)^2] \end{aligned} \tag{23}$$

where we assume $m_V = m_\rho = m_\omega = m_\Phi$. We do not calculate $SU(3)$ flavour symmetry breaking effects here. This will be justified below by the observation that the dependence of the result on m_V is very small.

In order to obtain the short distance corrections, we calculate the one-loop corrections $\delta\mathcal{A}$ to the operator $\mathcal{A}_0 = [\bar{u}_\nu \gamma^\mu \gamma_- u_l]\,[\bar{u}_d \gamma_\mu \gamma_- u_u]$ (and similarly for $d \to s$ in the case of the kaon). Neglecting all masses except for m_l and μ_{cut}, we obtain the correction

$$\left(\frac{\delta\Gamma}{\Gamma_0}\right)_{short\ dist.} \approx \frac{2\alpha}{\pi}\frac{1}{m_l^2 - \mu_{cut}^2}\left(m_l^2 \ln\frac{m_Z}{m_l} - \mu_{cut}^2 \ln\frac{m_Z}{\mu_{cut}}\right) \tag{24}$$

which has to be compared to Sirlin's logarithm $2\alpha/\pi \ln(m_Z/\mu_{cut})$ [16]. Of course, if we do not neglect m_μ, we should also take the meson mass m_M into account. However, the contributions depending on m_M cancel in the ratios R_M, whereas in the radiative correction to the decay rates themselves, μ_{cut} dominates anyway.

Note that for this leading logarithm, the correction to the quark level short distance amplitude $\delta\mathcal{A}$ is proportional to the Born amplitude $\mathcal{A}_0$. Thus the same logarithm is involved in the corrections to the hadronic amplitude without any model dependence resulting from hadronization.

While in the correction to the individual decay rates $M \to l\nu_l(\gamma)$ this leading logarithm dominates the short distance correction, it depends only very little on the lepton mass and thus cancels almost completely in the ratios δR_M. Therefore in the case of these ratios we go beyond the leading logarithm and calculate the full one-loop short distance correction.

The complete result for $\delta\mathcal{A}$ is no longer proportional to the Born amplitude $\mathcal{A}_0$, and furthermore it depends on the relative momentum of the two quarks. Therefore we project onto the $J^P = 0^-$ component and integrate over the relative momentum $u \times p$ of the quarks in the infinite momentum frame ($u = -1 \ldots +1$). The result can be written in the form

$$\left(\delta R_M\right)_{short\ dist.} = \frac{3}{2f_M}\int_{-1}^{+1} du\, \Phi_M(u) r_M(u) \tag{25}$$

Here $\Phi_M(u)$ is an unknown parton distribution function (pion or kaon wave function), whereas $r_M(u)$ is calculated from the short distance diagrams. We find that $r_\pi(u)$ and $r_K(u)$ depend only very little on u, and we can approximate them by their values at $u = 0$, where the wavefunction is presumably peaked:

$$\left(\delta R_M\right)_{short\ dist.} \approx r_M(u=0)\frac{3}{2f_M}\int_{-1}^{+1} du\, \Phi_M(u) = r_M(0) \tag{26}$$

where the last equation follows from a sum rule [17].

4 Numerical Results

Adding up long and short distance corrections, we obtain the full radiative correction. This depends on the choice of the matching scale μ_{cut} and on the hadronic parameters.

In Fig. 1 we display the correction to the decay rate $\Gamma(\pi \to \mu\nu_\mu(\gamma))$ in variation with μ_{cut}, using three different choices for the hadronic parameters. The solid line (I) corresponds to the central values given above. The dashed (II) and the dotted (III) lines are obtained by varying the hadronic parameters, viz. $F_A(0)$, the relative contributions of higher radial exitations in F_V, F_π and H_V and the width of the a_1. For the dotted curve (III), we have furthermore used the single vector dominance form of H_V and H_A according to (22) instead of the double domincance form of (21). It can be seen clearly that the single dominance form (III) leads to an unacceptably large dependence of the result on the matching scale μ_{cut} and therefore can be excluded. With the double dominance form, however, the dependence is rather moderate, indicating that our phenomenological model for the long distance part is indeed rather reasonable.

We choose a relatively high value of $\mu_{cut} = 1.5$ GeV as a central value for the matching scale, because we have included not only the lightest resonances (ρ, a_1), but also the radial

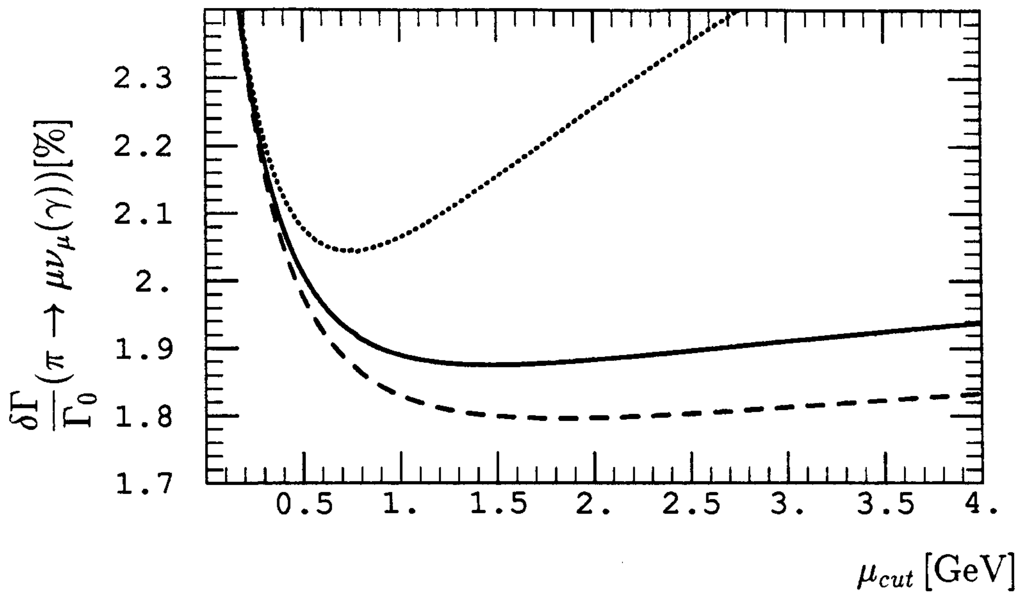

Figure 1: Radiative correction to $\Gamma(\pi \to \mu\nu_\mu$, using different choices for the hadronic parameters: Standard choice (I, solid) and variations (II, dashed and III, dotted)

excitations $\rho(1450)$ and $\rho(1700)$ as explicit degrees of freedom in our model. Thus we find from Fig. 1 the following $O(\alpha)$ correction to the decay rate

$$\frac{\delta\Gamma}{\Gamma_0}(\pi \to \mu\nu_\mu(\gamma)) = (1.88 \pm 0.04 \pm 0.08)\% + O(\alpha^2) + O(\alpha\alpha_s)$$

$$= (1.88 \pm 0.09)\% + \cdots \tag{27}$$

The first error given (0.04%) is the matching uncertainty, obtained by varying μ_{cut} by a factor of two (0.75...3 GeV), the second one (0.08%) is the uncertainty from the hadronic parameters.

A few comments are in order:

1. As we have already states above, this radiative correction is not a physical observable and therefore is not defined unambigously. The definition we have adopted is to include *all* $O(\alpha)$ corrections in the number 1.88% given. Part of this might be absorbed into f_π by definition, but we choose not to do so.

2. We employ m_Z as an ultra-violet cut-off for the short distance corrections, according to the general theorems by Sirlin [18] on short-distance electroweak corrections to semileptonic processes. But this implies that there is an arbitrariness of the order of $\alpha/(2\pi) \times O(1) \approx 0.1\%$ in the definition of the radiative correction, because a change of the cut-off scheme would induce a change of the result of this order. Note that the error of the $O(\alpha)$ correction which we have determined is of the same order of magnitude as this inherent ambiguity.

3. The error $\pm 0.09\%$ quoted above is the uncertainty of the $O(\alpha)$ correction only. In [6] the authors have summed up the leading $O(\alpha^n)$ corrections for the dominant contribution $2\alpha/\pi \, ln(m_Z/\mu_{cut})$ in the short distance part, using the renormalization group. This leads to an enhancement of the short distance correction of 0.13%. Furthermore they considered the leading QCD short distance correction, which decreases the short distance part by −0.03%. Similar $O(\alpha^n)$ effects should be considered in the long distance part.

Taking into account these higher order short distance corrections, and considering the uncertainties discussed above, we will use the following value in order to extract f_π:

$$\frac{\delta\Gamma}{\Gamma_0}(\pi \to \mu\nu_\mu(\gamma)) = (2.0 \pm 0.2)\% \tag{28}$$

We use the following input parameters [1] to extract f_π:

$$\begin{aligned} \Gamma(\pi \to \mu\nu_\mu(\gamma)) &= (2.5284 \pm 0.0023) \cdot 10^{-14}\,\text{MeV} \\ G_F &= (1.16639 \pm 0.00002) \cdot 10^{-5}\,\text{GeV}^{-2} \\ |V_{ud}| &= 0.9744 \pm 0.0010 \end{aligned} \tag{29}$$

With the radiative correction given in (28), we find

$$f_\pi = (92.14 \pm 0.09 \pm 0.09)\,\text{MeV} = (92.1 \pm 0.1)\,\text{MeV} \tag{30}$$

(The first error, ± 0.09, is due to V_{ud}, and the second one due to the radiative correction.) This is to be compared to the result obtained in [6]. They used a value for V_{ud} of $|V_{ud}| = 0.9750 \pm 0.0007$ instead of $|V_{ud}| = 0.9744 \pm 0.0010$, which we use. Transscribing their value for our V_{ud}, and dividing by $\sqrt{2}$ to translate into our convention for f_π, their results becomes:

$$f_\pi = (92.47 \pm 0.09 + 0.11 C_1)\,\text{MeV} \tag{31}$$

(the first error ± 0.09 is again due to V_{ud}), where they estimate C_1 to be in the range

$$C_1 = 0.0 \pm 2.4 \tag{32}$$

Our result in (30) implies $C_1 = -3.0 \pm 0.8$, which agrees with the estimate in [6] within the error bars.

Let us now come to the prediction for the ratio R_π of the electronic and muonic decay modes of the pion. In contrast to f_π, this is a physical observable and therefore free from ambiguities in its definition. In Fig. 2 we display the radiative correction δR_π of the ratio using the same standard parameter set (I) and variations (II) and (III) we used above for $\Gamma(\pi \to \mu\nu_\mu(\gamma))$. From this, we obtain the $O(\alpha)$ correction

$$\delta R_\pi = -(3.794 \pm 0.019 \pm 0.007)\% + O(\alpha^2) = -(3.794 \pm 0.020)\% + O(\alpha^2) \tag{33}$$

where the first error (0.019%) gives the matching uncertainty, estimated by varying μ_{cut} from 0.75 up to 3 GeV, and the second error (0.007%) arises from the uncertainties in the hadronic parameters.

However, comparing Figs. 1 and 2, the question arises if we underestimate the true model dependence by considering by considering the matching scale dependence of the ratio instead of that of the individual decay rates. From Fig. 1 it is obvious that the single vector meson

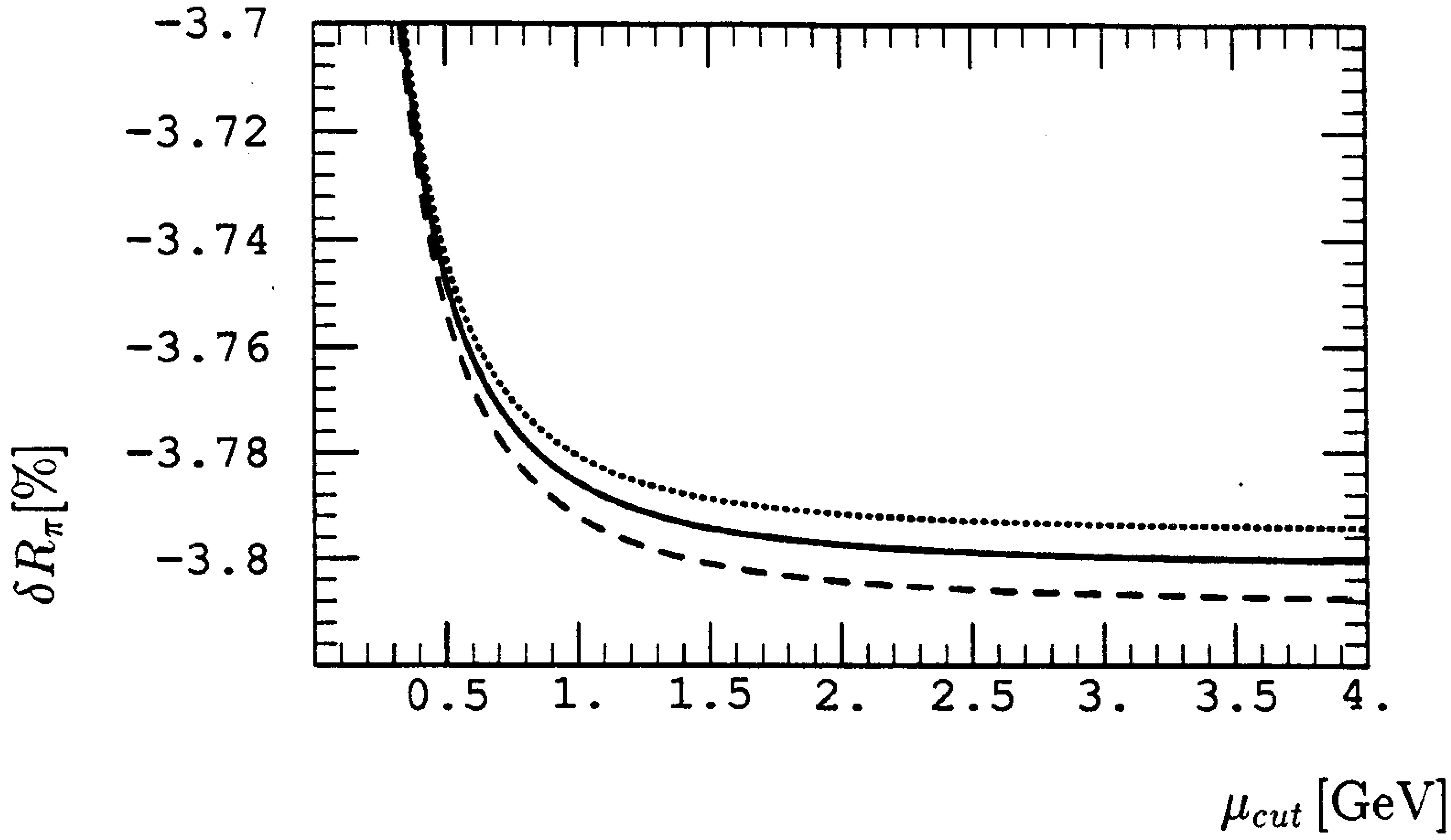

Figure 2: Radiative correction to the ratio R_π, using different choices for the hadronic parameters: Standard choice (I, solid) and variations (II, dashed and III, dotted)

dominance model corresponding to the dotted curve, is not a good approximation. Disregarding this model, the remaining scale uncertainty in the individual decay rates is of the order of $\pm 0.05\%$. In the ratio R_π these scale dependence cancels to a large extend, but this is even true for the paraemetrization of (III), which is certainly unrealistic.

Therefore we will now scrutinize the model dependence further in two ways: Firstly we examine from which scales the contributions to δR_π actually come. Secondly we compare the results from our model with the leading model independent contributions.

Consider Tab. 1. We display the contribution to the radiative corrections from photons with given Euklidean momenta $|k_E|$. We find that the contributions to the individual decay rates at large $|k_E^2|$ are quite sizeable. However, the contributions to the electronic and the muonic mode approach each other for large momenta, such that the contribution to the correction to the ratio R_π comes predominantly from very small scales. Uncertainties from the hadronics in the long distance regime and from QCD and wave function corrections in the short distance regime are large in the intermediate energy range of about $|k_E| = 500 \ldots 3000\,\text{MeV}$ only. The total contribution within this range is given by 0.026%, where we have added the absolute values in order to take care of cancelations. So by far the largest part of the radiative correction comes from the region below 500 MeV, where the model dependence is very small, see Figs. 1 and 2.

Next we will compare our model with model independent estimates. In [4] the author calculates the leading logarithmic corrections to the ratio R_π, which arise from hadronic structure dependend effects. He proves that this leading contribution is model independent, viz. independent on the form of the hadronic form factors. The only assumption needed is that the scale over which the form factor vary is given by a large hadronic scale of the order

Table 1: Contributions from photons with momenta within a given range to the various radiative corrections

Contribution from photons with $\|k_E\|$ in the range [MeV]:	Radiative correction to: (all numbers in units of %)					
	$\pi \to e\nu_e$	$\pi \to \mu\nu_\mu$	R_π	$K \to e\nu_e$	$K \to \mu\nu_\mu$	R_K
0...125	−3.893	−0.325	−3.569	−3.926	−0.515	3.411
125...250	−0.228	−0.077	−0.151	−0.450	−0.246	−0.204
250...500	−0.033	0.028	−0.061	−0.201	−0.115	−0.086
500...750	0.074	0.087	−0.013	−0.000	0.020	−0.020
750...1000	0.087	0.092	−0.005	0.047	0.055	−0.008
1000...1500	0.165	0.168	−0.003	0.121	0.126	−0.005
1500...3000	0.322	0.318	0.005	0.322	0.318	0.004
3000...m_Z	1.586	1.584	0.002	1.586	1.584	0.002

of m_ρ. The leading correction $\delta R_\pi^{(had)}$ can be seperated into three contributions

$$\delta R_\pi^{(had)} = \delta R_\pi^{(VMD)} + \delta R_\pi^{(v)} + \delta R_\pi^{(a)} \tag{34}$$

where $R_\pi^{(VMD)}$ is due to the vector meson dominance of the pion electromagnetic form factor, and $R_\pi^{(v/a)}$ correspond to virtual corrections proportional to the form factors $F_{V/A}(0)$. They are given by

$$\begin{aligned} \delta R_\pi^{(VMD)} &= \frac{3\alpha}{\pi}\frac{m_\mu^2}{m_\rho^2}\ln\frac{m_\rho^2}{m_\mu^2} \\ \delta R_\pi^{(v)} &= \frac{\alpha}{6\pi}\frac{F_V(0)}{\sqrt{2}m_\pi f_\pi}\left[m_\pi^2\ln\frac{m_\mu^2}{m_e^2} + 4m_\mu^2\ln\frac{m_\rho^2}{m_\mu^2}\right] \\ \delta R_\pi^{(a)} &= -\frac{\alpha}{6\pi}\frac{F_A(0)}{\sqrt{2}m_\pi f_\pi}\left[m_\pi^2\ln\frac{m_\mu^2}{m_e^2} + 7m_\mu^2\ln\frac{m_\rho^2}{m_\mu^2}\right] \end{aligned} \tag{35}$$

Note that $\delta R_\pi^{(v)}$ and $\delta R_\pi^{(a)}$ consist of two parts. The first one, being proportional to $\ln(m_\mu^2/m_e^2)$, modifies the coefficient C_{lms} of the lepton mass singularities (lms). The term $C_{lms}\alpha/\pi\ \ln(m_\mu/m_e)$, which in the effective point meson model comes with $C_{lms} = -3$, strongly dominates the total radiative correction. According to a theorem by Sirlin, the modification of C_{lms} by hadronic structure dependent effects in the virtual corrections is exactly canceled by hadronic structure dependent effects in the real radiative decay (viz. by the interference contribution INT between internal bremsstrahlung and structure dependent radiation), if *all* real photons are included in the decay rate. So in [6], where only π_{l2} decays are discussed, these terms are not considered at all. For K_{l2} decays, however, we do not count hard photons, and therefore this cancelation of the hadronic structure dependent corrections to the lepton mass singularity coefficient does not take place (see below).

Numerically, the leading hadronic structure effects in π_{l2} decays are

$$\delta R_\pi^{(VMD)} = 5.2 \cdot 10^{-4}$$

Table 2: The different contributions adding up to the total radiative corrections (the numbers in brackets are obtained assuming exact $SU(3)$ flavour symmetry

contribution	δR_π [%]	δR_K [%]
(1) effective point meson	-3.930	-3.786
(2) vector meson dominance in the point meson loops	0.053	0.048 (0.055)
(3) hadronic structure dependent loops proportional to $F_V(0)$	0.018	0.135
(4) hadronic structure dependent loops proportional to $F_A(0)$	-0.009	-0.134
(5) cutting off the long distance part at $\mu_{cut} = 1.5\,\text{GeV}$	0.000	0.003
(6) short distance corrections	0.007	0.006
(7) SD + INT: pure structure dependent (SD) radiation and its interference (INT) with the internal bremsstrahlung	0.066	—
(8) total	-3.794	-3.729 (-3.723)

$$\delta R_\pi^{(v)} = (1.2 + 1.0) \cdot 10^{-4} = 2.2 \cdot 10^{-4}$$

$$\delta R_\pi^{(a)} = -(0.5 + 0.8) \cdot 10^{-4} = -1.3 \cdot 10^{-4} \tag{36}$$

Now let us compare these numbers with the corresponding results from our model, see Tab. 2. In the first row (1) we give the results obtained with an effective point meson. In the second row (2) we display the change of the result, when switching on the vector meson dominance in the meson electromagnetic form factor. In rows (3) and (4) we give the contributions from those loop diagrams which correspond to the SD part in the real radiation. In (2)–(4), we have extended the loop integration up to m_Z, and so in row (5) we display the change when cutting off the long distance correction at $\mu_{cut} = 1.5\,\mathrm{GeV}$, and in row (6) we give the short distance correction.

Now comparing the model independent numbers 5.2, 2.2 and −1.3 with our numbers 5.3, 1.8 and −0.9 (hadronic structure dependent corrections in units of 10^{-4}), we find that the model dependent contribution in the long distance correction is extremely small, giving rise to an uncertainty which is certainly below $0.7 \cdot 10^{-4}$.

Let us now consider our result for the short distance correction, $0.7 \cdot 10^{-4}$, which includes contributions which depend on the pion wave function. However, we find (compare Eqns. (25–26)

$$\begin{aligned} r_\pi(u=0) &= 0.7 \cdot 10^{-4} \\ r_\pi(u=1) &= 0.9 \cdot 10^{-4} \end{aligned} \tag{37}$$

So for any choice of the pion wave function, the resulting short distance correction will be within the range $(0.7 \cdots 0.9) \cdot 10^{-4}$. This should also be compared with contribution from the leading lepton mass dependent logarithm,

$$\frac{2\alpha}{\pi} \frac{m_\mu^2}{m_\mu^2 - \mu_{cut}^2} \ln \frac{m_\mu}{\mu_{cut}} = 0.6 \cdot 10^{-4} \tag{38}$$

Thus in the short distance part, the model dependent contribution is of the order of $(0.1 \cdots 0.3) \cdot 10^{-4}$.

Finaly the uncertainty of the hadronic structure dependent contribution SD + INT in the radiative decay can also be neglected. From a variation of the form factors, we estimate an uncertainty of $\pm 0.4 \cdot 10^{-4}$.

And so by adding up linearly the moduli of the model dependent corrections, we obtain an uncertainty due to model dependence of ±0.014%. In view of this size of the model dependent corrections, we are convinced that our error estimate for the $O(\alpha)$ correction to R_π of ±0.020% given above in (33) is conservative and reliable.

However, we have to worry about corrections of higher order, $O(\alpha^n)$. Given the fact that the $O(\alpha)$ correction is dominated by the contribution from the lepton mass singularity, $-\frac{3\alpha}{\pi} \ln \frac{m_\mu}{m_e}$, in [6] the leading higher order corrections are estimated by summing up all such logs via the renormalization group, yielding a correction to R_π of

$$1 - \frac{\left(1 - \frac{2\alpha}{3\pi} \ln \frac{m_\mu}{m_e}\right)^{9/2}}{1 - \frac{3\alpha}{\pi} \ln \frac{m_\mu}{m_e}} = 5.5 \cdot 10^{-4} \tag{39}$$

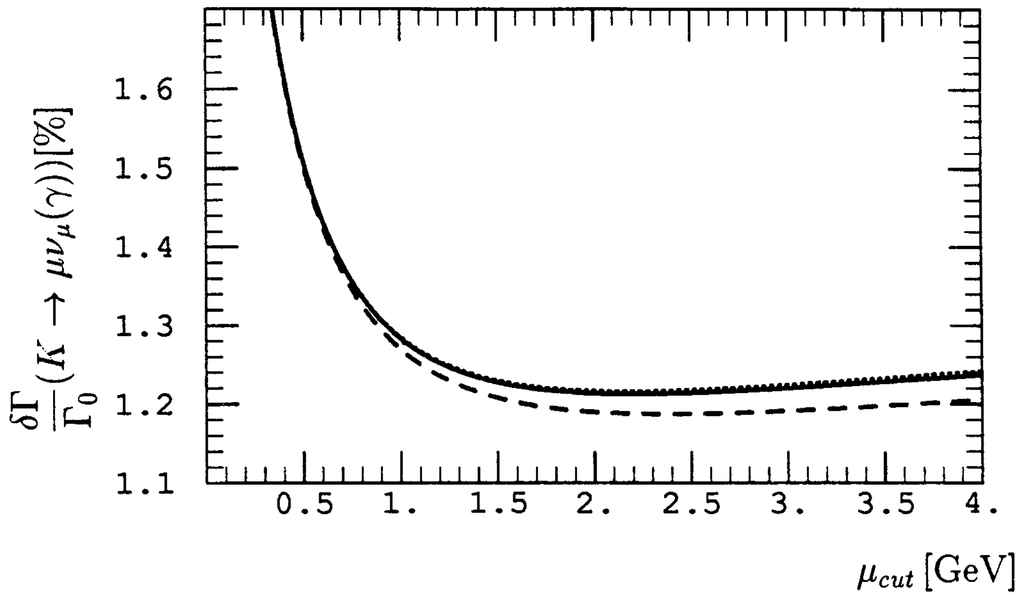

Figure 3: Radiative correction to $\Gamma(K \to \mu\nu_\mu)$, using different choices for the hadronic parameters: Standard choice (I, solid) and variations (II, dashed and III, dotted)

And so our final prediction for δR_π is

$$\delta R_\pi = (-3.794 \pm 0.020 + 0.055 \pm 0.01)\% = (-3.74 \pm 0.02)\% \tag{40}$$

In the sum, the first number (−3.794) is the central value and the second number the uncertainty of the $O(\alpha)$ correction. The third number (+0.055) is the leading higher order correction and ±0.01 our estimate of the next-to-leading correction.

For the ratio R_π this implies

$$R_\pi = R_\pi^{(0)}\left(1 + \delta R_\pi\right) = 1.2834 \cdot 10^{-4} \times \left(1 - 0.0374 \pm 0.0002\right) = (1.2354 \pm 0.0002) \cdot 10^{-4} \tag{41}$$

This agrees with the prediction $R_\pi = (1.2352 \pm 0.0005) \cdot 10^{-4}$ in [6] within their error estimate, but we have been able to further reduce the uncertainty.

Now we will consider K_{l2} decays. In Fig. 3 we show the radiative correction to $\Gamma(K \to \mu\nu_\mu)$ in variation with the matching scale, using central values for the hadronic parameters (I, solid) and reasonable variations (II and III, dashed and dash-dotted, respectively). From this we obtain

$$\frac{\delta\Gamma}{\Gamma_0}(K \to \mu\nu_\mu(\gamma)) = (1.23 \pm 0.13 \pm 0.02)\% + O(\alpha^2) + O(\alpha\alpha_s) \tag{42}$$

The first error given is the matching uncertainty and the second one is the uncertainty from the hadronic parameters.

Taking into account an additional +0.10% short distance correction, and uncertainties from higher order long distance corrections and from the very definition of the radiative correction as we did in the case of the pion decay, we use

$$\frac{\delta\Gamma}{\Gamma_0}(K \to \mu\nu_\mu(\gamma)) = (1.3 \pm 0.2)\% \tag{43}$$

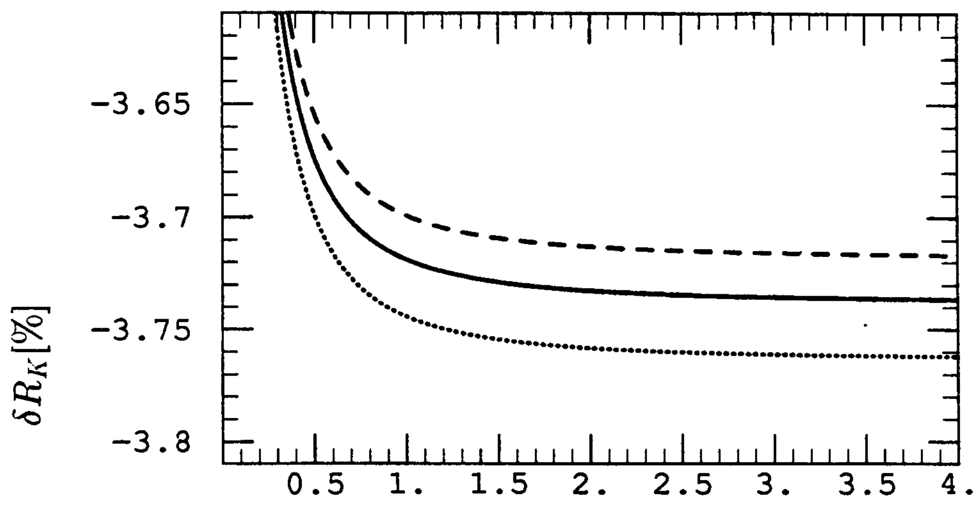

Figure 4: Radiative correction to the ratio R_K, using different choices for the hadronic parameters: Standard choice (I, solid) and variations (II, dashed and III, dotted)

to extract f_K from the exerimental decay rate. Using

$$\begin{aligned} \Gamma(K \to \mu\nu_\mu(\gamma)) &= (3.38 \pm 0.01) \cdot 10^{-14}\,\mathrm{MeV} \\ V_{us} &= 0.2205 \pm 0.0018 \end{aligned} \tag{44}$$

we obtain

$$f_K = (112.4 \pm 0.9 \pm 0.1)\,\mathrm{MeV} = (112.4 \pm 0.9)\,\mathrm{MeV} \tag{45}$$

where the first error is due to V_{us} and the second one to the radiative correction.

This implies

$$\frac{f_K}{f_\pi} = 1.22 \pm 0.01 \tag{46}$$

which agrees with the conventional value [19]. Note, however, that our error is entirely dominated by the uncertainty of V_{us}.

Finally we consider the ratio R_K. In Fig. 4 we display the radiative correction δR_K in variation with the matching scale, using the three parameter sets for the hadronics. We obtain

$$\delta R_K = -(3.729 \pm 0.023 \pm 0.025)\% + O(\alpha^2) \tag{47}$$

where the first error is the matching uncertainty and the second one the uncertainty from the hadronic parameters. In the case of R_K, the model independent logarithms are given by

$$\begin{aligned} \delta R_K^{(VMD)} &= \frac{3\alpha}{\pi}\frac{m_\mu^2}{m_\rho^2}\ln\frac{m_\rho^2}{m_\mu^2} = 5.2 \cdot 10^{-4} \\ \delta R_K^{(v)} &= \frac{\alpha}{6\pi}\frac{F_V(0)}{\sqrt{2}m_K f_K}\left[m_K^2 \ln\frac{m_\mu^2}{m_e^2} + 4m_\mu^2 \ln\frac{m_\rho^2}{m_\mu^2}\right] \end{aligned}$$

$$
\begin{aligned}
&= (14.8+1.0)\cdot 10^{-4} = 15.8\cdot 10^{-4} \\
\delta R_K^{(a)} &= -\frac{\alpha}{6\pi}\frac{F_A(0)}{\sqrt{2}m_K f_K}\left[m_K^2 \ln\frac{m_\mu^2}{m_e^2} + 7m_\mu^2 \ln\frac{m_\rho^2}{m_\mu^2}\right] \\
&= -(8.1+1.0)\cdot 10^{-4} = -9.1\cdot 10^{-4} \qquad (48)
\end{aligned}
$$

to be compared with our numbers (see Tab. 2) 4.8, 13.5 and −13.4. Adding up the model dependent contributions quadratically, we obtain an error estimate of $\pm 3.9\cdot 10^{-4}$.

Remember that we have included the $SU(3)$ flavour symmetry breaking in the electromagnetic form factor of the kaon, i.e. in row (2) of Tab. 2. But in the vector meson dominance of the photon coupling in the hadronic structure dependent loops, row (3) and (4), we used $m_\rho = m_\omega = m_\Phi = m_V = 768\,\text{MeV}$. However, as can be seen from (48), the contribution which depends on the vector meson mass m_V is very small, $O(1\cdot 10^{-4})$, in both cases, and the correction is strongly dominated by the modification of the ratio of lepton mass singularities. In fact we have checked that the result from our model in rows (3) and (4) changes only by 0.001% if we use increase m_V up to 1 GeV. So this approximation of flavour symmetry does not induce a significant uncertainty.

Thus we quote our final result for δR_K

$$\delta R_K = -(3.729 \pm 0.039 + 0.055 \pm 0.01)\% = (3.78 \pm 0.04)\% \qquad (49)$$

where in the sum, as for R_π, the first number is the $O(\alpha)$ correction, the second number is the uncertainty of the $O(\alpha)$ correction, the third number is the leading higher order correction and the last one is our estimate of the next-to-leading order corrections.

And so for R_K we predict

$$R_K = R_K^{(0)}\left(1+\delta R_K\right) = 2.569\cdot 10^{-5}\times\left(1 - 0.0378 \pm 0.0004\right) = (2.472 \pm 0.001)\cdot 10^{-5} \quad (50)$$

5 Summary and Conclusions

We have extracted the pseudoscalar decay constants f_π and f_K, and have predicted the ratios R_π and R_K of the electronic and muonic decay modes of the pion and the kaon, respectively. To this aim we have performed a matching calculation using a matching scale $\mu_{cut} = (0.75\ldots 3.0)\,\text{GeV}$. In the long distance part we have used a specific model which complies with the low energy theorems of QCD and which includes vector and axial vector resonances as explicit degrees of freedom. Regarding the hadronic structure dependent effects, we discussed both leading model independent effects and model dependent contributions. In the short distance part we used the parton model, resulting in a dominating model independent logarithm and a small additional contribution which depends on the parton distribution functions of the mesons. For the meson decay constants we have obtained

$$
\begin{aligned}
f_\pi &= (92.1 \pm 0.1)\,\text{MeV} \\
f_K &= (112.4 \pm 0.9)\,\text{MeV} \qquad (51)
\end{aligned}
$$

where we have by definition factored out all $O(\alpha)$ effects from the decay constants. This definition does not agree with the one used by Holstein [12], but it is identical to the one used by Marciano and Sirlin [6].

The errors given include the errors from the CKM matrix elements and an estimate of the model dependence, which is based on the dependence on the matching scale and on the hadronic parameters. Taking the full size of the mode dependent contribution as uncertainty, a very conservative error estimate would be $f_\pi = (92.1 \pm 0.3)$ MeV, whereas the error on f_K is dominated completeley by the uncertainty in V_{us} and not by model dependence.

For the ratios of the electronic and muonic decay modes, we predict

$$\begin{aligned} R_\pi = \frac{\Gamma(\pi \to e\nu_e(\gamma))}{\Gamma(\pi \to \mu\nu_\mu(\gamma))} &= (1.2354 \pm 0.0002) \cdot 10^{-4} \\ R_K = \frac{\Gamma(K \to e\nu_e(\gamma))}{\Gamma(K \to \mu\nu_\mu(\gamma))} &= (2.472 \pm 0.001) \cdot 10^{-5} \end{aligned} \tag{52}$$

In the case of the pion decays, we have included all radiative decays $\pi \to l\nu_l\gamma$, without any cut on the photon energy. In the case of the kaon decays, on the other hand, we have included only the internal bremsstrahlung part of the radiative decays and excluded completely the structure dependent radiation.

The central values include model dependent contributions, but the error bars given are based on the full size of these model dependent contributions, and therefore our final predictions for the ratios R_π and R_K are in fact model independent. Thus both π_{l2} and K_{l2} decays allow for low energy precision tests of the standard model, viz. of the electron-muon universality and of the (V, A) structure of the weak currents.

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